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Queues where customers of one queue act as servers of the other queue

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Abstract We consider a system comprised of two connected $M/M/\bullet/\bullet$ type queues, where customers of one queue act as servers for the other queue. One queue, Q_1, operates as a limited-buffer $M/M/1/N-1$ system. The other queue, Q_2, has an unlimited-buffer and receives service from the *customers* of Q_1. Such analytic models may represent applications like SETI@home, where idle computers of users are used to process data collected by space radio telescopes. Let L_1 denote the number of customers in Q_1. Then, two models are studied, distinguished by their service discipline in Q_2: In Model 1, Q_2 operates as an unlimited-buffer, single-server $M/M/1/\infty$ queue with Poisson arrival rate λ_2 and *dynamically changing* service rate $\mu_2 L_1$. In Model 2, Q_2 operates as a multi-server $M/M/L_1/\infty$ queue with varying number of servers, L_1, each serving at a Poisson rate of μ_2.

We analyze both models and derive the Probability Generating Functions of the system's steady-state probabilities. We then calculate the mean total number of customers present in each queue. Extreme cases are indicated.

Keywords Connected 2-queue systems · Customers act as servers · Markovian queues

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1 Introduction

Consider a system comprised of two connected queues as follows: one queue, Q_1, operates as a limited-buffer $M/M/1/N-1$ system with Poisson arrival rate λ_1, exponential service time with mean $1/\mu_1$ and a limited waiting room of size $N-1$. The other queue, Q_2, is *dependent* on Q_1 as follows: The customers present in Q_1 act as the *servers* of the customers in Q_2. Scenarios in which customers in a queue render service while waiting for their own service to start or to be completed are quite natural in networks comprised of nodes that can receive and provide service at the same time. An example related to computer networks is presented in Arazi, Ben-Jacob and Yechiali [1]. Another application arises in the field of "file sharing": Once a user activates a file sharing program, he operates simultaneously as a server for the other connected users and as a customer searching for a file. An additional application is the SETI@home project. This program, initiated by the Space Sciences Laboratory at the University of California, Berkeley, aims at searching for extraterrestrial intelligence, using radio telescopes. The process of analyzing the vast amount of collected data is assisted by volunteers, who install on their PCs a designated screen saver. Whenever such a PC is idle (for example, when it waits for a user's input), the screen saver is activated and SETI data is processed.

A model resembling our system, but different, was studied by Nunez Queija and Boxma [7]. A set of N servers attends two customer populations, each arriving as a Poisson process, where the first has a preemptive priority over the second and forms an $M/M/N/0$ queue with arrival and service rates λ_H and μ_H, respectively. The lower priority customers are served only when no higher priority customers are present and are served in a processor sharing fashion. That is, when $0 \le i \le N$ high priority customers and $j \ge 1$ other customers are present in the queues, each lower priority customer is served at a Poissonian rate of $(N-i)\mu_L/j$. A complete characterization of the joint distribution of the number of customers in the system in steady state is given.

In this work we consider two schemes of service for Q_2:

In Model 1 (Sect. 2) we assume that Q_2 operates as an infinite-buffer $M/M/1/\infty$ type system with Poisson arrival rate λ_2, but with dynamically changing service rate $\mu_2 L_1$, where the random variable L_1 denotes the number of customers present in Q_1. That is, the L_1 customers present in Q_1 join hands together and form *a single* server, having a combined service rate of $\mu_2 L_1$, for the customers in Q_2. In other words, the service rate at Q_2 changes according to the alternating queue size of Q_1. We formulate this model as a two-dimensional continuous-time Markov chain, study its steady-state behavior and show that the stability condition for Q_2 is simply $\lambda_2 < \mu_2 E[L_1]$. We then derive the (conditional) probability generating functions (PGFs) of the steady-state probabilities of the system's state. Deriving the generating functions requires the calculation of N 'boundary' probabilities. These probabilities are obtained by finding and characterizing the roots of a $(2N+1)$-degree polynomial being the determinant of a certain matrix whose entries are functions of the system's parameters.

Given the generating functions, we calculate numerically the mean total number of customers in Q_2. We further calculate $\mathrm{Cov}(L_1, L_2)$, showing numerically that it is ≤ 0, and use this fact to establish a lower bound for $E[L_2]$.

In Model 2 (Sect. 3) we assume a different service scheme for Q_2: Q_2 operates as an unlimited-buffer multi-server $M/M/L_1/\infty$ system with Poisson arrival rate λ_2 and service time, for each individual customer, exponentially distributed with mean $1/\mu_2$. That is, each customer present in Q_1 *individually* acts as a server for the customers in Q_2.

We show that the stability condition is the same as in Model 1, namely $\lambda_2 < \mu_2 E[L_1]$, and derive the PGFs of the steady-state probabilities of the system state. In this case, getting the generating functions requires the calculation of $\frac{N(N+1)}{2}$ probabilities, which are derived by utilizing the *same roots* used in Sect. 2 and, in addition, some of the balance equations.

The mean total number of customers in Q_2, $\mathrm{Cov}(L_1, L_2)$ and a lower bound for $E[L_2]$ are calculated. For both models extreme cases are indicated and a few numerical results are presented.

We note that since both models can be represented as quasi-birth-and-death (QBD) processes (see Latouche and Ramaswami [3]), their analysis can also be carried out using Matrix Geometric methods (see Neuts [6]), as presented in Perel and Yechiali [9]. Model 1 can be formulated as a Markov-modulated queue. Such modulation has been extensively studied in the literature. We mention a few papers, such as Takine [10], Mahabhashyam and Gautam [5], Ozawa [8], and refer the reader to the many references there. Specifically in [10], a single-server queue in a Markovian random environment with several customer classes is considered, where customer arrivals and service speed change with the fluctuating environment. In [5], the service speed of a single-server queue is determined by an external environment governed by a Markov process. A motivation for such formulation arises from computer systems applications. In [8], the stationary distribution of the sojourn time of a customer in a general QBD process is calculated via Matrix Geometric approach.

Model 2 is not a Markov-modulated queue since service rates at Q_2 may also change when the system resides in a given phase (number of customers in Q_1) of the non-homogeneous QBD process, and not only when a change of phase occurs, as is the case in Model 1.

2 Model 1

Consider two connected queueing systems operating as follows:

One queue, Q_1, is a limited-buffer $M(\lambda_1)/M(\mu_1)/1/N-1$ system as described in the Introduction. The other queue, Q_2, is an unlimited-buffer single-server $M(\lambda_2)/M(\mu_2 L_1)/1/\infty$ system, where L_1 is the number of customers in Q_1. That is, the service rate at Q_2 is changing dynamically, following the fluctuations in the queue size of Q_1.

2.1 Balance equations

Let L_j denote the total number of customers in Q_j, $j = 1, 2$. Then, the pair (L_1, L_2) defines a non-reducible continuous-time Markov process with transition rate diagram as shown in Fig. 1. Let $P_{nm} = P(L_1 = n, L_2 = m)$, $0 \le n \le N$ and $m = 0, 1, 2, \ldots$

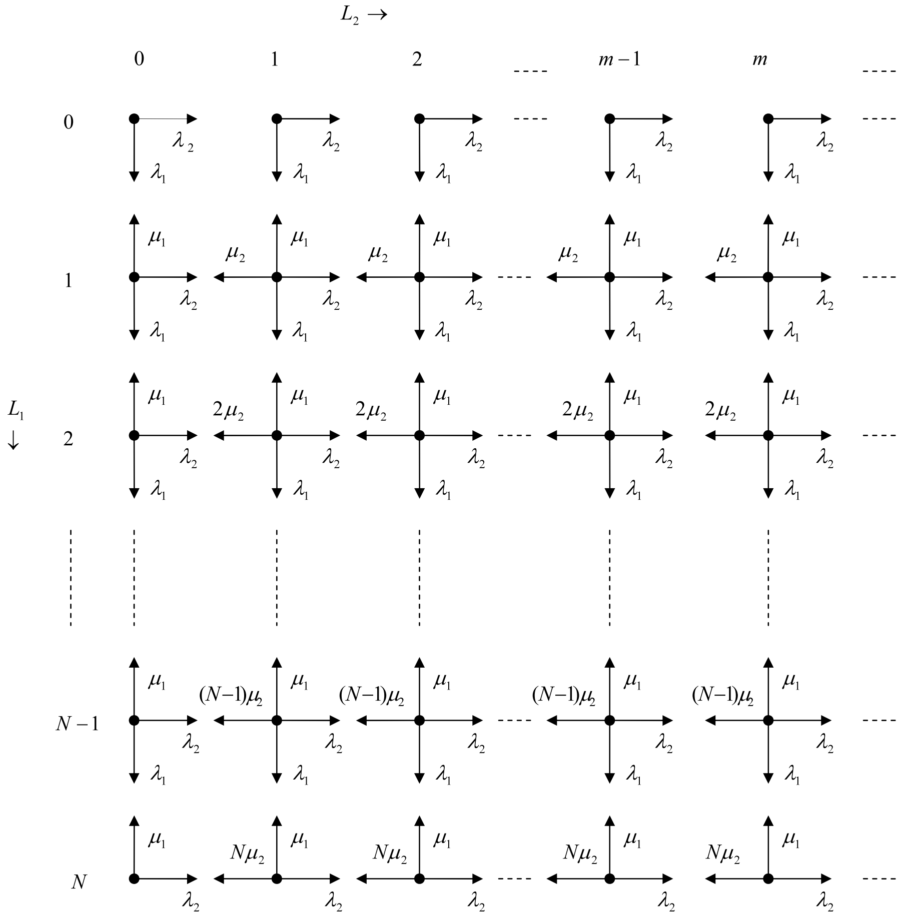

Fig. 1 Transition-rate diagram of (L_1, L_2) for Model 1

denote the system's stationary probabilities. Then, the set of balance equations is given by

$n=0$,

$$\begin{cases} m=0: \quad (\lambda_1+\lambda_2)P_{00}=\mu_1 P_{10}, \\ m\geq 1: \quad (\lambda_1+\lambda_2)P_{0m}=\lambda_2 P_{0,m-1}+\mu_1 P_{1m}, \end{cases} \tag{2.1}$$

$1\leq n\leq N-1$,

$$\begin{cases} m=0: \quad (\lambda_1+\lambda_2+\mu_1)P_{n0}=\lambda_1 P_{n-1,0}+\mu_1 P_{n+1,0}+n\mu_2 P_{n1}, \\ m\geq 1: \quad (\lambda_1+\lambda_2+\mu_1+n\mu_2)P_{nm}=\lambda_1 P_{n-1,m}+\lambda_2 P_{n,m-1}+\mu_1 P_{n+1,m} \\ \qquad\qquad +n\mu_2 P_{n,m+1}, \end{cases} \tag{2.2}$$

$n = N$,

$$\begin{cases} m = 0: & (\lambda_2 + \mu_1) P_{N0} = \lambda_1 P_{N-1,0} + N\mu_2 P_{N1}, \\ m \geq 1: & (\lambda_2 + \mu_1 + N\mu_2) P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1}. \end{cases} \tag{2.3}$$

Define

$$P_{n\bullet} = \sum_{m=0}^{\infty} P_{nm} \quad \text{for } 0 \leq n \leq N,$$

$$P_{\bullet m} = \sum_{n=0}^{N} P_{nm} \quad \text{for } m = 0, 1, 2, \ldots.$$

Summing (2.1), (2.2) and (2.3) over n for all $m \geq 0$ yields

$$\begin{aligned} m = 0, \quad & \lambda_2 P_{\bullet 0} = \mu_2 \sum_{n=0}^{\infty} n P_{n1} = \mu_2 P_{\bullet 1} E[L_1 | L_2 = 1], \\ 1 \leq m, \quad & \lambda_2 P_{\bullet m} + \mu_2 P_{\bullet m} E[L_1 | L_2 = m] = \lambda_2 P_{\bullet m-1} + \mu_2 P_{\bullet m+1} E[L_1 | L_2 \\ & \qquad = m+1]. \end{aligned}$$

This implies that for all $m \geq 0$,

$$\lambda_2 P_{\bullet m} = \mu_2 P_{\bullet m+1} E[L_1 | L_2 = m+1]. \tag{2.4}$$

By summing (2.4) over m, we get

$$\lambda_2 \sum_{m=0}^{\infty} P_{\bullet m} = \mu_2 \sum_{m=0}^{\infty} P_{\bullet m+1} E[L_1 | L_2 = m+1]. \tag{2.5}$$

Therefore,

$$\lambda_2 = \mu_2 (E[L_1] - P_{\bullet 0} E[L_1 | L_2 = 0]) = \mu_2 \left(E[L_1] - \sum_{n=1}^{N} n P_{n0} \right). \tag{2.6}$$

From (2.6) we get that, in steady state (see also Remark 2 in the sequel),

$$\frac{\lambda_2}{\mu_2} < E[L_1], \tag{2.7}$$

where $E[L_1] = \frac{\lambda_1}{\mu_1 - \lambda_1} - \frac{(N+1)\lambda_1^{N+1}}{\mu_1^{N+1} - \lambda_1^{N+1}}$ is the (well-known) mean queue size for the $M(\lambda_1)/M(\mu_1)/1/N-1$ queue.

Furthermore, by summing (2.1), (2.2) and (2.3) over m, we get

$$\begin{aligned} & n = 0, \quad \lambda_1 P_{0\bullet} = \mu_1 P_{1\bullet}, \\ & 1 \leq n \leq N-1, \quad \lambda_1 P_{n\bullet} + \mu_1 P_{n\bullet} = \lambda_1 P_{n-1,\bullet} + \mu_1 P_{n+1,\bullet}, \\ & n = N, \quad \lambda_1 P_{N-1,\bullet} = \mu_1 P_{N\bullet}. \end{aligned}$$

The above set of equations is the set of balance equations of the classical $M/M/1/N-1$ model, meaning that, for all $0 \le n \le N-1$, the following equation holds:

$$\lambda_1 P_{n\bullet} = \mu_1 P_{n+1\bullet}.$$

Therefore, for all $0 \le n \le N$,

$$P_{n\bullet} = \frac{(\frac{\lambda_1}{\mu_1})^n (1 - \frac{\lambda_1}{\mu_1})}{1 - (\frac{\lambda_1}{\mu_1})^{N+1}}. \tag{2.8}$$

2.2 Generating functions

Define, for each $0 \le n \le N$, the probability generating functions

$$G_n(z) = \sum_{m=0}^{\infty} P_{nm} z^m.$$

By multiplying each equation for m in (2.1), (2.2) and (2.3) by z^m, summing over all m and rearranging terms, we get

$$\begin{aligned} &n = 0,\\ &\big(\lambda_1 + \lambda_2(1-z)\big)G_0(z) = \mu_1 G_1(z), \end{aligned} \tag{2.9}$$

$$\begin{aligned} &1 \le n \le N-1,\\ &\big((\lambda_1 + \mu_1)\, z + (\lambda_2 z - n\mu_2)\,(1-z)\big)G_n(z) = \lambda_1 z G_{n-1}(z) + \mu_1 z G_{n+1}(z)\\ &\qquad\qquad - n\mu_2 P_{n0}(1-z), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &n = N,\\ &\big(\mu_1 z + (\lambda_2 z - N\mu_2)\,(1-z)\big)G_N(z) = \lambda_1 z G_{N-1}(z) - N\mu_2 P_{N0}(1-z). \end{aligned} \tag{2.11}$$

Thus, we get a system of linear equations of the form

$$A(z)\vec{G}(z) = -\mu_2(1-z)\vec{P},$$

where $A(z)$, $\vec{G}(z)$ and $\vec{P}$ are defined as follows:

$$\vec{G}(z) = (G_0(z), G_1(z), \ldots, G_N(z))^t,$$

$$\vec{P} = (0, P_{10}, 2P_{20}, \ldots, NP_{N0})^t,$$

$$A_{(N+1)\times(N+1)}(z) = \begin{pmatrix} \alpha_0(z) & -\mu_1 & 0 & \cdots & \cdots & 0 \\ -\lambda_1 z & \alpha_1(z) & -\mu_1 z & 0 & \cdots & 0 \\ 0 & \ddots & \alpha_2(z) & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & -\mu_1 z \\ 0 & \cdots & \cdots & 0 & -\lambda_1 z & \alpha_N(z) \end{pmatrix},$$

where

$$n=0, \quad \alpha_0(z)=\lambda_1+\lambda_2(1-z),$$
$$1\le n\le N-1, \quad \alpha_n(z)=(\lambda_1+\mu_1)z+(\lambda_2 z-n\mu_2)(1-z),$$
$$n=N, \quad \alpha_N(z)=\mu_1 z+(\lambda_2 z-N\mu_2)(1-z).$$

To obtain $G_n(z)$ we use Cramer's rule. That is, for all $0\le n\le N$,

$$G_n(z)=\frac{|A_n(z)|}{|A(z)|},$$

where $|A|$ is the determinant of the matrix A, and $A_n(z)$ is a matrix obtained from $A(z)$ by replacing the nth column by $-\mu_2(1-z)\vec{P}$. This leads to an expression of $G_n(z)$ in terms of N unknown probabilities $\{P_{n0}, 1\le n\le N\}$ appearing in $\vec{P}$. In order to get $\vec{P}$ we need to find N equations relating these N variables. We do that in the next section by characterizing and using the roots of $|A(z)|$. Since $G_n(z)$ is a probability generating function defined for all $0\le z\le 1$, each root of $|A(z)|$ in that interval is a root of $|A_n(z)|$ for every $0\le n\le N$.

2.3 Derivation of $P_{01}, P_{02}, \dots, P_{0N}$ and $E[L_2]$

Theorem 2.1 *For any $\lambda_1>0$, $\mu_1,\lambda_2\ge 0$, $\mu_2>0$ and $N\ge 1$, $|A(z)|$ is a polynomial of degree $2N+1$ possessing $N-1$ distinct roots in the open interval $(0,1)$, a single root at $z=1$, and N roots in the open interval $(1,\infty)$. Another root exists in the open interval $(0,1)$, provided that $\lambda_2>\mu_2 E[L_1]$, where $E[L_1]=\sum_{n=0}^{N} nP_{n\bullet}$.*

Proof Let $q_0(z)=1$. Define the minors of the diagonal of $A(z)$ starting from the higher left-side corner as follows:

$$q_1(z)=\alpha_0(z), \qquad q_2(z)=\begin{vmatrix}\alpha_0(z) & -\mu_1\\ -\lambda_1 z & \alpha_1(z)\end{vmatrix}, \quad \dots, \quad q_{N+1}(z)=|A(z)|. \tag{2.12}$$

The polynomials $q_n(z)$, $1\le n\le N+1$, satisfy the following equations:

$$\begin{aligned}
&q_1(z)=\alpha_0(z)q_0(z),\\
&q_2(z)=\alpha_1(z)q_1(z)-\lambda_1\mu_1 z q_0(z),\\
&q_n(z)=\alpha_{n-1}(z)q_{n-1}(z)-\lambda_1\mu_1 z^2 q_{n-2}(z) \quad \text{for } 3\le n\le N+1.
\end{aligned} \tag{2.13}$$

From (2.12) and (2.13) we conclude that

1. $q_0(z)=1$ and therefore has no roots.
2. $q_n(z)$ and $q_{n+1}(z)$ have no joint roots in $(0,\infty)$. Otherwise, suppose they have a joint root, then it must also be a root for $q_{n-1}(z), q_{n-2}(z), \dots, q_0(z)$ which contradicts 1.
3. Sign $(q_n(0))=(-1)^{n+1}$ for $1\le n\le N+1$.
4. Sign $(q_n(\infty))=(-1)^n$ for all n.

5. $q_n(1)=\lambda_1^n$ for $0\le n\le N$ and $q_{N+1}(1)=0$.
6. Sign $(\alpha_n(0))=-1$ for $1\le n\le N$.
7. Given $\tilde{z}>0$ is a root of $q_n(z)$, then $\text{sign}(q_{n-1}(\tilde{z})q_{n+1}(\tilde{z}))=-1$.
8. $q_n(z)$ is a polynomial of degree $2n-1$ for $1\le n\le N+1$.
9. For $n\le N$ the polynomial $q_n(z)$ has $2n-1$ distinct roots, where $n-1$ of them are in the open interval (0, 1) and the other n are in the open interval $(1,\infty)$.

From the above we conclude that $q_1(z)$ has only one root, $z_{1,1}=1+\lambda_1/\lambda_2>1$, and $q_2(0)<0, q_2(1)=\lambda_1^2>0, q_2(z_{1,1})<0, q_2(\infty)>0$. Therefore, the 3 roots of $q_2(z)$ satisfy $z_{2,1}\in(0,1), z_{2,2}\in(1,z_{1,1}), z_{2,3}\in(\mathrm{z}_{1,1},\infty)$. Similarly, $q_3(z)$ is of degree 5 and therefore it can have no more than 5 roots. Also, $q_3(0)>0$, $q_3(z_{2,1})<0$, $q_3(1)=\lambda_1^3>0$, $q_3(z_{2,2})<0$, $q_3(z_{2,3})>0$, $q_3(\infty)<0$. This implies that $q_3(z)$ has exactly 5 distinct roots satisfying $z_{3,1}\in(0,z_{2,1}), z_{3,2}\in(z_{2.1},1)$, $z_{3,3}\in(1,z_{2,2}), z_{3,4}\in(z_{2,2},z_{2,3}), z_{3,5}\in(z_{2,3},\infty)$. In general, for $2\le n\le N$, given $2n-3$ distinct roots of $q_{n-1}(z)$, the roots of $q_n(z)$ satisfy $z_{n,1}\in(0,z_{n-1,1})$, $z_{n,2}\in(z_{n-1,1},z_{n-1,2}),\dots,z_{n,n-1}\in(z_{n-1,n-2},1), z_{n,n}\in(1,z_{n-1,n-1}),\dots,z_{n,2n-1}\in(z_{n-1,2n-3},\infty)$. $q_{N+1}(z)$ has $2N+1$ roots, where the first $N-1$ are within the interval (0, 1) satisfying $z_{N+1,1}\in(0,z_{N,1}), z_{N+1,2}\in(z_{N,1},z_{N,2}),\dots,z_{N+1,N-1}\in(z_{N,N-2},z_{N,N-1})$.

As for the Nth root, $z_{N+1,N}$, we observe that, since $z_{N,N-1}\in(z_{N-1,N-2}, z_{N-1,N-1})$, where $z_{N-1,N-1}<1$, we have that $q_{N-1}(z_{N,N-1})>0$ and therefore

$$q_{N+1}(z_{N,N-1})=-\lambda_1\mu_1(z_{N,N-1})^2q_{N-1}(z_{N,N-1})<0.$$

$q_{N+1}(1)=0$, and we need to check whether another root (besides the $N-1$ already exhibited) exists in $(z_{N,N-1},1)$.

We will show that under stationary condition, such a root does not exist. In such a case, the $N-1$ distinct roots of $q_{N+1}(z)$ in (0, 1) will provide $N-1$ equations relating the N unknown probabilities $\{P_{n0}\}$ for $1\le n\le N$.

Define

$$\begin{aligned}
h_1(z)&=\lambda_2,\\
h_2(z)&=(\lambda_2z-\mu_2)\big(\lambda_1+\lambda_2(1-z)\big)+\lambda_2z(\lambda_1+\mu_1),\\
h_n(z)&=\lambda_1^{n-1}z^{n-2}\big(\lambda_2z-(n-1)\mu_2\big)\\
&\qquad+\big((\lambda_1+\mu_1)z+\big(\lambda_2z-(n-1)\mu_2\big)(1-z)\big)h_{n-1}(z)\\
&\qquad-\lambda_1\mu_1z^2h_{n-2}(z),\quad 3\le n\le N,\\
h_{N+1}(z)&=\lambda_1^Nz^{N-1}(\lambda_2z-N\mu_2)+\big(\mu_1z+(\lambda_2z-N\mu_2)(1-z)\big)h_N(z)\\
&\qquad-\lambda_1\mu_1z^2h_{N-1}(z).
\end{aligned}$$

By induction over n it can be shown that

$$q_n(z)=\lambda_1^nz^{n-1}+(1-z)h_n(z),\quad 1\le n\le N. \tag{2.14}$$

$$q_{N+1}(z)=(1-z)h_{N+1}(z). \tag{2.15}$$

Furthermore, from the discussion above it follows that another root exists in $(z_{N,N-1},1)$ if and only if $h_{N+1}(1)>0$.

Substituting $z = 1$ in $h_n(z)$ $(1 \le n \le N+1)$ gives

$$\begin{aligned}
&h_1(1) = \lambda_2, \\
&h_2(1) = (\lambda_2 - \mu_2)\lambda_1 + \lambda_2(\lambda_1 + \mu_1), \qquad (2.16)\\
&h_n(1) = \lambda_1^{n-1}\big(\lambda_2 - (n-1)\mu_2\big) + (\lambda_1 + \mu_1)h_{n-1}(1) - \lambda_1\mu_1 h_{n-2}(1), \quad 3 \le n \le N. \\
&h_{N+1}(1) = \lambda_1^N(\lambda_2 - N\mu_2) + \mu_1 h_N(1) - \lambda_1\mu_1 h_{N-1}(1). \qquad (2.17)
\end{aligned}$$

Solving the recursive system defined in (2.16) yields a general formula for $h_n(1)$, that is,

$$\begin{aligned}
h_n(1) &= \frac{\lambda_2(2(n+1)\lambda_1^n(\lambda_1-\mu_1)^2 + 2(\mu_1^{n+1} - \lambda_1^{n+1})(\lambda_1-\mu_1))}{2(\lambda_1-\mu_1)^3} \\
&\quad + \frac{\mu_2(-n(n+1)\lambda_1^n(\lambda_1-\mu_1)^2 + 2\lambda_1\mu_1(\mu_1^n - \lambda_1^n) + 2n\lambda_1^{n+1}(\lambda_1-\mu_1))}{2(\lambda_1-\mu_1)^3}.
\end{aligned} \tag{2.18}$$

Substituting (2.18) in (2.17) and rearranging terms yields a closed formula for $h_{N+1}(1)$,

$$\begin{aligned}
&h_{N+1}(1) \\
&\quad = \frac{\lambda_2((\lambda_1^{N+1} - \mu_1^{N+1})(\lambda_1-\mu_1)) - \mu_2((N+1)\lambda_1^{N+1}(\lambda_1-\mu_1) - \lambda_1(\lambda_1^{N+1} - \mu_1^{N+1}))}{(\lambda_1-\mu_1)^2}.
\end{aligned}$$

Since another root for $|A(z)|$ exists if and only if $h_{N+1}(1) > 0$,

$$\frac{\lambda_2((\lambda_1^{N+1} - \mu_1^{N+1})(\lambda_1-\mu_1)) - \mu_2((N+1)\lambda_1^{N+1}(\lambda_1-\mu_1) - \lambda_1(\lambda_1^{N+1} - \mu_1^{N+1}))}{(\lambda_1-\mu_1)^2} > 0.$$

This implies that another root exists if and only if

$$\frac{\lambda_2}{\mu_2} > \frac{\lambda_1}{\mu_1 - \lambda_1} - \frac{(N+1)\lambda_1^{N+1}}{\mu_1^{N+1} - \lambda_1^{N+1}} = E[L_1].$$

This completes the proof. □

Remark 1 The condition given in Theorem 2.1 regarding the existence of "another root" in the open interval (0, 1) contradicts the condition (2.7) for the system's stability. Thus, when the system is stable, meaning $\lambda_2 < \mu_2 E[L_1]$, we use only the $N-1$ roots in (0, 1) to find the N unknown probabilities $(P_{10}, P_{20}, \dots, P_{N0})$. These $N-1$ distinct roots provide us with $N-1$ equations, and together with (2.6) we have a set of N (independent) equations in the N unknowns $(P_{10}, P_{20}, \dots, P_{N0})$.

Note For $N = 2$ or $N = 3$, it is possible to show directly, by solving analytically (we omit the calculations), that the above-mentioned set is independent. For $N = 4$, our numerical calculations confirm this assertion. For larger values of N, it is conjectured (see also Avi-Itzhak and Mitrani [2], Levy and Yechiali [4] and Yechiali [11]) that the set is indeed independent.

Remark 2 One can possibly generalize the model and analyze the system assuming that the service rate at Q_2 is $\mu_2 f(n)$ (instead of $\mu_2 n$) whenever $L_1 = n$, where $f(0) = 0$, $f(n) \geq 0$ for all $n \geq 0$ and $f(n) > 0$ for at least one value of $1 \leq n \leq N$. This will lead to (2.6) being modified to

$$\lambda_2 = \mu_2 \left(E[f(L_1)] - \sum_{n=1}^{N} f(n) P_{n0} \right).$$

Since $L_1 \geq 0$ is bounded from above, $E[f(L_1)]$ is finite for every finite $f(n)$. Moreover, since a necessary and sufficient condition for stability of the system is $P_{\bullet 0} = \sum_{n=0}^{N} P_{n0} > 0$, it follows that, in steady state, $\sum_{n=1}^{N} f(n) P_{n0} > 0$. Thus, $\lambda_2 < \mu_2 E[f(L_1)]$. In addition, in such a case, one gets

$$\alpha_n(z) = (\lambda_1 + \mu_1) z + \big(\lambda_2 z - f(n)\mu_2\big)(1 - z), \quad 1 \leq n \leq N - 1,$$
$$\alpha_N(z) = \mu_1 z + \big(\lambda_2 z - f(N)\mu_2\big)(1 - z).$$

Now, in order to proceed and investigate the existence and number of roots of $|A(z)|$, it is necessary to specify $f(n)$.

The mean total number of customers in Q_2, $E[L_2]$, is obtained by summing the derivatives of $G_n(z)$ over n at $z = 1$. That is,

$$E[L_2] = \sum_{n=0}^{N} G_n'(1) = \sum_{n=0}^{N} E[L_2 | L_1 = n] P(L_1 = n).$$

Also, by multiplying (2.9) by z, summing it with (2.10) and (2.11) and rearranging terms, we get

$$\sum_{n=0}^{N} (\lambda_2 z - n\mu_2) G_n(z) = -\mu_2 \sum_{n=0}^{N} n P_{n0}.$$

Differentiating both sides of the above and setting $z = 1$ yields

$$\sum_{n=0}^{N} \lambda_2 G_n(1) + \sum_{n=0}^{N} (\lambda_2 - n\mu_2) G_n'(1) = \lambda_2 + \lambda_2 E[L_2] - \mu_2 E[L_1 \cdot L_2] = 0.$$

Therefore,

$$E[L_1 \cdot L_2] = \frac{\lambda_2 + \lambda_2 E[L_2]}{\mu_2} = \frac{\lambda_2}{\mu_2} + \frac{\lambda_2 E[L_2]}{\mu_2} = \frac{\lambda_2}{\mu_2}\big(1 + E[L_2]\big).$$

Thus,

$$\begin{aligned}
\mathrm{Cov}(L_1, L_2) = E[L_1 \cdot L_2] - E[L_1] \cdot E[L_2] &= \frac{\lambda_2}{\mu_2} + \frac{\lambda_2}{\mu_2} E[L_2] - E[L_1] \cdot E[L_2] \\
&= \frac{\lambda_2}{\mu_2} - E[L_2]\left(E[L_1] - \frac{\lambda_2}{\mu_2} \right).
\end{aligned}$$

In steady state, since $\frac{\lambda_2}{\mu_2} < E[L_1]$, $\mathrm{Cov}(L_1, L_2) < \frac{\lambda_2}{\mu_2}$. Moreover, we argue that $\mathrm{Cov}(L_1, L_2) \le 0$, since increasing values of L_1 reduce the magnitude of L_2 (see numerical examples in Sect. 2.4). Hence, we can derive a lower bound for $E[L_2]$,

$$E[L_2] \ge \frac{\frac{\lambda_2}{\mu_2}}{E[L_1] - \frac{\lambda_2}{\mu_2}} = \frac{\lambda_2}{\mu_2 E[L_1] - \lambda_2}.$$

Clearly, by Little's Law, $E[W_2] = \frac{E[L_2]}{\lambda_2}$.

2.4 An example and numerical results

We present an analytic solution for the case $N = 1$ and numerical results for the cases where $N = 2, 3$ and 4.

$N = 1$: No waiting room for customers in Q_1.

2.4.1 Balance equations

$$n = 0 \quad \begin{cases} m = 0: & (\lambda_1 + \lambda_2) P_{00} = \mu_1 P_{10}, \\ m \ge 1: & (\lambda_1 + \lambda_2) P_{0m} = \lambda_2 P_{0,m-1} + \mu_1 P_{1m}, \end{cases}$$

$$n = 1 \quad \begin{cases} m = 0: & \lambda_2 + \mu_1) P_{10} = \lambda_1 P_{00} + \mu_2 P_{11} \\ m \ge 1: & (\lambda_2 + \mu_1 + \mu_2) P_{1m} = \lambda_1 P_{0m} + \lambda_2 P_{1,m-1} + \mu_2 P_{1,m+1}. \end{cases}$$

By summing these equations and rearranging terms, we obtain

$$\frac{\lambda_2}{\mu_2} = E[L_1] - P_{10}.$$

Therefore, using (2.7) for $N = 1$,

$$P_{10} = E[L_1] - \frac{\lambda_2}{\mu_2} = \frac{\lambda_1}{\lambda_1 + \mu_1} - \frac{\lambda_2}{\mu_2},$$

$$P_{00} = \frac{\mu_1}{\lambda_1 + \lambda_2} P_{10} = \frac{\mu_1}{\lambda_1 + \lambda_2} \left(\frac{\lambda_1}{\lambda_1 + \mu_1} - \frac{\lambda_2}{\mu_2} \right).$$

The stability condition is $\lambda_2 < \mu_2 \frac{\lambda_1}{\lambda_1 + \mu_1}$.

2.4.2 Generating functions

We have

$$\big[\lambda_1 + \lambda_2(1 - z)\big] G_0(z) = \mu_1 G_1(z), \tag{2.19}$$

$$\big[\mu_1 z + (\lambda_2 z - \mu_2)(1 - z)\big] G_1(z) = \lambda_1 z G_0(z) - \mu_2 P_{10}(1 - z). \tag{2.20}$$

Therefore,

$$G_0(z) = \frac{\mu_1}{\lambda_1 + \lambda_2(1-z)} G_1(z). \tag{2.21}$$

Substituting (2.21) into (2.20) yields

$$\big[\mu_1 z + (\lambda_2 z - \mu_2)(1-z)\big] G_1(z) = \frac{\lambda_1 \mu_1 z}{\lambda_1 + \lambda_2(1-z)} G_1(z) - \mu_2 P_{10}(1-z).$$

By rearranging terms and substituting P_{10}, we get

$$G_1(z) = \frac{\mu_2(\frac{\lambda_1}{\lambda_1+\mu_1} - \frac{\lambda_2}{\mu_2})(\lambda_1 + \lambda_2(1-z))}{(\mu_2 - \lambda_2 z)(\lambda_1 + \lambda_2(1-z)) - \mu_1 \lambda_2 z}. \tag{2.22}$$

Substituting (2.22) into (2.21) yields

$$G_0(z) = \frac{\mu_1 \mu_2(\frac{\lambda_1}{\lambda_1+\mu_1} - \frac{\lambda_2}{\mu_2})}{(\mu_2 - \lambda_2 z)(\lambda_1 + \lambda_2(1-z)) - \mu_1 \lambda_2 z}. \tag{2.23}$$

Note Indeed, following Theorem 2.1, the probabilities P_{00} and P_{10} are derived directly from the balance equations since $q_1(z)$ possesses no roots in (0, 1).

The mean total number of customers in Q_2, $E[L_2]$, is obtained by summing $G_0'(z)$ and $G_1'(z)$ at $z = 1$. After rearranging terms, we get

$$\begin{aligned} E[L_2] &= G_0'(1) + G_1'(1) \\ &= \mu_2 \lambda_2 \left(\frac{\lambda_1}{\lambda_1 + \mu_1} - \frac{\lambda_2}{\mu_2}\right) \frac{\frac{(\lambda_1+\mu_1)^2 + \mu_1\mu_2}{(\lambda_1+\mu_1)^2 \mu_2^2}}{(\frac{\lambda_1}{\lambda_1+\mu_1} - \frac{\lambda_2}{\mu_2})^2} = \frac{\lambda_2(\frac{1}{\mu_2} + \frac{\mu_1}{(\lambda_1+\mu_1)^2})}{(\frac{\lambda_1}{\lambda_1+\mu_1} - \frac{\lambda_2}{\mu_2})}. \end{aligned}$$

Therefore, $\mathrm{Cov}(L_1, L_2)$ is given by

$$\begin{aligned} \mathrm{Cov}(L_1, L_2) &= \frac{\lambda_2}{\mu_2} - E[L_2]\left(E[L_1] - \frac{\lambda_2}{\mu_2}\right) \\ &= \frac{\lambda_2}{\mu_2} - \frac{\lambda_2(\frac{1}{\mu_2} + \frac{\mu_1}{(\lambda_1+\mu_1)^2})}{(\frac{\lambda_1}{\lambda_1+\mu_1} - \frac{\lambda_2}{\mu_2})} \left(\frac{\lambda_1}{\lambda_1 + \mu_1} - \frac{\lambda_2}{\mu_2}\right) \\ &= \frac{\lambda_2}{\mu_2} - \frac{\lambda_2}{\mu_2} - \frac{\lambda_2 \mu_1}{(\lambda_1 + \mu_1)^2} = -\frac{\lambda_2 \mu_1}{(\lambda_1 + \mu_1)^2} < 0. \end{aligned}$$

2.4.3 Numerical results for $N = 2$, 3 and 4

We use the following set of parameters for cases where $N = 2, 3$ and 4:

Table 1 Numerical results for $N = 2, 3$ and 4

	P_{10}	P_{20}	P_{30}	P_{40}	$E[L_1]$	Lower bound for $E[L_2]$	$E[L_2]$	$\mathrm{Cov}(L_1, L_2)$
$N = 2$	0.02220	0.02399	–	–	14/19	19/2	18.66281	−0.643004
$N = 3$	0.054203	0.062544	0.056476	–	66/65	65/34	5.28405	−1.17598
$N = 4$	0.056316	0.065796	0.062385	0.049994	262/211	211/182	4.299212	−1.80555

$\lambda_1 = 2$, $\mu_1 = 3$, $\lambda_2 = 2$ and $\mu_2 = 3$. Notice that the stability condition $\lambda_2 < \mu_2 E[L_1]$ is valid for all values of N considered. The results are presented in Table 1.

2.5 Extreme cases

(i) $\lambda_2 \to 0$, or $\mu_2 \to \infty$

In both cases, using (2.9), (2.10) and (2.11), rearranging terms and comparing coefficients of z^m, we get that $P_{nm} = 0$ for all $1 \le n \le N$ and for all $m \ge 1$. Therefore, $G_n(z) = P_{n0}$, for all $1 \le n \le N$. It then also follows that $G_0(z) = P_{00}$. That is, in both cases, the system shrinks to an $M(\lambda_1)/M(\mu_1)/1/N-1$ *single* queue.

(ii) $\lambda_1 \to \infty$, or $\mu_1 \to 0$

In these cases, there are always N customers present in Q_1 (all busy together serving the customers of Q_2), implying that $P_{n\bullet} = 0$ for all $0 \le n \le N-1$ and $P_{N\bullet} = 1$. Furthermore, $P_{N0} = 1 - \frac{\lambda_2}{N\mu_2}$ and $G_N(z) = \frac{1-\rho}{1-\rho z}$, where $\rho = \frac{\lambda_2}{N\mu_2}$. That is, the system shrinks to a regular $M(\lambda_2)/M(N\mu_2)/1/\infty$ *single* queue. In this case $E[L_2] = \lambda_2/(N\mu_2 - \lambda_2) =$ lower bound for $E[L_2]$.

3 Model 2

In this model Q_1 operates as in Model 1, but Q_2 operates as an unlimited-buffer $M(\lambda_2)/M(\mu_2)/L_1/\infty$ system. That is, each customer present in Q_1 individually acts as a server for customers in Q_2.

3.1 Balance equations

The pair (L_1, L_2) defines a continuous-time Markov process with transition rate diagram as shown in Fig. 2. The set of balance equations is given as follows:

$n = 0$,

$$\begin{cases} m = 0: & (\lambda_1 + \lambda_2) P_{00} = \mu_1 P_{10}, \\ m \ge 1: & (\lambda_1 + \lambda_2) P_{0m} = \lambda_2 P_{0,m-1} + \mu_1 P_{1m}, \end{cases} \tag{3.1}$$

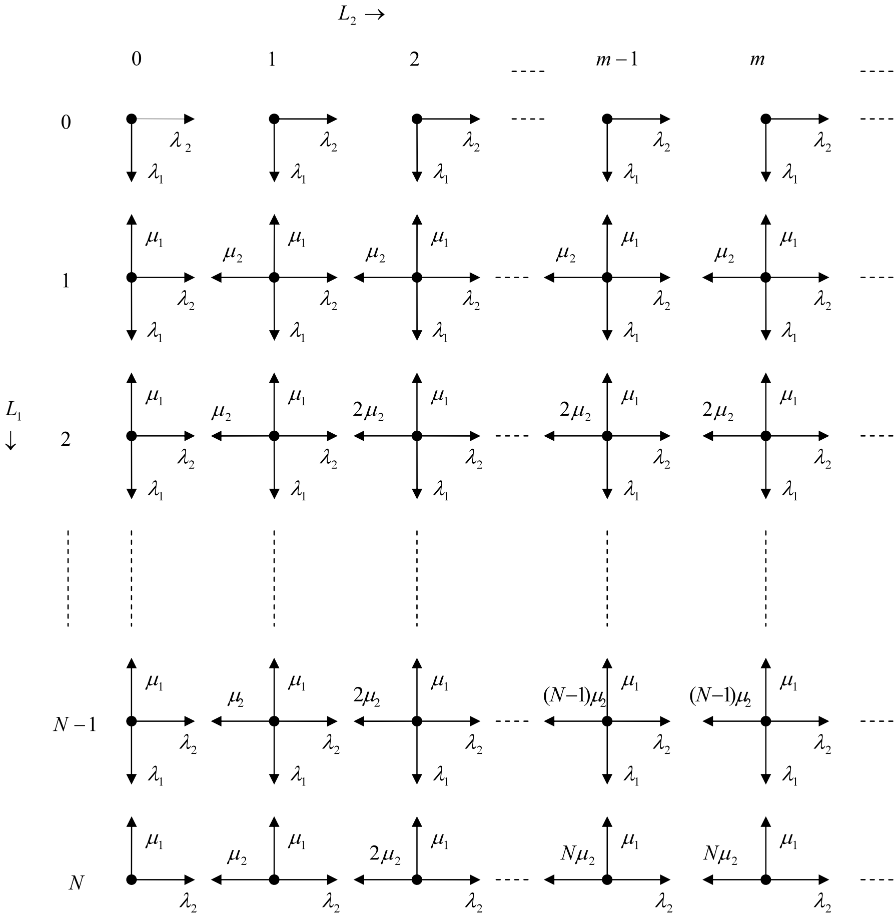

Fig. 2 Transition-rate diagram of (L_1, L_2) for Model 2

$$1 \le n \le N-1,$$

$$\begin{cases} m=0: \quad (\lambda_1+\lambda_2+\mu_1)P_{n0} = \lambda_1 P_{n-1,0} + \mu_1 P_{n+1,0} + \mu_2 P_{n1}, \\ 1 \le m \le n-1: \quad (\lambda_1+\lambda_2+\mu_1+m\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} \\ \qquad\qquad + \mu_1 P_{n+1,m} \\ \qquad\qquad + (m+1)\mu_2 P_{n,m+1}, \\ m \ge n: \quad (\lambda_1+\lambda_2+\mu_1+n\mu_2)P_{nm} = \lambda_1 P_{n-1,m} + \lambda_2 P_{n,m-1} + \mu_1 P_{n+1,m} \\ \qquad\qquad + n\mu_2 P_{n,m+1}, \end{cases} \tag{3.2}$$

$$n = N,$$

$$\begin{cases} m=0: \quad (\lambda_2+\mu_1)P_{N0} = \lambda_1 P_{N-1,0} + \mu_2 P_{N1}, \\ 1 \le m \le N-1: \quad (\lambda_2+\mu_1+m\mu_2)P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} \\ \qquad\qquad + (m+1)\mu_2 P_{N,m+1}, \\ m \ge N: \quad (\lambda_2+\mu_1+N\mu_2)P_{Nm} = \lambda_1 P_{N-1,m} + \lambda_2 P_{N,m-1} + N\mu_2 P_{N,m+1}. \end{cases} \tag{3.3}$$

Let $P_{n\bullet}$ and $P_{\bullet m}$ be defined as in Model 1, Sect. 2.1. Then, by summing each set of (3.1), (3.2) and (3.3) over n, we get

$$\begin{aligned} 0 \le m \le N-1, \quad \lambda_2 P_{\bullet m} &= \mu_2 P_{\bullet m+1} E[L_1 \,|\, L_2 = m+1] \\ &\quad - \mu_2 \sum_{n=m+1}^{N} (n-m-1) P_{n,m+1}, \qquad (3.4) \\ N \le m, \quad \lambda_2 P_{\bullet m} &= \mu_2 P_{\bullet m+1} E[L_1 \,|\, L_2 = m+1]. \end{aligned}$$

Now, summing (3.4) over m, we get

$$\lambda_2 = \mu_2 \left(E[L_1] - \sum_{n=1}^{N} \sum_{m=0}^{n-1} (n-m) P_{n,m+1} \right). \qquad (3.5)$$

From (3.5) it follows that, in steady state,

$$\frac{\lambda_2}{\mu_2} < E[L_1] = \frac{\lambda_1}{\mu_1 - \lambda_1} - \frac{(N+1)\lambda_1^{N+1}}{\mu_1^{N+1} - \lambda_1^{N+1}}. \qquad (3.6)$$

Notice that result (3.6) is the same as in Model 1, and $P_{n\bullet}$ is again given by (2.8).

3.2 Generating functions

As in Model 1, define, for each $0 \le n \le N$, the Probability Generating Function

$$G_n(z) = \sum_{m=0}^{\infty} P_{nm} z^m. \qquad (3.7)$$

By multiplying each equation for m in (3.1), (3.2) and in (3.3) by z^m, summing over m and rearranging terms, we get

$$\begin{aligned} &n = 0, \\ &\big(\lambda_1 + \lambda_2(1-z)\big) G_0(z) = \mu_1 G_1(z), \end{aligned} \qquad (3.8)$$

$$\begin{aligned} &1 \le n \le N-1, \\ &\big((\lambda_1 + \mu_1) z + (\lambda_2 z - n\mu_2)(1-z)\big) G_n(z) = \lambda_1 z G_{n-1}(z) + \mu_1 z G_{n+1}(z) \\ &\qquad\qquad - \mu_2 (1-z) \sum_{m=0}^{n-1} (n-m) P_{nm} z^m, \end{aligned} \qquad (3.9)$$

$$\begin{aligned} &n = N, \\ &\big(\mu_1 z + (\lambda_2 z - N\mu_2)(1-z)\big) G_N(z) = \lambda_1 z G_{N-1}(z) \\ &\qquad\qquad - \mu_2 (1-z) \sum_{m=0}^{N-1} (N-m) P_{Nm} z^m. \end{aligned} \qquad (3.10)$$

Thus we get a system of linear equations of the form

$$A(z)\vec{G}(z) = -\mu_2(1-z)\vec{P}(z),$$

where $A(z)$ and $\vec{G}(z)$ are the same as in Model 1, but $\vec{P}(z)$ is given by

$$\big(\vec{P}(z)\big)_n = \begin{cases} 0, & n = 0, \\ \sum_{m=0}^{n-1}(n-m)P_{nm}z^m, & 1 \le n \le N. \end{cases}$$

To obtain $G_n(z)$, we again use Cramer's rule, i.e. for all $0 \le n \le N$,

$$G_n(z) = \frac{|A_n(z)|}{|A(z)|},$$

where $|A|$ is the determinant of the matrix A, and $A_n(z)$ is a matrix obtained from $A(z)$ by replacing the nth column by $-\mu_2(1-z)\vec{P}(z)$. This leads to an expression of $G_n(z)$ in terms of $\frac{N(N+1)}{2}$ unknown probabilities appearing in $\vec{P}(z)$.

3.3 Derivation of $P_{10}, P_{20}, P_{21}, \ldots, P_{N0}, P_{N1}, \ldots, P_{N,N-1}$ and $E[L_2]$

In order to get $P_{10}, P_{20}, P_{21}, \ldots, P_{N0}, P_{N1}, \ldots, P_{N,N-1}$ we need to find $\frac{N(N+1)}{2}$ equations relating these $\frac{N(N+1)}{2}$ variables. We will do that by using the roots of $|A(z)|$. By Theorem 2.1, $|A(z)|$ has $N-1$ distinct roots in the open interval (0, 1) (for $\lambda_1 > 0, \mu_1, \lambda_2 \ge 0, \mu_2 > 0$ and provided that $\lambda_2 < \mu_2 E[L_1]$, where $E[L_1] = \sum_{n=0}^{N} nP_{n\bullet}$). Another $\frac{N(N-1)}{2}$ equations are taken from the balance equations for states $(n, m), 2 \le n \le N, 0 \le m \le n-2$. The last equation is (3.5). Thus we have $\frac{N(N+1)}{2}$ equations relating these $\frac{N(N+1)}{2}$ variables as requested.

The mean total number of customers in Q_2, $E[L_2]$, is given by

$$E[L_2] = \sum_{n=0}^{N} G_n'(1).$$

Repeating the process described in Sect. 2.3 for (3.8), (3.9) and (3.10) yields

$$\begin{aligned}\sum_{n=0}^{N}\lambda_2 G_n(1) + \sum_{n=0}^{N}(\lambda_2 - n\mu_2)G_n'(1) &= \lambda_2 + \lambda_2 E[L_2] - \mu_2 E[L_1 \cdot L_2] \\ &= -\mu_2 \sum_{n=2}^{N}\sum_{m=1}^{n-1} m(n-m)P_{nm}.\end{aligned}$$

Therefore,

$$\begin{aligned}E[L_1 \cdot L_2] &= \frac{\lambda_2 + \lambda_2 E[L_2] + \mu_2 \sum_{n=2}^{N}\sum_{m=1}^{n-1} m(n-m)P_{nm}}{\mu_2} \\ &= \frac{\lambda_2}{\mu_2}(1 + E[L_2]) + \sum_{n=2}^{N}\sum_{m=1}^{n-1} m(n-m)P_{nm}.\end{aligned}$$

Thus,

$$\begin{aligned}\mathrm{Cov}(L_1,L_2) &= E[L_1\cdot L_2]-E[L_1]\cdot E[L_2]\\ &= \frac{\lambda_2}{\mu_2}+\sum_{n=2}^{N}\sum_{m=1}^{n-1}m(n-m)P_{nm}+\frac{\lambda_2}{\mu_2}E[L_2]-E[L_1]\cdot E[L_2]\\ &= \frac{\lambda_2}{\mu_2}+\sum_{n=2}^{N}\sum_{m=1}^{n-1}m(n-m)P_{nm}-E[L_2]\left(E[L_1]-\frac{\lambda_2}{\mu_2}\right).\end{aligned}$$

In steady state $\frac{\lambda_2}{\mu_2}<E[L_1]$, therefore $\mathrm{Cov}(L_1,L_2)<\frac{\lambda_2}{\mu_2}+\sum_{n=2}^{N}\sum_{m=1}^{n-1}m(n-m)P_{nm}$. As in Sect. 2.3 we argue that $\mathrm{Cov}(L_1,L_2)\le 0$ and using this fact we can derive a lower bound for $E[L_2]$,

$$E[L_2]\ge\frac{\frac{\lambda_2}{\mu_2}+\sum_{n=2}^{N}\sum_{m=1}^{n-1}m(n-m)P_{nm}}{E[L_1]-\frac{\lambda_2}{\mu_2}}=\frac{\lambda_2+\mu_2\sum_{n=2}^{N}\sum_{m=1}^{n-1}m(n-m)P_{nm}}{\mu_2E[L_1]-\lambda_2}.$$

3.4 Numerical results for $N=2,3$

We use the same set of parameters as in Sect. 2.4, namely $\lambda_1=2$, $\mu_1=3$, $\lambda_2=2$ and $\mu_2=3$ for $N=2$ and 3. The results are presented in Table 2.

3.5 Extreme cases

(i) $\lambda_2\to 0$, or $\mu_2\to\infty$

These cases are identical to their counterpart in Model 1 presented in Sect. 2.5.

(ii) $\lambda_1\to\infty$, or $\mu_1\to 0$

These cases are also similar to the corresponding cases in Model 1. The only difference is that now the system shrinks to an $M(\lambda_2)/M(\mu_2)/N/\infty$ multi-server queue.

Remark 3 We note, again, that both models can be formulated in terms of a Matrix Geometric system (see [9]). In such a case the numerical calculations of the roots of $|A(z)|$ are replaced by numerical procedures for calculating the matrix R (see [6] and [3]). Furthermore, the sojourn time distribution of Q_2-customers in Model 1 can be obtained using the methods presented in [8]. Clearly, the sojourn time of a Q_1-customer is the sojourn time of a customer in the regular $M/M/1/N-1$ queue.

Table 2 Numerical results for $N=2$ and $N=3$

	P_{10}	P_{20}	P_{21}	P_{30}	P_{31}	P_{32}	$E[L_1]$	Lower bound for $E[L_2]$	$E[L_2]$	$\mathrm{Cov}(L_1,L_2)$
$N=2$	0.02018	0.02002	0.00996	–	–	–	14/19	9.642	18.7683	−0.65041
$N=3$	0.04464	0.04374	0.03716	0.03515	0.02942	0.01513	66/65	2.1673	5.61544	−1.16527

4 Conclusions

This paper presents a model, never studied before in the queueing literature, of a system of two connected queues where customers of one queue act as the servers of the other queue. Two models are analyzed: (i) when all customers of Q_1 act as one group, forming a single server for Q_2 (Model 1), and (ii) when each customer in Q_1 operates individually as a server for Q_2 (Model 2). It is shown that the stability condition is the same for both models, but the mean total queue size in Q_2, $E[L_2]$, is smaller in Model 1 than in Model 2. This follows since the service scheme applied for Q_2 in Model 1 is more efficient than in Model 2. That is, in Model 1, when $L_1 > L_2$, the departure rate of customers from Q_2 is $L_1\mu_2$ and *all* L_1 customers present in Q_1 contribute to reduce the size of L_2. However, in Model 2, when $L_1 > L_2$, only L_2 customers of Q_1 act as servers while the rest $L_1 - L_2$ remain idle. When $L_1 \leq L_2$, the departure rate is the same in both models. We also calculate $\mathrm{Cov}(L_1, L_2)$, indicate extreme cases and present several numerical examples.

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