6 Symmetric and asymmetric[1]

6a A simple counterexample

Consider an asymmetric first price auction with n players. That is, payoff functions $\mathbf{\Pi}_1 = \dots = \mathbf{\Pi}_n = \mathbf{\Pi}$ are standard, given by (5a3) with $\mathbf{G}$ of (3e1) and $\mathbf{L}$ of (3d1). The asymmetry is in signal distributions $P_{S_1}, \dots, P_{S_n}$; they need not be the same. Accordingly, an equilibrium is not expected to be symmetric.

Let $(\mu_1, \dots, \mu_n)$ be an equilibrium, then μ_1 (as well as any μ_k) is a best response to others, $\mu_2, \dots, \mu_n$ (recall (5a6)); results of 2c are applicable (as was noted on page 56). Especially, by Corollary 2c5,

- μ_k is a weakly increasing strategy except, maybe, never-wining actions.

The reservation about never-wining actions was eliminated in 3b1, 3b2 using symmetry. Now, having no symmetry, we must be in earnest about the reservation.

The following conjectures may seem to be natural, but are generally wrong:

- If signal distributions are nonatomic then action distributions (in an equilibrium) are also nonatomic. (WRONG)
- An action (in an equilibrium) never exceeds the signal. (WRONG)
- An equilibrium is unique. (WRONG)

Here is a counterexample. Let $n = 3$, $S_1 \sim \mathrm{U}(1,2)$, $S_2 \sim \mathrm{U}(3,4)$, $S_3 \sim \mathrm{U}(3,4)$. Let a_0 be an arbitrary number of $[0,3]$. Consider such a triple of strategies:

$$\begin{aligned} A_1 &= a_0 \quad \text{always (irrespective of } S_1) \, ; \\ A_2 &= 3 + \frac{1}{2}(S_2 - 3) \, ; \\ A_3 &= 3 + \frac{1}{2}(S_3 - 3) \, . \end{aligned}$$

It is an equilibrium, and it refutes the three conjectures.

Of course, the counterexample is not at all deep. However, it emphasizes that seemingly natural properties of strategies must be proven, not just declared.

6b Hopeful and hopeless

Striving to exclude pathologies we may restrict ourselves to signal distributions satisfying the assumption

(6b1) $$F_{S_1}(s) = 0 \iff \dots \iff F_{S_n}(s) = 0 \quad \text{for all } s \in [0, \infty) \, .$$

That is, the low end $s_k^{\min}$ of the support of S_k does not depend on k,

(6b2) $$s_1^{\min} = \dots = s_n^{\min} \; (= s^{\min})$$

and in addition, $s^{\min}$ is an atom either for all S_k or for no one of them.

[1] Section 6 is especially influenced by the paper
BERNARD LEBRUN, "First price auctions in the asymmetric N bidder case", International Economic Review **40**:1 (1999), 125–142.

6b3. Lemma. Assuming (6b1) we have $A_k \le S_k$ almost surely, for all k.

Proof. Consider sets $E_k = \{a \in \mathbb{R} : F_{A_k}(a) = 0\}$. Each E_k is of the form $(-\infty, a_k^{\min}]$ or $(-\infty, a_k^{\min})$; the latter happens if $a_k^{\min}$ is an atom of A_k. Sets $E_1, \dots, E_n$ are linearly ordered; consider the greatest one, let it be E_1. Every action in E_1 is never-winning for players $2, \dots, n$. Every action outside E_1 has a positive chance to win for each player $1, \dots, n$. However, $A_1 \notin E_1$ (almost surely),[2] therefore A_1 always has a chance to win; therefore $A_1 \le S_1$ (a.s.);[3] therefore $S_1 \notin E_1$ (a.s.). Condition (6b1) ensures that $S_k \notin E_1$ (a.s.) for all k (think, why). The inequality $A_k > S_k$ can hold only when A_k is a never-winning action (otherwise μ_k is not optimal), therefore, only when $A_k \in E_1$. However, the three relations $S_k \notin E_1$, $A_k \in E_1$ and $A_k > S_k$ are evidently inconsistent. □

6b4. Exercise. Assuming (6b2) we have $A_k \le S_k$ almost surely, for all k.
Prove it.
Hint: a small insertion to the proof of 6b3 is needed.

6b5. Corollary. Assuming (6b2), the expected profit is strictly positive for every player and every signal except, maybe, an atom at $s^{\min}$.

Proof. Recall (2a12): $\mathbf{\Pi}_1^{\max}(s_1) = \sup_{a_1} \mathbf{\Pi}_1(a_1, s_1)$. Let $s_1 > s^{\min}$; we have to prove that $\mathbf{\Pi}_1(a_1, s_1) > 0$ for some a_1. Let $a_1 \in (s^{\min}, s_1)$; it suffices to prove that $\mathbf{\Pi}_1(a_1, s_1) > 0$. However, $\mathbf{\Pi}_1(a_1, s_1) = (s_1 - a_1)W(a_1)$, and the winning probability

$$W(a_1) \ge \mathbb{P}\big(A_2 < a_1, \dots, A_n < a_1 \big) \ge \mathbb{P}\big(S_2 < a_1, \dots, S_n < a_1 \big) = \\ = \mathbb{P}\big(S_2 < a_1 \big) \dots \mathbb{P}\big(S_n < a_1 \big) > 0 \,,$$

since $A_2 \le S_2, \dots, A_n \le S_n$ by 6b4. □

6b6. Corollary. Assuming (6b2) we have $A_k < S_k$ almost surely, unless $S_k = s^{\min}$.

6b7. Note. Atoms of $S_1, \dots, S_n$ at $s^{\min}$ can lead to pathologies. If, say, μ_1 and μ_2 are such that $S_1 = s^{\min} \implies A_1 = s^{\min}$ and $S_2 = s^{\min} \implies A_2 = s^{\min}$, then $\mu_3, \dots, \mu_n$ may use arbitrary actions $A_k \in [0, s^{\min}]$ when $S_k = s^{\min}$.

In an equilibrium, μ_1 is a best response to $\mu_2, \dots, \mu_n$, that is, to the corresponding winning probability function (recall (5c13)) $a_1 \mapsto W_1(a_1)$; we have (recall (5c14), (5c15))

$$\begin{aligned} & W_1(a_1-) = F_{A_2}(a_1-) \dots F_{A_n}(a_1-) \,, \\ & W_1(a_1+) = F_{A_2}(a_1+) \dots F_{A_n}(a_1+) \,, \\ & W_1(a_1-) = W_1(a_1+) \implies W_1(a_1-) = W_1(a_1) = W_1(a_1+) \,, \\ & W_1(a_1-) < W_1(a_1+) \implies W_1(a_1-) < W_1(a_1) < W_1(a_1+) \end{aligned} \tag{6b8}$$

for $0 < a_1 < \infty$, and $W_1(0) = 0$. However, generally $W_1(a_1) \ne F_{A_2}(a_1) \dots F_{A_n}(a_1)$. The same for W_k, the winning probability of player k. In fact, $W_1, \dots, W_n$ are well-defined for arbitrary strategies $\mu_1, \dots, \mu_n$ (not just an equilibrium).

[2] Check it separately for the two cases.
[3] Otherwise μ_1 cannot be optimal; when $A_1 > S_1$, it is strictly better to quit.

6b9. Lemma. Assuming (6b2) we have $\mathbf{\Pi}_k^{\max}(s^{\min}) = 0$ for all k.

Proof. Assume the contrary, say, $\mathbf{\Pi}_1^{\max}(s^{\min}) > 0$. Take a_1 such that $\mathbf{\Pi}_1(s^{\min}, a_1) > 0$, then $a_1 < s^{\min}$ and $W_1(a_1) > 0$, therefore $W_1(s^{\min}-) > 0$ and $\mathbb{P}\big(A_k < s^{\min} \big) > 0$ for $k = 2, \dots, n$. Also $\mathbb{P}\big(A_1 < s^{\min} \big) > 0$, since otherwise $\mathbf{\Pi}_1^{\max}(s) \le s - s^{\min}$ for P_{S_1}-almost all s, in contradiction to $\mathbf{\Pi}_1^{\max}(s^{\min}) > 0$. So,

$$\mathbb{P}\big(A_k < s^{\min} \big) > 0 \quad \text{for } k = 1, \dots, n \,,$$

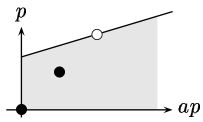
therefore

$$\mathbf{\Pi}_k^{\max}(s^{\min}) > 0 \quad \text{for } k = 1, \dots, n \,.$$

Similarly to the proof of 6b3, we introduce sets E_k, but now we consider the least one, let it be E_1. We have $\mathbb{P}\big(A_k < a_1^{\min} \big) = 0$ for all k. For μ_1-almost all pairs (s_1, a_1) we have $\mathbf{\Pi}_1^{\max}(s^{\min}) \le \mathbf{\Pi}_1^{\max}(s_1) = (s_1 - a_1) W_1(a_1)$, therefore $\mathbf{\Pi}_1^{\max}(s^{\min}) \le (s^{\min} - a_1^{\min}) W_1(a_1^{\min}+)$. So,

$$W_1(a_1^{\min}-) = 0 \,, \qquad W_1(a_1^{\min}+) > 0 \,,$$

which shows that $a_1^{\min}$ is an atom of A_2 (as well as $A_3, \dots, A_n$). Thus, $a_1^{\min} \notin E_2 \supset E_1$, hence $a_1^{\min}$ is an atom of A_1, too. However, $a_1^{\min}$ can be an optimal action for $s_1 = a_1^{\min}$ and no other s_1 (similarly to 2b9); a contradiction.



□

6b10. Exercise. $\mathbb{P}\big(A_k < s^{\min} \big) = 0$ for at least two players.

Prove it.

What about other players? (Recall 6b7.)

6b11. Exercise. Assuming (6b2) we have $S_k > s^{\min} \implies A_k \ge s^{\min}$ (almost surely).

Prove it.

Hint: recall 6b5.

6b12. Exercise. Assuming (6b2) we have $S_k > s^{\min} \implies A_k > s^{\min}$ (almost surely).

Prove it.

Hint: otherwise, $s^{\min}$ is an optimal action for some signal $s > s^{\min}$, and $W_1(s^{\min}) > 0$ by 6b5, but $W_1(s^{\min}-) = 0$ by 6b10; now recall 2b9.

6c Nonatomicity of actions

6c1. Exercise. If a_1 is an atom of A_1 and $W_1(a_1+) > 0$ then $W_2(a_1-) < W_2(a_1+)$, ... , $W_n(a_1-) < W_n(a_1+)$.

Prove it.

Hint: recall (6b8) and forget about equilibria (and best responses); the statement holds for arbitrary strategies.

6c2. Exercise. Let μ_1 be a best response to $\mu_2, \dots, \mu_n$, and $0 < W_1(a_1-) < W_1(a_1+)$. Then $\mathbb{P}\big(a_1 - \varepsilon \le A_1 \le a_1 \big) = 0$ for some $\varepsilon > 0$.

Prove it.

Hint: recall 2b7 and 2b8(a).

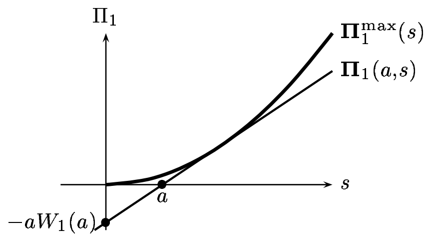
6c3. Lemma. Under (6b2), $A_1, \dots, A_n$ have no atoms on $(s^{\min}, \infty)$.

Proof. Assume the contrary: a_1 is an atom of A_1, and $a_1 > s^{\min}$. Then $W_1(a_1+) > 0$, moreover, $W_k(a_1-) > 0$ for all k (which follows from 6b3), and 6c1 shows that $W_2, \dots, W_n$ are discontinuous at a_1. We apply 6c2 to μ_2 (rather than μ_1) and get $\mathbb{P}\big(a_1 - \varepsilon \le A_2 \le a_1 \big) = 0$ for some $\varepsilon > 0$. The same holds for $A_3, \dots, A_n$. Thus, $W_1(a_1 - \varepsilon) = W_1(a_1) > 0$, in contradiction to optimality of μ_1 (since $a_1 - \varepsilon$ is strictly better than a_1, irrespective of s). □

Note that 6c3 does not assume nonatomicity of signals.

6d Toward differential equations

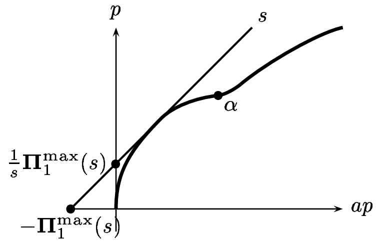
The framework of this subsection is best response (rather than equilibrium). Let μ_1 be a best response to $\mu_2, \dots, \mu_n$, that is, to the corresponding W_1 (recall (6b8)). Each action a is represented by a linear function $s \mapsto \mathbf{\Pi}_1(a, s) = (s - a)W_1(a)$, and their supremum is the convex function $s \mapsto \mathbf{\Pi}_1^{\max}(s)$ (recall 2a).



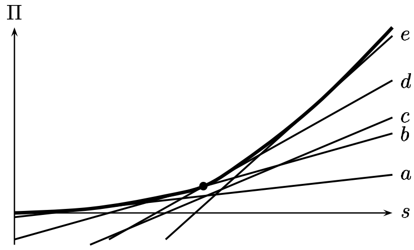
We know that $\mathbf{\Pi}_1^{\max}(s)$ is the integral of the winning probability $\mathbf{p}^{\text{win}}(s)$ treated as a function of a signal. Now, however, we have no way to get $\mathbf{p}^{\text{win}}$ apriori. Thus, we are more interested in relating $\mathbf{\Pi}_1^{\max}$ to the winning probability W_1 treated as a function of an action. It is easy to express $\mathbf{\Pi}_1^{\max}$ in terms of W_1,

$$\mathbf{\Pi}_1^{\max}(s) = \sup_a \, (s - a)W_1(a) \, . \tag{6d1}$$

Can we express W_1 in terms of $\mathbf{\Pi}_1^{\max}$? Generally, not; different functions W_1 can lead to the same $\mathbf{\Pi}_1^{\max}$, which is easier to see on the plane (expected loss, winning probability) (recall 2b):



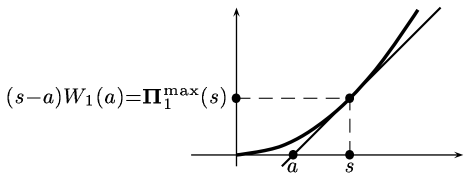
The function $\mathbf{\Pi}_1^{\max}$ determines uniquely the convex envelope of the curve rather than the curve itself. The point α (see the picture) describes an action a that is never optimal (irrespective of s); its $W_1(a)$ cannot be determined by $\mathbf{\Pi}_1^{\max}$. We can see the same on the (s, Π) plane:



Actions a, b, d, e are optimal for some signals, but c is not.

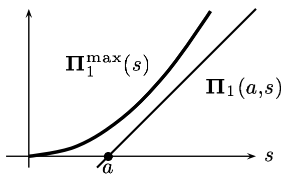
If a is an optimal action for some signal, then

(6d2) $$W_1(a) = \min_{s \in (a,\infty)} \frac{\mathbf{\Pi}_1^{\max}(s)}{s-a} \,.$$



Therefore (6d2) holds for P_{A_1}-almost all a. If, however, a is not an optimal action (for any signal) then[4]

(6d3) $$W_1(a) < \frac{\mathbf{\Pi}_1^{\max}(s)}{s-a} \quad \text{for all } s \in (a, \infty)\,.$$



We define

(6d4) $$V_1(a) = \inf_{s \in (a,\infty)} \frac{\mathbf{\Pi}_1^{\max}(s)}{s-a} \,,$$

[4]And nevertheless the infimum may happen to be equal to $W_1(a)$, though typically it is greater than $W_1(a)$.

then

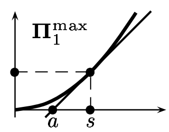
$$(6d5) \qquad \begin{aligned} W_1(a) &\le V_1(a) \quad \text{for all } a, \\ W_1(a) &= V_1(a) \quad \text{for } P_{A_1}\text{-almost all } a. \end{aligned}$$

The function W_1 is monotone, it increases on $[0,\infty)$ from 0 to 1, which basically exhausts its properties.[5] However, W_1 is related by (6d5) to another function, V_1, constructed by (6d4) in a purely geometric way from a given function $\mathbf{\Pi}_1^{\max}$ such that[6]

$$(6d6) \qquad \begin{gathered} \mathbf{\Pi}_1^{\max} \text{ is a convex increasing function on } [0,\infty)\,, \\ \mathbf{\Pi}_1^{\max}(0) = 0\,, \quad \text{and} \quad \frac{d}{ds}\mathbf{\Pi}_1^{\max}(s) \to 1 \text{ for } s \to \infty\,. \end{gathered}$$

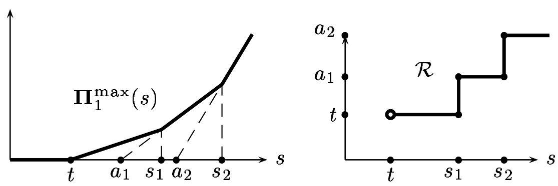
Due to convexity of $\mathbf{\Pi}_1^{\max}$, the function V_1 is much more special than W_1, as we'll see. First, introduce such a set of pairs (s,a):

$$(6d7) \qquad \mathcal{R} = \{(s,a) \in [0,\infty) \times [0,\infty) : (s-a)V_1(a) = \mathbf{\Pi}_1^{\max}(s) > 0\}\,.$$

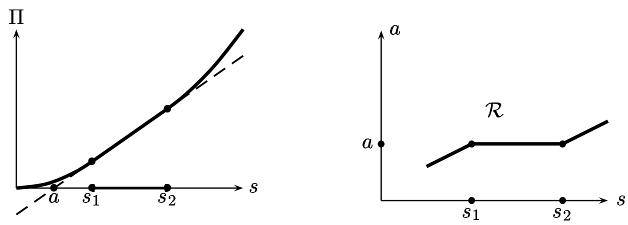


We have a weakly increasing relation between s and a, and $\mathcal{R}$ is a connected line.

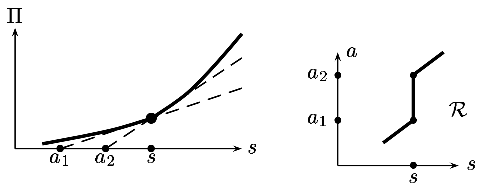
6d8. Example. Let $\mathbf{\Pi}_1^{\max}$ be piecewise linear. Then $\mathcal{R}$ consists of a finite number of horizontal and vertical segments.



In general, horizontal ($a = \text{const}$) segments of $\mathcal{R}$ correspond to linear segments of $\mathbf{\Pi}_1^{\max}$ (nothing but bunches, recall 3c), except for $\mathbf{\Pi}_1^{\max}(s) = 0$.



Similarly, vertical ($s = \text{const}$) segments of $\mathcal{R}$ correspond to jumps of the derivative of $\mathbf{\Pi}_1^{\max}$.



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[5]The last line of (6b8) is the only additional property, and is relevant at discontinuity points only.

[6]The latter holds since $\frac{d}{ds}\mathbf{\Pi}_1^{\max}(s) = \mathbf{p}^{\text{win}}(s)$ (except discontinuity points, if any).

In the latter case we have $V_1(a) = \mathbf{\Pi}_1^{\max}(s)/(s-a)$ for all $a \in (a_1, a_2)$, thus $\ln V_1(a) = \text{const} - \ln(s-a)$ and

$$\text{(6d9)} \qquad \frac{V_1'(a)}{V_1(a)} = \frac{d}{da} \ln V_1(a) = -\frac{d}{da} \ln(s-a) = \frac{1}{s-a} \quad \text{for } a \in (a_1, a_2)\,.$$

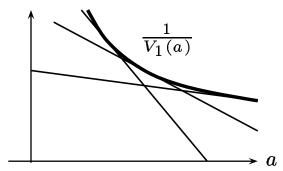
We may guess that the equality $V_1'(a)/V_1(a) = 1/(s-a)$ holds also for nondegenerate cases, when s is not constant. There are several ways to show that the guess is true. Here is one of them. First, note that in the degenerate case $s = \text{const}$ we have $1/V_1(a) = (s-a)/\mathbf{\Pi}_1^{\max}(s)$, a linear function on (a_1, a_2). In general, it follows from (6d4) that

$$\text{(6d10)} \qquad \frac{1}{V_1(a)} = \sup_{s \in (t,\infty)} \frac{s-a}{\mathbf{\Pi}_1^{\max}(s)} \quad \text{for } a \in (t, \infty)\,,$$

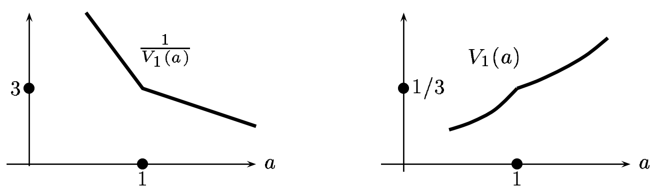
where

$$\begin{aligned} \text{(6d11)} \quad t = \inf\{s \in [0,\infty) : \mathbf{\Pi}_1^{\max}(s) > 0\} &= \max\{s \in [0,\infty) : \mathbf{\Pi}_1^{\max}(s) = 0\} = \\ = \inf\{a \in [0,\infty) : V_1(a) > 0\} &= \sup\{a \in [0,\infty) : V_1(a) = 0\} = \\ = \inf\{a \in [0,\infty) : W_1(a) > 0\} &= \sup\{a \in [0,\infty) : W_1(a) = 0\}\,, \end{aligned}$$

$0 \le t < \infty$. (Think, why the threshold of signals is equal to the threshold of actions.) Being the supremum of linear functions, the function $a \mapsto 1/V_1(a)$ is convex on (t, ∞).



It follows that the function $1/V_1$ is differentiable on (t, ∞) except (maybe) a finite or countable set, and its derivative is an increasing function. Therefore the function V_1 is differentiable except (maybe) a finite or countable set. Though, its derivative V_1' need not be monotone,[7] it may behave like that:



If $(s, a) \in \mathcal{R}$ then the supremum in (6d10) for a is reached at s. If in addition, $1/V_1$ is differentiable at a, then its derivative corresponds to the tangent line:

$$\frac{d}{da_1}\bigg|_{a_1=a} \frac{1}{V_1(a_1)} = \frac{d}{da_1}\bigg|_{a_1=a} \frac{s-a_1}{\mathbf{\Pi}_1^{\max}(s)} = -\frac{1}{\mathbf{\Pi}_1^{\max}(s)}\,.$$

[7]Rather, 2b5(b) is satisfied,

$$V_1''(a) \le 2\frac{V_1'^2(a)}{V_1(a)}$$

(think, why), provided that V_1'' exists.

Combining it with

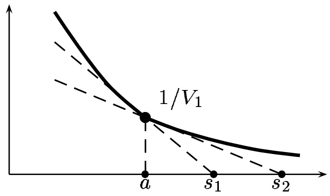
$$\frac{1}{V_1(a)} = \frac{s-a}{\mathbf{\Pi}_1^{\max}(s)}$$

we get

$$\frac{d}{da_1}\bigg|_{a_1=a} \ln \frac{1}{V_1(a_1)} = -\frac{1}{s-a}\,,$$

which proves (6d9).

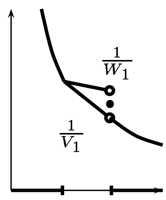
If $(s,a) \in \mathcal{R}$ but $1/V_1$ is not differentiable at a, it means that (s,a) belongs to a horizontal segment of $\mathcal{R}$.



In such a case $V_1'(a)$ does not exists, however, one-sided derivatives exist:[8]

(6d12) $$\frac{V_1'(a-)}{V_1(a)} = \frac{1}{s_1-a}\,, \qquad \frac{V_1'(a+)}{V_1(a)} = \frac{1}{s_2-a}\,.$$

Now, understanding the auxiliary function V_1, we return to the winning probability function W_1. The two functions coincide P_{A_1}-almost everywhere (recall 6d5). Do not think that they must coincide everywhere on the support of A_1.



6d13. Lemma. $V_1(a) = W_1(a)$ whenever a is an interior point of the support of A_1.

Proof. The support of A_1 contains a neighborhood $(a-\varepsilon, a+\varepsilon)$ of a. The functions V_1 and W_1 coincide P_{A_1}-almost everywhere, therefore they coincide on a dense subset E of $(a-\varepsilon, a+\varepsilon)$. However, V_1 is continuous (since $1/V_1$ is convex, therefore continuous), and W_1 is monotone. We have

$$W_1(a) \le \lim_{b\to a, b\in E\cap(a,a+\varepsilon)} W_1(b) = \lim_{b\to a, b\in E\cap(a,a+\varepsilon)} V_1(b) = V_1(a)\,,$$
$$W_1(a) \ge \lim_{b\to a, b\in E\cap(a-\varepsilon,a)} W_1(b) = \lim_{b\to a, b\in E\cap(a-\varepsilon,a)} V_1(b) = V_1(a)\,,$$

thus $W_1(a) = V_1(a)$. □

[8] It may be interpreted as $V_1'(a+) = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}(V_1(a+\varepsilon) - V_1(a))$ or, equally well, as $V_1'(a+) = \lim_{\varepsilon\to 0+} V_1'(a+\varepsilon)$; in the latter formulation ε must be such that $V_1'(a+\varepsilon)$ exists. In general it may happen that the former limit exists but the latter does not. However, for our functions existence of both is ensured by convexity of $1/V_1$.

6d14. Lemma. If $a \in (t, \infty)$ is such that $V_1(a) = W_1(a)$ then the following two conditions are equivalent for every s:

(a) a is an optimal action for s;

(b) $(s, a) \in \mathcal{R}$.

Proof. Condition (a) means that $\mathbf{\Pi}_1(a, s) = \mathbf{\Pi}_1^{\max}(s)$, that is, $(s - a)W_1(a) = \mathbf{\Pi}_1^{\max}(s)$. Condition (b) means that $(s - a)V_1(a) = \mathbf{\Pi}_1^{\max}(s) > 0$ (recall (6d7)). ☐

For such a, the set of all $s \in (a, \infty)$ such that a is optimal for s is either a single point, or a closed interval (bunch), or an empty set. The latter may happen only if $\mathbb{P}\big(A_1 < a \big) = 1$ (think, why).

6d15. Proposition. Let $a \in (t, \infty)$ be an interior point of the support of A_1. If a is optimal for one and only one s, then W_1 is differentiable at a, and

$$\frac{W_1'(a)}{W_1(a)} = \frac{1}{s - a} \, .$$

Otherwise, the set of all $s \in (a, \infty)$ such that a is optimal for s is an interval $[s_1, s_2]$, $s_1 < s_2$, and W_1 has one-sided derivatives[9] at a, and

$$\frac{W_1'(a-)}{W_1(a)} = \frac{1}{s_1 - a} \, , \qquad \frac{W_1'(a+)}{W_1(a)} = \frac{1}{s_2 - a} \, .$$

Proof. By Lemma 6d13, $V_1(a) = W_1(a)$. By Lemma 6d14, a is optimal for s if and only if $(s, a) \in \mathcal{R}$. Also, $\mathbb{P}\big(A_1 < a \big) < 1$. Thus, the set of s such that a is optimal for s is either a single point $s \in (a, \infty)$ or a closed bounded interval $[s_1, s_2] \subset (a, \infty)$.

By Lemma 6d13 again, functions V_1 and W_1 coincide on a neighborhood of a. Therefore we may replace V_1 with W_1 in (6d9) or (6d12). ☐

We want to describe these s, s_1, s_2 (for a given a) in terms of distributions. If everything is continuous and strictly increasing then clearly

$$s = F_{S_1}^{-1}\big(F_{A_1}(a)\big) \, .$$

What happens in general? Introduce the quantile function S_1^* (known also as the inverse distribution function $F_{S_1}^{-1}$); it is an increasing function $S_1^* : (0,1) \to \mathbb{R}$ such that $S_1^*(p)$ is a p-quantile of S_1, in other words,

$$\mathbb{P}\big(S_1 < S_1^*(p) \big) \le p \le \mathbb{P}\big(S_1 \le S_1^*(p) \big) \quad \text{for all } p \in (0,1) \, .$$

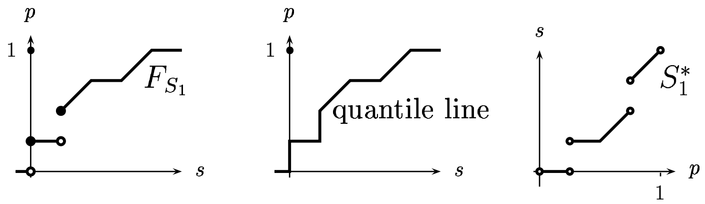
If (and only if) S_1 have gaps then S_1^* is discontinuous. We could assume it to be right-continuous, or alternatively, left-continuous, but I prefer not to specify it at jumping points.

[9] Recall the footnote to (6d12).

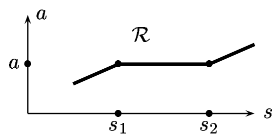
Anyway, the arbitrariness does not influence on-sided limits $S_1^*(p-)$ and $S_1^*(p+)$. We cannot say that $p = F_{S_1}(s) \iff S_1^*(p) = s$, but the following equivalences hold in every case:

(6d16)
$$
\begin{aligned}
p < F_{S_1}(s-) &\iff S_1^*(p+) < s\,;\\
p > F_{S_1}(s+) &\iff S_1^*(p-) > s\,;\\
p \le F_{S_1}(s+) &\iff S_1^*(p-) \le s\,;\\
p \ge F_{S_1}(s-) &\iff S_1^*(p+) \ge s\,;\\
F_{S_1}(s-) \le p \le F_{S_1}(s+) &\iff S_1^*(p-) \le s \le S_1^*(p+)\,.
\end{aligned}
$$

The set of all pairs (s,p) satisfying the two equivalent relations in the last line of (6d16) is a connected line on the (s,p) plane (the quantile line), and describes a weakly monotone relation between s and p.



Being equipped with the quantile functions S_1^*, we return to S_1, A_1 and $\mathcal{R}$. Recall the case of a bunch:

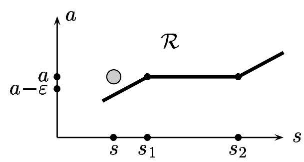


6d17. Lemma. Let $a \in (t,\infty)$ be such that $F_{A_1}(a-\varepsilon) < F_{A_1}(a) < F_{A_1}(a+\varepsilon)$ for every $\varepsilon > 0$, and $\{s : (s,a) \in \mathcal{R}\} = [s_1, s_2]$, $s_1 \le s_2$.[10] Then $s_1 = S_1^*\big(F_{A_1}(a-)-\big)$ and $s_2 = S_1^*\big(F_{A_1}(a+)+\big)$.

Proof. Denote $p_1 = F_{A_1}(a-)$, $p_2 = F_{A_1}(a+)$. We have $s < S_1^*(p_1-) \iff F_{S_1}(s+) < p_1$ and $s > S_1^*(p_2+) \iff F_{S_1}(s-) > p_2$. In order to prove that $s_1 = S_1^*(p_1-)$ and $s_2 = S_1^*(p_2+)$, it suffices to prove that

$$
\begin{aligned}
s < s_1 &\iff F_{S_1}(s+) < p_1\,,\\
s > s_2 &\iff F_{S_1}(s-) > p_2\,.
\end{aligned}
$$

Clearly, $A_1 < a \implies S_1 \le s_1$ (almost surely), thus $p_1 = F_{A_1}(a-) \le F_{S_1}(s_1+)$ and $s < s_1 \impliedby F_{S_1}(s+) < p_1$. Similarly, $A_1 > a \implies S_1 \ge s_2$, thus $p_2 = F_{A_1}(a+) \ge F_{S_1}(s_2-)$ and $s > s_2 \impliedby F_{S_1}(s-) > p_2$. It remains to prove " $\implies$ " implications. Let $s < s_1$, then the point (s,a) does not belong to the closed set $\mathcal{R}$, therefore its ε-neighborhood does not intersect $\mathcal{R}$ if ε is small enough.



[10]Both cases, $s_1 = s_2$ and $s_1 < s_2$ are covered at once.

Clearly, $S_1 \le s \implies A_1 \le a - \varepsilon$ (a.s). Therefore $F_{S_1}(s+) \le F_{A_1}(a-\varepsilon) < F_{A_1}(a-) = p_1$. So, $s < s_1 \implies F_{S_1}(s+) < p_1$.

Similarly, if $s > s_2$ then $S_1 \ge s \implies A_1 \ge a + \varepsilon$, thus $1 - F_{S_1}(s-) \le 1 - F_{A_1}(a+\varepsilon-)$, so, $F_{S_1}(s-) \ge F_{A_1}(a+\varepsilon-) > F_{A_1}(a+) = p_2$. □

6d18. Theorem. [11] Let $a \in (t, \infty)$ be an interior point of the support of A_1. Then W_1 has one-sided derivatives[12] at a, and

$$\frac{W_1'(a-)}{W_1(a)} = \frac{1}{s_1 - a}, \qquad \frac{W_1'(a+)}{W_1(a)} = \frac{1}{s_2 - a},$$

where

$$s_1 = S_1^*\big(F_{A_1}(a-)-\big), \qquad s_2 = S_1^*\big(F_{A_1}(a+)+\big)$$

(here S_1^* stands for the quantile function of S_1).

Proof. Follows immediately from Proposition 6d15 and Lemma 6d16. □

6d19. Corollary. Under the conditions of Theorem 6d18, the function W_1 is differentiable at a if and only if $S_1^*\big(F_{A_1}(a-)-\big) = S_1^*\big(F_{A_1}(a+)+\big)$.

6d20. Exercise. If $S_1^*\big(F_{A_1}(a-)-\big) = S_1^*\big(F_{A_1}(a+)+\big)$, does it follow that F_{A_1} is continuous at a and S_1^* is continuous at $F_{A_1}(a)$?

6d21. Exercise. Let $a \in (t, \infty)$ be such that the support of A contains $[a, a+\varepsilon]$ for some $\varepsilon > 0$. Then W_1 has the right-hand side derivative at a, and

$$\frac{W_1'(a+)}{W_1(a)} = \frac{1}{s_2 - a} \qquad \text{where} \quad s_2 = S_1^*\big(F_{A_1}(a+)+\big).$$

The same for the other case, $[a-\varepsilon, a]$.

Prove it.

What happens for $a = t$?

6e Two players

Consider an asymmetric first price auction with 2 players, assuming (6b2). Let (μ_1, μ_2) be an equilibrium. We have

$$s^{\min} \le A_1 \le S_1, \quad s^{\min} \le A_2 \le S_2 \quad \text{(almost surely)}$$

by 6b4 and 6b10;

$$S_1 > s^{\min} \implies s^{\min} < A_1 < S_1, \qquad S_2 > s^{\min} \implies s^{\min} < A_2 < S_2 \quad \text{(a.s.)}$$

[11] Recall the framework (stated in the beginning of 6d): μ_1 is a best response to $\mu_2, \dots, \mu_n$, that is, to the corresponding W_1. And, of course, (S_1, A_1) is distributed μ_1.

[12] Recall the footnote to (6d12).

by 6b6 and 6b12;

$$\mathbf{\Pi}_1^{\max}(s^{\min}) = \mathbf{\Pi}_2^{\max}(s^{\min}) = 0$$

by 6b9;

$$\mathbf{\Pi}_1^{\max}(s) > 0\,, \quad \mathbf{\Pi}_2^{\max}(s) > 0 \quad \text{for all } s \in (s^{\min}, \infty)$$

by 6b5;

$$A_1, A_2 \text{ have no atoms on } (s^{\min}, \infty)$$

by 6c3; and

$$W_1(a) = F_{A_2}(a)\,, \quad W_2(a) = F_{A_1}(a) \quad \text{for all } a \in (s^{\min}, \infty)$$

(think, why).

Denote $a_k^{\max} = \sup\{a : F_{A_1}(a) < 1\}$,[13] then

$$a_1^{\max} = a_2^{\max}\,;$$

indeed, if, say, $a_1^{\max} < a_2^{\max}$ then μ_2 is not optimal (since $a_1^{\max}$ is strictly better than A_2 whenever $A_2 > a_1^{\max}$).

6e1. Exercise. A_1, A_2 have no gaps.

Prove it.

Hint: use 2b10(b).

So, A_1 and A_2 have the same support

$$[s^{\min}, a^{\max}]\,;$$

here $a^{\max} = a_1^{\max} = a_2^{\max}$. We apply Theorem 6d18, taking into account that the threshold t introduced by (6d11) is nothing but $s^{\min}$ (for both players).

6e2. Theorem. Functions F_{A_1}, F_{A_2} are continuous on $(s^{\min}, \infty)$, have one-sided derivatives on that interval, and

$$\frac{F'_{A_1}(a-)}{F_{A_1}(a)} = \frac{1}{S_2^*(F_{A_2}(a)-) - a}\,, \qquad \frac{F'_{A_1}(a+)}{F_{A_1}(a)} = \frac{1}{S_2^*(F_{A_2}(a)+) - a}\,,$$

$$\frac{F'_{A_2}(a-)}{F_{A_2}(a)} = \frac{1}{S_1^*(F_{A_1}(a)-) - a}\,, \qquad \frac{F'_{A_2}(a+)}{F_{A_2}(a)} = \frac{1}{S_1^*(F_{A_1}(a)+) - a}$$

for all $a \in (s^{\min}, a^{\max})$.

[13] A finite number or $+\infty$. Or is the latter impossible?

We see that A_1, A_2 have densities[14]

$$f_{A_1} = F'_{A_1} , \quad f_{A_2} = F'_{A_2}$$

except for possible atoms at $s^{\min}$; the density f_{A_k} is continuous except, maybe, a finite or countable number of jumps downwards. Each jump of f_{A_k} corresponds to a gap of S_{3-k}, except for the last jump, at $a^{\max}$.

6e3. Corollary. If S_2 has no gaps then f_{A_1} is continuous on $(s^{\min}, a^{\max})$, and

$$\frac{f_{A_1}(a)}{F_{A_1}(a)} = \frac{1}{S_2^*(F_{A_2}(a)) - a} \quad \text{for all } a \in (s^{\min}, a^{\max}) .$$

The same for S_1 and f_{A_2}.

6e4. Exercise. The density f_{A_1} has a jump at $a^{\max}$ if and only if S_2 is bounded.[15]
Prove it.
Hint: recall 6d21.

We turn to a *symmetric* action with two players;

$$S_1^*(p) = S_2^*(p) \quad \text{for all } p \in (0,1) ;$$

its symmetric equilibrium was found in 3d; now we address the question, whether other, asymmetric equilibria exist, or not.

If the distribution P_S $(= P_{S_1} = P_{S_2})$ has no gaps, then we have

$$\text{(6e5)} \qquad \frac{d}{da}\bigl(\ln F_{A_1}(a) - \ln F_{A_2}(a)\bigr) = \frac{1}{S^*(F_{A_2}(a)) - a} - \frac{1}{S^*(F_{A_1}(a)) - a}$$

for all $a \in (s^{\min}, a^{\max})$, which shows that the function $a \mapsto F_{A_1}(a)/F_{A_2}(a)$ increases whenever it exceeds 1 and decreases whenever it is less than 1. Gaps of P_S (if any) do not invalidate the statement, since (6e5) still holds everywhere except, maybe, a finite or countable set, of no influence to the integral of the derivative.[16] Here is an exact formulation. If $[a,b] \subset [s^{\min}, a^{\max}]$ is such that $F_{A_1}(x) \ge F_{A_2}(x)$ for all $x \in [a,b]$, then $\frac{F_{A_1}(a)}{F_{A_2}(a)} \le \frac{F_{A_1}(b)}{F_{A_2}(b)}$.

Assume that $F_{A_1}(a) > F_{A_2}(a)$ for some $a \in (s^{\min}, a^{\max}]$. Recall that $F_{A_1}(a^{\max}) = F_{A_2}(a^{\max}) = 1$, and F_{A_1}, F_{A_2} are continuous on $(s^{\min}, a^{\max})$. Consider the least $b \in [a, a^{\max}]$ such that $F_{A_1}(b) = F_{A_2}(b)$, then $F_{A_1}(\cdot) \ge F_{A_2}(\cdot)$ on $[a,b]$, therefore $\frac{F_{A_1}(a)}{F_{A_2}(a)} \le \frac{F_{A_1}(b)}{F_{A_2}(b)}$, which is a contradiction. So, $F_{A_1}(a) \le F_{A_2}(a)$ for all $a \in (s^{\min}, a^{\max}]$. Similarly, $F_{A_2}(a) \le F_{A_1}(a)$. The following result is thus proven.

6e6. Theorem. A symmetric first price auction with two players has one and only one equilibrium, and the equilibrium is symmetric.

---

[14]In order to be a density, the derivative F' must return the distribution function F by integration, which means that F must be absolutely continuous. In fact, convexity of $1/F_1$, $1/F_2$ ensures absolute continuity of F_1, F_2.

[15]Though, if S_2 is unbounded, it could happen that $a^{\max} = +\infty$. Or maybe that is excluded by assuming $\mathbb{E} S_1 < \infty$, $\mathbb{E} S_2 < \infty$? I do not know.

[16]Once again, absolute continuity is relevant.

6f Equilibrium, supports, cells

We return to an asymmetric first price auction with n players, assuming (6b2) and an equilibrium. We have

$$A_k \le S_k \quad \text{for all } k \text{ (almost surely)}$$

by 6b4; though, for some (but not all) k it may happen that $A_k < s^{\min}$ when $S_k = s^{\min}$. Further,

$$S_k > s^{\min} \quad \Longrightarrow \quad s^{\min} < A_k < S_k \quad \text{for all } k \text{ (a.s.)}$$

by 6b6 and 6b12;

$$\mathbf{\Pi}_k^{\max}(s^{\min}) = 0$$

by 6b9;

$$\mathbf{\Pi}_k^{\max}(s) > 0 \quad \text{for all } s \in (s^{\min}, \infty) \text{ and all } k$$

by 6b5;

$$A_1, \dots, A_n \text{ have no atoms on } (s^{\min}, \infty)$$

by 6c3; and

$$W_k(a) = \frac{F(a)}{F_{A_k}(a)} \quad \text{for all } a \in (s^{\min}, \infty)$$
$$\text{where} \quad F(a) = F_{A_1}(a) \dots F_{A_n}(a)$$

(think, why).

Unfortunately it is far from being evident, whether all A_k have the same support, or not, even for a symmetric auction, as far as the equilibrium is not assumed to be symmetric. Assume for a while that the auction is symmetric, and all A_k have the same support, and moreover, it is an interval, that is,

(6f1) $$\text{the support of } A_k \text{ is } [s^{\min}, a^{\max}] \quad \text{for all } k$$

for some $a^{\max} \in (s^{\min}, \infty)$. Theorem 6d18 gives

$$\frac{W_k'(a)}{W_k(a)} = \frac{1}{S^*(F_{A_k}(a)) - a}$$

for all k and all $a \in (s^{\min}, a^{\max})$ except maybe a finite or countable set (of discontinuity points). We have[17] $W_k'(a)/W_k(a) = \frac{d}{da} \ln W_k(a) = \frac{d}{da} \ln F(a) - \frac{d}{da} \ln F_{A_k}(a)$, hence

$$\frac{d}{da} \ln F_{A_l}(a) - \frac{d}{da} \ln F_{A_k}(a) = \frac{1}{S^*(F_k(a)) - a} - \frac{1}{S^*(F_l(a)) - a}$$

for all k, l. It shows that the function $F_{A_k}(a)/F_{A_l}(a)$ increases whenever it exceeds 1, and decreases whenever it is less than 1. Similarly to 6e we conclude that $F_{A_k} = F_{A_l}$ on the whole $(s^{\min}, a^{\max}]$, which proves the following result.

[17] However, it is not evident that functions F_{A_k} are differentiable. We know that W_k are absolutely continuous, therefore $\ln W_k$ are absolutely continuous, therefore $\sum_k \ln W_k = (n-1) \ln F$ is absolutely continuous, as well as $\ln F_k = \ln F - \ln W_k$ are, which means that F_k are absolutely continuous. See also 6g10.

6f2. Proposition. A symmetric first price auction has one and only one equilibrium satisfying (6f1), and the equilibrium is symmetric.

What happens beyond (6f1)? Generally, I do not know. The rest of Sect. 6 assumes that

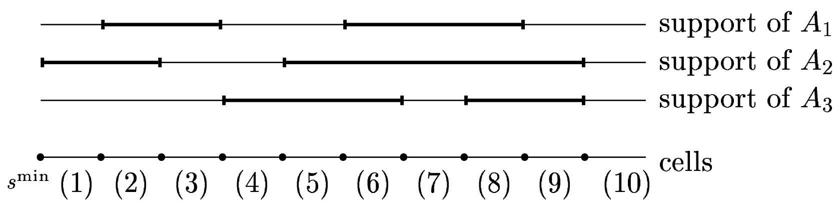
(6f3) the support of A_k consists of a finite number of intervals

for each k. Note that supports need not be equal. The assumption excludes such supports as, say,

(an infinite sequence of intervals accumulating to a point), or a Cantor-type set

(with a dense set of gaps).

Accordingly to (6f3), every player may have any finite number of gaps in his A_k, and gaps of different players may overlap arbitrarily. The boundary of the support of A_k is a finite set of endpoints of all its intervals. The union of all these boundaries is a finite set; it divides $[s^{\min}, \infty)$ into a finite number of intervals; these will be called *cells.* An example:



Here, the support of A_1 consists of 2 intervals, and its boundary contains 4 points. Also A_2 and A_3 contribute 4 boundary points each. Some of these points coincide; the union contains 9 points, and we have 10 cells, the last one being unbounded from above.

Given a cell and a player, we have two possible cases; the support of the action A_k of the player contains either all points of the cell, or no one of them. Accordingly we say that the player is active or passive on that cell.[18]

6f4. Exercise. The number of active players on a cell cannot be equal to 1.

Prove it.

Hint: if only player 1 is active then W_1 is constant on the cell.

6f5. Exercise. The number of active players on a cell cannot be equal to 0, except for the last (unbounded) cell.

Prove it.

Hint: use 2b10(b).

[18] On the example shown above, say, on the cell number 4, only player 3 is active; on the cell number 8, all players are active; on the cell number 10, all players are passive. The next exercise shows that such a case is in fact impossible. You may also note that all players must be active on the first cell. Still, a lot of possibilities persist.

6g Convexity on cells

The following purely analytical fact will be essential.

6g1. Lemma. Let $f_1, \dots, f_n : (a,b) \to (0,\infty)$ be increasing functions such that

$$\frac{f_1 \dots f_n}{f_k} \text{ is convex on } (a,b)$$

for all $k = 1, \dots, n$, and $n \ge 2$. Then

$$f_1 \dots f_n \text{ is convex on } (a,b)\,.$$

Proof. Assume for a while that functions $f_1, \dots, f_n$ are smooth, namely, twice continuously differentiable. We need to deduce the inequality $(f_1 \dots f_n)''/(f_1 \dots f_n) \ge 0$, that is,

$$\sum_{1\le k\le n} \frac{f_k''}{f_k} + 2 \sum_{1\le k<l\le n} \frac{f_k' f_l'}{f_k f_l} \ge 0 \tag{6g2}$$

from n similar inequalities, one of these being

$$\sum_{1\le k\le n-1} \frac{f_k''}{f_k} + 2 \sum_{1\le k<l\le n-1} \frac{f_k' f_l'}{f_k f_l} \ge 0\,.$$

The sum of these n given inequalities is

$$(n-1) \sum_{1\le k\le n} \frac{f_k''}{f_k} + 2(n-2) \sum_{1\le k<l\le n} \frac{f_k' f_l'}{f_k f_l} \ge 0\,;$$

we divide the sum by $n-1$; taking into account that $f_k' \ge 0$ for all k, we get (6g2), which proves the lemma for smooth functions.

We turn to the general (not just smooth) case.[19] Convexity of $f_1 \dots f_n / f_k$ implies that the derivative $(f_1 \dots f_n/f_k)'$ exists everywhere except for a finite (maybe empty) or countable set of jumps, and returns the original function by integration. The same holds for the product $\prod_{k=1,\dots,n} \frac{f_1 \dots f_n}{f_k} = (f_1 \dots f_n)^{n-1}$, therefore for $f_1 \dots f_n$ and further, for the quotient $(f_1 \dots f_n)/(f_1 \dots f_n/f_k) = f_k$. In other words, f_k is locally absolutely continuous, f_k' is locally bounded and has left and right limits $f_k'(x-), f_k'(x+)$ for all $x \in (a,b)$.

In order to generalize our argument for non-smooth functions, we rewrite it in a form free of second derivatives, using integration by parts:

$$\int \biggl(\frac{(f_1 \dots f_n)''}{f_1 \dots f_n} - \frac{1}{n-1} \sum_{1\le k\le n} \frac{(f_1 \dots f_n/f_k)''}{f_1 \dots f_n/f_k} - \frac{2}{n-1} \sum_{1\le k<l\le n} \frac{f_k' f_l'}{f_k f_l} \biggr) \varphi(x)\, dx = 0$$

turns into

$$- \int (f_1 \dots f_n)' \Bigl(\frac{\varphi}{f_1 \dots f_n} \Bigr)' dx + \frac{1}{n-1} \sum_{1\le k\le n} \int \Bigl(\frac{f_1 \dots f_n}{f_k} \Bigr)' \Bigl(\frac{\varphi}{f_1 \dots f_n/f_k} \Bigr)' dx - \\ - \frac{2}{n-1} \sum_{1\le k<l\le n} \int \frac{f_k' f_l'}{f_k f_l} \varphi\, dx = 0 \tag{6g3}$$

[19]Maybe, the rest of the proof is of interest for mathematicians only.

for an arbitrary smooth function φ whose support lies strictly inside (a,b). The standard approximation generalizes the latter equality to a wider class of functions $f_1,\dots,f_n$. Namely, instead of a continuous second derivative, it is enough to be absolutely continuous and have a locally bounded first derivative.[20] Similarly, instead of smoothness of φ it is enough if φ is absolutely continuous and has a bounded derivative.

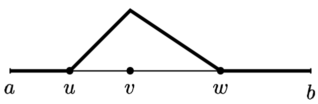
If $\varphi \ge 0$ on (a,b) then

$$\int \Big(\frac{f_1\dots f_n}{f_k}\Big)'\Big(\frac{\varphi}{f_1\dots f_n/f_k}\Big)'\,dx = -\int \frac{\varphi}{f_1\dots f_n/f_k}\,d\Big(\frac{f_1\dots f_n}{f_k}\Big)' \le 0\,,$$

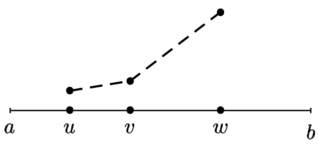
since $(f_1\dots f_n/f_k)'$ increases (due to convexity of $f_1\dots f_n/f_k$). Combining it with (6g3) we get $\int (f_1\dots f_n)'\big(\frac{\varphi}{f_1\dots f_n}\big)'\,dx \le 0$. However, $\frac{\varphi}{f_1\dots f_n}$ is as arbitrary as φ, and so,

$$\text{(6g4)}\qquad \int (f_1\dots f_n)'\varphi'\,dx \le 0$$

for every $\varphi \ge 0$ of the class considered. For any given u,v,w such that $a<u<v<w<b$ we define φ as the piecewise linear 'triangle' function



then (6g4) becomes just the three-point inequality that expresses convexity of $f_1\dots f_n$.



□

6g5. Note. If $f_1\dots f_n$ are as in Lemma 6g1 then each f_k is the quotient of two convex functions,

$$f_k = \frac{f_1\dots f_n}{f_1\dots f_n/f_k}\,,$$

therefore one-sided derivatives $f_k'(x-), f_k'(x+)$ exist (and are finite) for all $x\in(a,b)$, and $f_k'(x)$ is well-defined and continuous for all $x\in(a,b)$ except for a finite (maybe empty) or countable set, and f_k' returns f_k by integration.[21]

6g6. Note. If $f_1\dots f_n$ as in Lemma 6g1 are strictly increasing on (a,b) then $f_1\dots f_n$ is strictly convex on (a,b).

Sketch of the proof: $\int \frac{f_1' f_2'}{f_1 f_2}\varphi\,dx > 0$, since $\{x\in(a,b) : f_1'(x-)=0 \text{ or } f_1'(x+)=0\}$ is a closed nowhere dense set; the same for f_2.

(It is enough if only two functions among $f_1,\dots,f_n$ are strictly increasing.)

[20] Local boundedness of f_k' may be replaced with a weaker condition of local square integrability. Such a function is the limit of a sequence $f_{k,1}, f_{k,2},\dots$ of smooth functions such that $f_{k,i}\to f_k$ for $i\to\infty$ uniformly on compact subsets of (a,b), and $\int_{a+\varepsilon}^{b-\varepsilon} |f_k'(x)-f_{k,i}'(x)|^2\,dx \to 0$ for $i\to\infty$. That is enough for the limiting procedure.

[21] Well, all that was already said in the proof of Lemma 6g1.

We return to the framework of 6f, assuming (6f3). We simplify the notation, denoting F_{A_k} by F_k.

6g7. Lemma. For every $k = 1, \dots, n$, the function $1/W_k$ is convex on every cell.

Proof. Consider a cell (a, b). If player k is active on (a, b) then $1/W_k = 1/V_k$ on (a, b) by 6d13, and $1/V_k$ is convex by (6d10).

Let players $1, \dots, m$ be active on (a, b) and players $m + 1, \dots, n$ be passive; here $m \in \{2, \dots, n - 1\}$.[22] Functions $F_{m+1}, \dots, F_n$ are constant on (a, b), therefore

$$\frac{1}{W_k} = \frac{F_k}{F_1 \dots F_n} = \text{const} \cdot \frac{1}{F_1} \dots \frac{1}{F_m} \quad \text{on } (a, b) \text{ for } k = m + 1, \dots, n\,;$$
$$\frac{1}{W_k} = \frac{F_k}{F_1 \dots F_n} = \text{const} \cdot \frac{\frac{1}{F_1} \dots \frac{1}{F_m}}{\frac{1}{F_k}} \quad \text{on } (a, b) \text{ for } k = 1, \dots, m\,.$$

Using the first part of the proof we see that functions $1/W_k$, as well as $\frac{(1/F_1)\dots(1/F_m)}{1/F_k}$, are convex on (a, b) for $k = 1, \dots, m$. We apply Lemma 6f1 to functions $1/F_1, \dots, 1/F_m$; though, they decrease, but we may reverse the argument by considering functions $x \mapsto 1/F_k(-x)$ increasing on $(-b, -a)$. We conclude that the function $(1/F_1) \dots (1/F_m)$ is convex, therefore $1/W_k$ is convex for $k = m + 1, \dots, n$. □

6g8. Exercise. If player k is passive on a cell (a, b), $b \ne \infty$, then $1/W_k$ is strictly convex on (a, b).

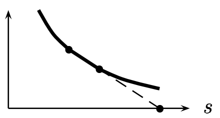
Prove it.

Hint: use 6g6, 6f4, 6f5.

6g9. Exercise. If player k is active on a cell (a, b) and S_k has no atoms on $(s^{\min}, \infty)$, then $1/W_k$ is strictly convex on (a, b).

Prove it.

Hint:



6g10. Exercise. Each action A_k has a density $f_k = F_k'$ on $(s^{\min}, \infty)$;[23] f_k is continuous on $(s^{\min}, \infty)$ except for a finite (maybe empty) or countable set; one-sided limits $f_k(a-)$, $f_k(a+)$ exist[24] for every $a \in (s^{\min}, \infty)$.[25]

Prove it.

Hint: recall 6g5.

[22] If all players are active, convexity is already proven. If all players are passive then $1/W_k$ is constant, therefore convex. The case $m = 1$ is excluded by 6f4.

[23] However, A_k may have atoms outside $(s^{\min}, \infty)$ if S_k has an atom at $s^{\min}$.

[24] For now, $f_k(a+) = \infty$ is allowed if a separates two adjacent cells. Such a situation will be excluded soon. In fact, only $f_k(s^{\min}+)$ can be infinite.

[25] See also 6h8.

6h Jumps between cells, and global convexity

Still, the framework is that of 6f, assuming (6b2), an equilibrium, and in addition (6f3).

6h1. Exercise. If $a \in (s^{\min}, \infty)$ is such that $W_1(a) = V_1(a)$, then

$$\Big(\frac{1}{W_1} \Big)'(a-) \le \Big(\frac{1}{W_1} \Big)'(a+) \,.$$

Prove it.
Hint: $W_1 \le V_1$ (recall (6d5)), and $1/V_1$ is convex.

Let (a,b) and (b,c) be two adjacent cells. Due to 6h1, $(1/W_k)'(b-) \le (1/W_k)'(b+)$ for all k such that player k is active on at least one of the two cells.

Now we'll transfer the property from active to passive players using an argument somewhat similar to 6g but much, much simpler. First, some elementary mathematics.

6h2. Exercise. Let $x_1, \dots, x_n \in \mathbb{R}$ satisfy

$$(x_1 + \dots + x_n) - x_k \ge 0$$

for all $k = 1, \dots, n$, and $n \ge 2$. Then

$$x_1 + \dots + x_n \ge 0 \,.$$

Prove it.
Hint: $n(x_1 + \dots + x_n) \ge x_1 + \dots + x_n$ (though, you may find another argument).

6h3. Lemma. Let (a,b) and (b,c) be two adjacent cells, then

$$\Big(\frac{1}{W_k} \Big)'(b-) \le \Big(\frac{1}{W_k} \Big)'(b+)$$

for $k = 1, \dots, n$.

Proof. Let each of players $m+1, \dots, n$ be passive on both cells (a,b) and (c,d), while each of players $1, \dots, m$ active on at least one of the two cells; here $m \in \{2, \dots, n\}$, since $m = 1$ is excluded by 6f4. We know that the inequality holds for $k = 1, \dots, m$.

Consider numbers

$$x_k = (\ln F_k)'(b+) - (\ln F_k)'(b-) \,;$$

existence of these one-sided derivatives is ensured by existence of $(\ln W_k)'(b-)$ and $(\ln W_k)'(b+)$, since

$$\ln F_k = (\ln F_1 + \dots + \ln F_n) - \ln W_k = \frac{1}{n-1} (\ln W_1 + \dots + \ln W_n) - \ln W_k \,.$$

Note that $x_k = 0$ for $k = m+1, \dots, n$, since these F_k are constant on (a,c). Also,

$$(\ln W_k)'(b+) - (\ln W_k)'(b-) = x_1 + \dots + x_n \quad \text{for } k = m+1, \dots, n \,;$$
$$(\ln W_k)'(b+) - (\ln W_k)'(b-) = (x_1 + \dots + x_n) - x_k \quad \text{for } k = m+1, \dots, n \,.$$

For $k = 1, \dots, m$ we have $(1/W_k)'(b-) \le (1/W_k)'(b+)$, thus[26] $\big(\ln(1/W_k)\big)'(b-) \le \big(\ln(1/W_k)\big)'(b+)$ and $(\ln W_k)'(b-) \ge (\ln W_k)'(b+)$, that is,

$$(x_1 + \dots + x_m) - x_k \le 0 \quad \text{for } k = 1, \dots, m\,.$$

Applying 6h2 to numbers $(-x_1), \dots, (-x_m)$, we conclude that

$$x_1 + \dots + x_m \le 0\,,$$

therefore $(\ln W_k)'(b+) \le (\ln W_k)'(b-)$ for $k = m+1, \dots, n$, and so, $(1/W_k)'(b+) \ge (1/W_k)'(b-)$ for $k = m+1, \dots, n$. □

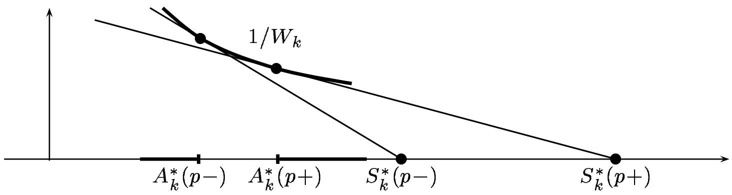
6h4. Proposition. Functions $1/W_k$ are convex on $(s^{\min}, \infty)$ for $k = 1, \dots, n$.

Proof. Follows immediately from 6g7 and 6h3. □

6h5. Exercise. The function $1/W_k$ is strictly convex on every gap (if any) of A_k.
Prove it.
Hint: use 6g8 and 6h3.

6h6. Exercise. Every gap of A_k on $(s^{\min}, \infty)$ corresponds to a gap of S_k. Namely, if $p \in (0,1)$ is such that $s^{\min} < A_k^*(p-) < A_k^*(p+)$ then $s^{\min} < S_k^*(p-) < S_k^*(p+)$.
Prove it.

Hint:



6h7. Theorem. If a signal S_k has no gaps then the corresponding action A_k has no gaps on $(s^{\min}, \infty)$.[27]

Proof. Follows immediately from 6h6. □

By the way, our assumptions do not forbid atoms of signals.

6h8. Theorem. If signals $S_1, \dots, S_n$ have no gaps then actions $A_1, \dots, A_n$ have continuous densities $f_1, \dots, f_n$ on $(s^{\min}, a^{\max})$, where $a^{\max} = \sup\{a : F_1(a) \dots F_n(a) < 1\}$.[28]

Proof. Consider an arbitrary point $a \in (s^{\min}, a^{\max})$ and numbers

$$x_k = (\ln F_k)'(a+) - (\ln F_k)'(a-)$$

(used before, in the proof of Lemma 6h3). It suffices to prove that $x_k = 0$ for all k. It may happen that for some players k, the support of A_k does not contain a; these players are irrelevant, and will be ignored. Thus, 6h1 is applicable to all players, giving[29]

$$(x_1 + \dots + x_n) - x_k \le 0 \quad \text{for } k = 1, \dots, n\,;$$

[26] You see, $(1/W_k)(b-) = (1/W_k)(b+)$.
[27] Below $s^{\min}$ everything may happen, if S_k has an atom at $s^{\min}$.
[28] However, A_k may have atoms on $[0, s^{\min}]$, if S_k has an atom at $s^{\min}$.
[29] Similarly to the proof of 6h3.

therefore

$$x_1 + \dots + x_n \le 0$$

by 6h2 (applied to $(-x_1), \dots, (-x_n)$). Let

$$\begin{aligned} F_k(a) &< 1 \quad \text{for } k = 1, \dots, m, \\ F_k(a) &= 1 \quad \text{for } k = m+1, \dots, n; \end{aligned}$$

here $m \in \{2, \dots, n\}$.[30]

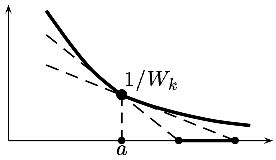
We have

$$x_k \le 0 \quad \text{for } k = m+1, \dots, n,$$

since $(\ln F_k)'(a+) = 0$ for $k = m+1, \dots, n$. On the other hand,

$$(x_1 + \dots + x_n) - x_k = 0 \quad \text{for } k = 1, \dots, m,$$

since $(x_1 + \dots + x_n) - x_k = (\ln W_k)'(a+) - (\ln W_k)'(a-)$, which cannot be (strictly) positive, since A_k cannot have an atom at a, and S_k cannot have a gap.



We have

$$x_1 + \dots + x_n \le x_1 + \dots + x_m = m(x_1 + \dots + x_n),$$

therefore $x_1 + \dots + x_n \ge 0$, thus $x_1 + \dots + x_n = 0$, and then $x_1 = 0, \dots, x_m = 0$. Hence $x_{m+1} + \dots + x_n = x_1 + \dots + x_n = 0$ and $x_{m+1} = 0, \dots, x_n = 0$. □

6i Symmetric auctions: uniqueness

Consider a symmetric first price auction with n players. Condition (6b2) is satisfied automatically. We know (recall 6f2) that every equilibrium satisfying (6f1) is symmetric. Of course, such an equilibrium is unique. The question is, whether every equilibrium satisfies (6f1), or not.

Generally, I do not know. However, assume (6f3), then Theorem 6h7 ensures that actions have no gaps, provided that signals have no gaps. We have to prove that all actions $A_1, \dots, A_n$ have the same support, that is, $a_k^{\max} = a^{\max}$ for all k; here $a_k^{\max} = \sup\{a : F_k(a) < 1\}$ and $a^{\max} = \max_k a_k^{\max} = \sup\{a : F_1(a) \dots F_n(a) < 1\}$; as before, $F_k = F_{A_k}$.

Assume the contrary; say, $a_1^{\max} < a^{\max}$. By Theorem 6h8, densities F_k' are continuous at $a_1^{\max}$. Clearly, $F_1'(a_1^{\max}+) = 0$, and it follows that $F_1'(a_1^{\max}-) = 0$. However,[31]

$$(\ln F_k)'(a_1^{\max}-) - \underbrace{(\ln F_1)'(a_1^{\max}-)}_{=0} = \frac{1}{S^*\big(F_k(a_1^{\max})-\big) - a_1^{\max}} - \frac{1}{S^*(1-) - a_1^{\max}}$$

[30] $m \ne 0$ since $a < a^{\max}$; $m \ne 1$ by 6f4.

[31] Similarly to the argument before 6f2.

for all k such that $a_k^{\max} \ge a_1^{\max}$. It may happen that $a_k^{\max} < a_1^{\max}$ for some players k; these players are irrelevant and will be ignored. Thus, for all k the left-hand side is ≥ 0, while the right-hand side evidently is ≤ 0. It means that both sides vanish. So, $(\ln F_k)'(a_1^{\max}) = 0$ for all k, therefore $(\ln W_k)'(a_1^{\max}) = 0$ for all k. However, that can happen only if $F_k(a_1^{\max}) = 1$ for all k. The contradiction proves the final result of Sect. 6.

6i1. Theorem. A symmetric first price auction with no gaps of signals has one and only one[32] equilibrium satisfying (6f3), and the equilibrium is symmetric.

[32]Except for arbitrary never-winning actions, if $s^{\min}$ is an atom of signals.