Lectures on disordered models (notes under construction!)

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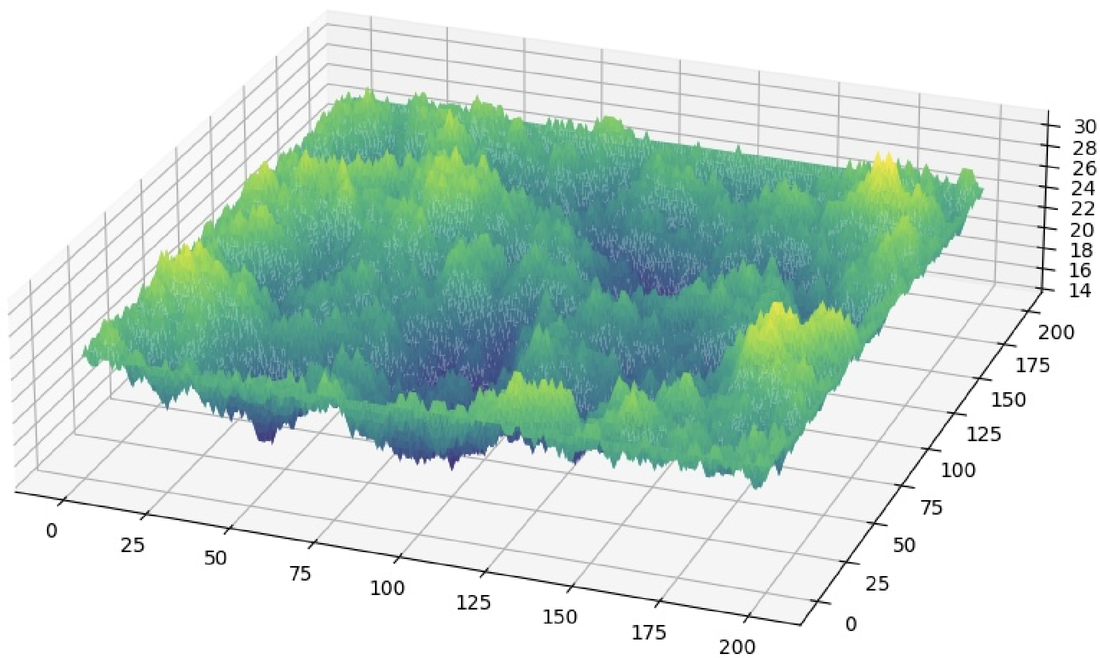

Figure 1: A minimal surface in independent disorder with $d = 2$, $n = 1$.

Consider putting some pictures on the title page, before the table of contents. Do we need an abstract?

In slide show of quantitative disorder effects, fix μ to κ (apriori measure) in first slide describing spin glasses

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Contents

1 Introduction

In these lectures we discuss some of the recent rigorous progress in the analysis of disordered models. Our focus is on disordered spin systems, first-passage percolation and minimal surfaces in random environments. Within these topics, we discuss the existence or absence of long-range order in disordered spin systems and questions of localization and delocalization of interfaces in disordered media.

These notes were initially written for the School on Disordered media, held in January 2024 at the Rényi Institute in Budapest, Hungary. We thank the organizers Ágnes Backhausz, Gábor Pete, Balázs Ráth and Bálint Tóth for their kind invitation to deliver a mini-course on these topics there.

2 Disordered spin systems

Spin systems may alter their properties when placed in non-homogeneous environments. In this section, we consider this effect for the case of a random environment (termed the *disorder*), formed from independent, local, random samples, and our focus is on the existence or absence of long-range order. We emphasize that the disorder is *quenched*; in other words, to sample a configuration of the system, one first samples an instance of the disorder and then samples a configuration from the model's disorder-dependent Hamiltonian.

Our systems are defined on the d-dimensional lattice $\mathbb{Z}^d$ and we write $u \sim v$ to indicate that $u, v \in \mathbb{Z}^d$ are nearest-neighbors (i.e., differ in exactly one coordinate and by exactly one). We also denote the set of edges by $E(\mathbb{Z}^d)$.

To illustrate the topic, we mainly focus on the random-field spin systems described by the following formal Hamiltonians and disorder choices:

1. Random-field Ising model: Configurations are described by $\sigma : \mathbb{Z}^d \to \{-1, 1\}$. The disorder consists of $(\eta_v^{\text{RF-Ising}})_{v\in\mathbb{Z}^d}$, independent standard Gaussian random variables (i.e., of mean 0 and variance 1). The disorder strength is denoted $\lambda > 0$. The formal Hamiltonian is

$$H^{\text{RF-Ising},\eta^{\text{RF-Ising}},\lambda}(\sigma) := -\sum_{u\sim v}\sigma_u\sigma_v - \lambda\sum_v \eta_v^{\text{RF-Ising}}\sigma_v. \tag{1}$$

2. Random-field Potts model: Let $q \geq 2$ integer denote the number of states. Configurations are described by $\sigma : \mathbb{Z}^d \to \{1, 2 \dots, q\}$. The disorder consists of $(\eta_{v,k}^{\text{RF-Potts}})_{v\in\mathbb{Z}^d, k\in\{1,\dots,q\}}$,

independent standard Gaussian random variables. The disorder strength is denoted $\lambda > 0$. The formal Hamiltonian is

$$H^{\text{RF-Potts},\eta^{\text{RF-Potts}},\lambda}(\sigma) := -\sum_{u\sim v} 1_{\sigma_u=\sigma_v} - \lambda \sum_v \sum_{k=1}^q \eta_{v,k}^{\text{RF-Potts}} 1_{\sigma_v=k}. \tag{2}$$

The case $q = 2$ is equivalent to the random-field Ising model (the Hamiltonians differ only by the addition of a disorder dependent term).

3. Random-field Spin $O(n)$ model: Let $n \geq 1$ integer denote the number of components. Configurations are described by $\sigma : \mathbb{Z}^d \to \mathbb{S}^{n-1}$. The disorder consists of $(\eta_v^{\text{RF}-O(n)})_{v\in\mathbb{Z}^d}$, independent standard Gaussian random vectors in $\mathbb{R}^n$ (i.e., of mean 0 and identity covariance matrix). The disorder strength is denoted $\lambda > 0$. The formal Hamiltonian is

$$H^{\text{RF}-O(n),\eta^{\text{RF}-O(n)},\lambda}(\sigma) := -\sum_{u\sim v} \sigma_u \cdot \sigma_v - \lambda \sum_v \eta_v^{\text{RF}-O(n)} \cdot \sigma_v. \tag{3}$$

Here, we endow $\mathbb{R}^n$ with the Euclidean inner product $x \cdot y := \sum_{i=1}^n x_i y_n$ and norm $\|x\|^2 := x \cdot x$, and denote by $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \colon \|x\| = 1\}$ the $(n-1)$-dimensional Euclidean sphere. The case $n = 1$ is again equivalent to the random-field Ising model.

We've restricted to Gaussian disorder for simplicity, but note that other disorder choices (typically having a rotationally-symmetric distribution around 0), are also of interest and are discussed in the literature.

To obtain a probability measure (termed a finite-volume Gibbs measure) from the formal Hamiltonian, one uses the following standard prescription: Fix a temperature $T > 0$, a finite $\Lambda \subset \mathbb{Z}^d$ and a configuration τ (the boundary values). Given a Hamiltonian $H^\#$, we write $H^\#_{\Lambda,\tau}$ for the Hamiltonian which includes only the terms that contain a spin in Λ, and where the spins σ_v with $v \notin \Lambda$ are replaced by τ_v. For instance, for the random-field Ising model,

$$H_{\Lambda,\tau}^{\text{RF-Ising},\eta^{\text{RF-Ising}},\lambda}(\sigma) := -\sum_{\substack{u\sim v\\ u,v\in\Lambda}} \sigma_u\sigma_v - \sum_{\substack{u\sim v\\ u\in\Lambda, v\notin\Lambda}} \sigma_u\tau_v - \lambda \sum_{v\in\Lambda} \eta_v^{\text{RF-Ising}}\sigma_v. \tag{4}$$

Then, the finite-volume Gibbs measure on configurations is given by

$$d\mathbb{P}_{\Lambda,\tau}^{\#,T}(\sigma) := \frac{1}{Z_{\Lambda,\tau}^{\#,T}} e^{-\frac{1}{T} H^\#_{\Lambda,\tau}(\sigma)} \prod_{v\in\Lambda} d\kappa^\#(\sigma_v) \prod_{v\in\mathbb{Z}^d\setminus\Lambda} d\delta_{\tau_v}(\sigma_v) \tag{5}$$

where $Z^\#_{\Lambda,\tau}$ (the partition function) is chosen so that $\mathbb{P}^\#_{\Lambda,\tau}$ is a probability measure and where $\kappa^\#$ denotes the apriori (or single site) measure on spin states for the model. The apriori measure is the counting measure on $\{-1, 1\}$ for the random-field Ising and Edwards–Anderson spin glass models, the counting measure on $\{1, \ldots, q\}$ for the random-field q-state Potts model and the uniform measure on $\mathbb{S}^{n-1}$ for the random-field spin $O(n)$ model.

We use the notation $\langle\cdot\rangle_{\Lambda,\tau}^{\#,T}$ for the expectation operator corresponding to the measure $\mathbb{P}_{\Lambda,\tau}^{\#,T}$.

We will mostly be interested in the properties of the models at low temperatures. In fact, in the presence of disorder, it turns out that the relevant phenomena already arise at *zero temperature*, and, mostly for simplicity, we will focus solely on this case. The zero-temperature measure, or *finite-volume ground state*, $\mathbb{P}^{\#,0}_{\Lambda,\tau}$ is defined as the limit in distribution of $\mathbb{P}^{\#,T}_{\Lambda,\tau}$ as $T \downarrow 0$. It is supported on the *minimizers* of the Hamiltonian $H^{\#}_{\Lambda,\tau}$, which we term *finite-volume ground configurations*[1]. In fact, in our examples above it is easily seen that there is a unique minimizer almost surely, so that $\mathbb{P}^{\#,0}_{\Lambda,\tau}$ is a delta measure (but note that there exist random λ, Λ, τ for which there are multiple minimizers).

An important role is played by the Gibbs measures of the model: These are the measures which arise as limits in distribution of $\mathbb{P}^{\#,T}_{\Lambda_n,\tau_n}$ for some sequence of domains $\Lambda_n \subset \mathbb{Z}^d$ which inrease to $\mathbb{Z}^d$ and some sequence of configurations τ_n, and also the convex combinations of these limits. The set of Gibbs measures is naturally random, depending on the realization of the disorder. Gibbs states at zero temperature are called *ground states.* They are supported on *ground configurations*, configurations σ which locally minimize the formal Hamiltonian $H^{\#}$ in the sense that if σ' differs from σ in finitely many vertices than $H^{\#}(\sigma') - H^{\#}(\sigma) \geq 0$ (noting that this energy difference is well defined, at least in the above examples, as only finitely many terms differ in the sums defining the Hamiltonians).

The above spin systems may be considered as perturbations of the corresponding pure (i.e., non-disordered) spin systems obtained by setting $\lambda = 0$ in the formal Hamiltonians. For instance, the random-field Ising model may be thought of as a perturbation of the Ising model, defined by the formal Hamiltonian

$$H^{\text{Ising}}(\sigma) := -\sum_{u\sim v} \sigma_u \sigma_v. \tag{6}$$

Our focus will then naturally be on the way in which the added disorder alters the properties of the underlying spin system.

2.1 The Imry-Ma phenomenon: Absence and preservation of long-range order in the presence of a random field

The pure (non-disordered) Ising, Potts and spin $O(n)$ models are well known to undergo a magnetization phase transition (see, e.g., [10]):

1. (Ising model). For $L \geq 0$ integer, let

$$\Lambda_L := \{-L, \dots, L\}^d \tag{7}$$

and consider the Ising model in Λ_L with $+$-boundary conditions, i.e., with $\tau \equiv +1$. Then in all dimensions $d \geq 2$ there exists a critical temperature $T_c^{\text{Ising}}(d)$ such that

$$\lim_{L\to\infty} \langle \sigma_0 \rangle^{\text{Ising},T}_{\Lambda_L,+} \begin{cases} = 0 & T > T_c^{\text{Ising}}(d) \\ > 0 & T < T_c^{\text{Ising}}(d) \end{cases}. \tag{8}$$

*** and it is further known that the limit is also zero at the critical temperature. Mention also the exponential rate of decay to zero at high temperatures? Divide into two parts, with the first part having supremum over boundary conditions? ***

[1] In the literature, the term finite-volume ground states is often also used for these minimizers

2. (Potts model). When placing the Potts model under 1-boundary conditions (i.e., $\tau \equiv 1$) then in all dimensions $d \geq 2$ there exists a critical temperature $T_c^{\text{Potts}}(d)$ such that

$$\lim_{L\to\infty} \langle 1_{\sigma_0=1}\rangle^{\text{Potts},T}_{\Lambda_L,1} \begin{cases} = \frac{1}{q} & T > T_c^{\text{Potts}}(d) \\ > \frac{1}{q} & T < T_c^{\text{Potts}}(d) \end{cases}. \tag{9}$$

3. ($O(n)$ model with $n \geq 2$). The pure $O(n)$ models with $n \geq 2$ have a continuous symmetry - for all rotations R in $\mathbb{R}^n$, all domains Λ, boundary values τ and configurations σ, the Hamiltonians satisfy $H^{O(n)}_{\Lambda,R\tau}(R\sigma) = H^{O(n)}_{\Lambda,\tau}(\sigma)$ where $R\rho : \mathbb{Z}^d \to \mathbb{S}^{n-1}$ is the rotated configuration defined by $(R\rho)_v := R(\rho_v)$. The Mermin–Wagner theorem thus dictates the absence of a magnetization phase transition in dimension $d = 2$ at all positive temperatures $T > 0$:

$$\lim_{L\to\infty} \sup_{\tau:\mathbb{Z}^d\to\mathbb{S}^{n-1}} \|\langle\sigma_0\rangle^{O(n),T}_{\Lambda_L,\tau}\| = 0. \tag{10}$$

An important fact, which will not be discussed here, is that a phase transition does occur in dimension $d = 2$: the famed Berezinskii–Kosterlitz–Thouless transition from a high-temperature regime with exponential decay of the above supremum to a low-temperature regime with power-law decay. In dimensions $d \geq 3$ a magnetization phase transition occurs: When placing the $O(n)$ model under $\to$-boundary conditions (i.e., $\tau \equiv (1, 0, \ldots, 0)$) then in all dimensions $d \geq 3$ there exists a critical temperature $T_c^{O(n)}(d)$ such that

$$\lim_{L\to\infty} \|\langle\sigma_0\rangle^{O(n),T}_{\Lambda_L,\to}\| \begin{cases} = 0 & T > T_c^{O(n)}(d) \\ > 0 & T < T_c^{O(n)}(d) \end{cases}. \tag{11}$$

How does the addition of the random field affect these phase transitions? The added disorder naturally competes with the ferromagnetic interaction of the pure Hamiltonian and, at least intuitively, should weaken the long-range order. One may consider several parameter regimes according to the temperature T and disorder strength λ.

For a sufficiently high threshold temperature $T_0^{\#}(d)$, it follows from Dobrushin's uniqueness criterion *** ref *** that the model is disordered for all temperatures $T > T_0^{\#}(d)$ and all disorder strengths $\lambda \geq 0$ *** in the sense of exponential decay? ***. Moreover, for the random-field Ising model, it has been shown that one may take $T_0^{\text{RF-Ising}}(d) = T_c^{\text{Ising}}(d)$ *** ref Ding–Sun–Song [6] ***. It is apparently open to obtain a similar result for the random-field Potts models with $q \geq 3$ and the random-field $O(n)$ models with $n \geq 2$. *** check that it is indeed still open ***

There are several results in the literature showing that there exists a threshold disorder strength $\lambda_0^{\#}(d)$ such that the models are also disordered when the disorder strength $\lambda > \lambda_0^{\#}(d)$ at all temperatures T, including zero temperature! *** reference such results. For the XY model, reference Feldman. Is the general $O(n)$ case also done? Is the XY case also done at positive temperatures? ***

Exercise: Prove the above assertion at zero temperature for the random-field Ising and Potts models. *** can use a percolation argument with the points of large disorder. Can make this a guided exercise and reference [2, Appendix A] ***

Given the above results, interest is naturally directed towards the regime of low temperature and weak disorder strength. This was famously addressed in the physics literature by the work of Imry–Ma, who argued that the magnetized phase will be lost, in the presence of arbitrarily weak disorder, in dimension $d = 2$ for the random-field Ising and Potts models (and more general systems), and in all dimensions $d \leq 4$ for the random-field $O(n)$ model with $n \geq 2$. This prediction was famously made rigorous by the work of Aizenman–Wehr, who greatly extended its scope. Imry–Ma further predicted that the magnetized phase will be retained by the disordered system in higher dimensions (dimensions $d \geq 3$ for the random-field Ising and Potts models and dimensions $d \geq 5$ for the random-field $O(n)$ models with $n \geq 2$). For the random-field Ising model, this claim was under significant debate in the physics literature, with Parisi–Surlas *** presenting arguments against it. The debate was famously resolved by the rigorous works of Imbrie *** (at zero temperature) and Bricmont–Kupiainen *** (at all temperatures) who showed that the Imry–Ma prediction is correct: the magnetized phase is retained already in three dimensions.

*** Open problem: Long-range order for the random-field spin $O(n)$ model in dimensions $d \geq 5$ (even at zero temperature and even for the random-field XY model). ***

*** Can add here the $d \geq 3$ work of Ding–Liu–Xia that the critical temperature can be arbitrarily close to the pure Ising model if the disorder strength is sufficiently small. There is a related work of Ding–Huang–Xia in $d = 2$ at the critical temperature to find the critical scaling of the disorder strength with the size of the box. ***

The next sections discuss the Imry–Ma prediction in more detail. We first present a recent short proof of the existence of the magnetized phase in dimensions $d \geq 3$ due to Ding–Zhuang [7]. Then, we discuss quantitative aspects of the absence of phase transition in lower dimensions, presenting the work of Dario–Harel–Peled [5] and highlighting the many remaining open questions.

2.1.1 Long-range order in the random-field Ising and Potts models

In this section we present the argument of Ding–Zhuang [7] for the existence of long-range order in the random-field Ising model in dimensions $d \geq 3$, at low temperature and weak disorder. The argument can be thought of as a version of the famous Peierls argument for showing long-range order, adapted to disordered spin systems. It extends a technique of Fisher–Fröhlich–Spencer [8] which was introduced in an earlier attempt to settle the problem (this latter work gave strong support to the long-range order prediction by showing that it would occur if there were "no domain walls within domain walls"; see also *** Chalker ***).

The argument also adapts to the random-field Potts model, and gave the first proof of existence of a magnetized phase there.

Theorem 2.1. *For every $d \geq 3$ there exists $T_0 > 0$ and $\lambda_0 > 0$ such that for all $0 \leq T < T_0$ and $0 \leq \lambda < \lambda_0$,*

$$\lim_{L\to\infty} \mathbb{E}\left[\langle\sigma_0\rangle_{\Lambda_L,+}^{\text{RF-Ising},\eta^{\text{RF-Ising}},T}\right] > 0. \tag{12}$$

To present the argument in its simplest form, we discuss only the zero temperature case random-field Ising model, leaving the extension to the other cases as an exercise *** add the exercise ***.

Fix $d \geq 3$. Let λ_0 be chosen sufficiently small and positive for the following arguments and fix a disorder strength $0 \leq \lambda < \lambda_0$. Fix $L \geq 0$ integer. For brevity, in the proof, we remove λ and L from most of the notation and write η for $\eta^{\text{RF-Ising}}$. We let σ^η be the, almost-surely unique, finite-volume ground configuration of the Ising model in Λ_L with $+$-boundary values. Also denote the finite-volume ground energy by

$$\mathrm{GE}^\eta := H^{\text{RF-Ising},\eta}_{\Lambda_L,+}(\sigma^\eta). \tag{13}$$

We denote the edge boundary of a set $A \subset \mathbb{Z}^d$ by

$$\partial A := \{\{u,v\} \in E(\mathbb{Z}^d) \colon |\{u,v\} \cap A| = 1\}. \tag{14}$$

For an integer $\ell \geq 1$ we let

$$\begin{aligned} \mathcal{C}_\ell &:= \{A \subset \mathbb{Z}^d \colon A \text{ finite and connected, } A^c \text{ connected}, 0 \in A, |\partial A| = \ell\}, \\ \mathcal{C} &:= \cup_{\ell=0}^\infty \mathcal{C}_\ell \end{aligned} \tag{15}$$

The first observation is that if $\sigma^\eta_0 = -1$ then there exists a (random) set $A \in \mathcal{C}$, $A \subset \Lambda_L$, such that $\sigma^\eta \equiv -1$ on the interior vertex boundary of A and $\sigma^\eta \equiv 1$ on the exterior vertex boundary of A. Suppose A is such a set. Define a new configuration and random field by flipping the configuration and random field on A,

$$\begin{aligned} \sigma^{\eta,A}_v &:= \begin{cases} -\sigma^\eta_v & v \in A \\ \sigma^\eta_v & v \notin A \end{cases}, \\ \eta^A_v &:= \begin{cases} -\eta_v & v \in A \\ \eta_v & v \notin A \end{cases}. \end{aligned} \tag{16}$$

The discrete ± 1 symmetry of the random-field Ising model then leads to the energy gap

$$H^{\text{RF-Ising},\eta}(\sigma^\eta) - H^{\text{RF-Ising},\eta^A}(\sigma^{\eta,A}) \geq 2|\partial A|. \tag{17}$$

This implies that also

$$\mathrm{GE}^\eta - \mathrm{GE}^{\eta^A} \geq 2|\partial A| \tag{18}$$

The argument will be (eventually) concluded by proving that, for each ℓ,

$$\mathbb{P}\left(\exists A \in \mathcal{C}_\ell \text{ such that } |\,\mathrm{GE}^\eta - \mathrm{GE}^{\eta^A}\,| \geq 2|\partial A|\right) \leq C_d \exp\left(-c_d \frac{\ell^{\frac{d-2}{d-1}}}{\lambda^2}\right) \tag{19}$$

(with $C_d, c_d > 0$ depending only on d).

The proof of (19) makes use of the concentration properties of the distribution of the ground energy. The first and fundamental ingredient is the following consequence of the Gaussian isoperimetric inequality of Borell and Tsirelson–Ibragimov–Sudakov *** ref? ***.

Theorem 2.2 (Concentration of maximum of Gaussian process)**.** *Let T be a compact set. Let $(X_t)_{t\in T}$ be a continuous Gaussian process (not necessarily centered). Denote $M_t := \max_{t\in T} X_t$. Then $\mathbb{E}(M_t) < \infty$ and for every $u > 0$,*

$$\mathbb{P}(|M_t - \mathbb{E}(M_t)| \geq u) \leq 2e^{-\frac{u^2}{2\sigma_T^2}} \tag{20}$$

with $\sigma_T^2 := \sup_{t\in T} \mathrm{Var}(X_t)$.

This result is applied conditionally. For each finite $A \subset \mathbb{Z}^d$, write η_{A^c} for the restriction of η to A^c. Observe that conditionally on η_{A^c}, GE^η is the minimum of a Gaussian process on the compact set $T = \{-1,1\}^{\Lambda_L}$, whose maximal variance is $\lambda^2|A \cap \Lambda_L| \le \lambda^2|A|$. Theorem 2.2 thus implies that, almost surely,

$$\mathbb{P}\left(\left|\,\mathrm{GE}^\eta - \mathbb{E}(\mathrm{GE}^\eta \mid \eta_{A^c})\right| \ge u \mid \eta_{A^c}\right) \le 2e^{-\frac{u^2}{2\lambda^2|A|}}. \tag{21}$$

This will be applied through the following useful corollary.

Corollary 2.3. *There exist $C, c > 0$ such that for each $A \subset \mathbb{Z}^d$ and $u > 0$,*

$$\mathbb{P}\left(\left|\,\mathrm{GE}^{\eta,L,\lambda} - \mathrm{GE}^{\eta^A,L,\lambda}\,\right| \ge u\right) \le Ce^{-c\frac{u^2}{\lambda^2|A|}}, \tag{22}$$

and also for each $A, A' \subset \mathbb{Z}^d$ and $u > 0$,

$$\mathbb{P}\left(\left|\,\mathrm{GE}^{\eta^{A'},L,\lambda} - \mathrm{GE}^{\eta^A,L,\lambda}\,\right| \ge u\right) \le Ce^{-c\frac{u^2}{\lambda^2|A\Delta A'|}}, \tag{23}$$

where $A\Delta A'$ is the symmetric difference of A and A'.

Proof. The essential point is that, almost surely, $\mathbb{E}(\mathrm{GE}^\eta \mid \eta_{A^c}) = \mathbb{E}(\mathrm{GE}^{\eta^A} \mid \eta_{A^c})$, which follows from the fact that η^A has the same distribution as η and $\eta^A = \eta$ on A^c. It thus follows from (21) that, almost surely,

$$\mathbb{P}\left(\left|\,\mathrm{GE}^{\eta,L,\lambda} - \mathrm{GE}^{\eta^A,L,\lambda}\,\right| \ge u \mid \eta_{A^c}\right) \le Ce^{-c\frac{u^2}{\lambda^2|A|}}, \tag{24}$$

The inequality (22) follows by taking the expectation of (24). Inequality (23) follows from (22) by replacing η with $\eta^{A'}$ (which has the same distribution as η). □

To understand (19) better, observe first that the same bound holds for a fixed deterministic finite set $A \subset \mathbb{Z}^d$ by (22) and the isoperimetric inequality

$$|A| \le C_d|\partial A|^{d/(d-1)}. \tag{25}$$

Indeed,

$$\mathbb{P}(\mathrm{GE}^\eta - \mathrm{GE}^{\eta^A} \ge 2|\partial A|) \le C\exp\left(-c\frac{|\partial A|^2}{\lambda^2|A|}\right) \le C\exp\left(-c_d\frac{|\partial A|^{\frac{d-2}{d-1}}}{\lambda^2}\right) \tag{26}$$

where we use the convention that the values of the positive C, c, C_d, c_d may change from expression to expression, with C, C_d only increasing and c, c_d only decreasing (but C, c remain absolute constants and C_d, c_d depend only on d).

However, the estimate (26) does not suffice to establish (19) via a union bound, since the number of subsets $A \in \mathcal{C}$ with $|\partial A| \le \ell$ is at least $c_d \exp(C_d\ell)$ (this may be argued directly. One may also consult [9] or [3, Theorem 6 and Theorem 7], noting the equivalence in [4, Appendix A]). Instead, the estimate (19) is derived from the concentration bound (23) using a coarse-graining technique (or chaining argument) introduced by Fisher–Fröhlich–Spencer [8] in a closely-related context. We proceed to elaborate on this technique.

Given a set $A \subset \mathbb{Z}^d$ and integer $N \geq 1$, let A_N be the N-coarse-grained version of A defined as the union of all cubes $B \subset \mathbb{Z}^d$, of the form $v + \{0, 1, \dots, N-1\}^d$ with $v \in N\mathbb{Z}^d$, which satisfy $|A \cap B| \geq \frac{1}{2}|B|$. We consider all possible coarse grainings of sets in $\mathcal{C}_\ell$,

$$\mathcal{C}_\ell^N := \{A_N \colon A \in \mathcal{C}_\ell\}. \tag{27}$$

The following basic inputs are established in [8] *** for $d = 3$ and maybe special value of the parameter; see also [4] for extensions ***

Proposition 2.4. *For each integer $\ell, N \geq 1$,*

$$|\mathcal{C}_\ell^N| \leq C_d e^{C_d \frac{\ell}{N^{d-1}} \log(N+1)} \tag{28}$$

and, for each $A \in \mathcal{C}_\ell$,

$$|A_{2N} \Delta A_N| \leq C_d N \ell. \tag{29}$$

*** Very roughly, $|\partial A_N| \approx |\partial A|$ so that A_N may be regarded as a set with surface volume $|\partial A|/N^{d-1}$ after shrinking the lattice $\mathbb{Z}^d$ by a factor N. This is complicated, however, by the fact that A_N need not be connected or have connected complement ***

One may then prove (19) via the following chaining argument. Write the telescopic expansion

$$\mathrm{GE}^{\eta} - \mathrm{GE}^{\eta^A} = \sum_{k=0}^{K-1} \left(\mathrm{GE}^{\eta^{A_{2^{k+1}}}} - \mathrm{GE}^{\eta^{A_{2^k}}} \right) \tag{30}$$

where we note that $A_{2^0} = A_1 = A$ and where we choose K sufficiently large that $A_{2^K} = \emptyset$ (so that $\eta^{A_{2^K}} = \eta$). Specifically, choosing K so that 2^K has order $\ell^{\frac{1}{d-1}}$ suffices by the isoperimetric inequality (25). Then, for each choice of positive coefficients $(\alpha_k)_{k=0}^{K-1}$ summing to 1 we have, using Proposition 2.4,

$$\begin{aligned}
&\mathbb{P}\left(\exists A \in \mathcal{C}_\ell \text{ such that } |\,\mathrm{GE}^{\eta} - \mathrm{GE}^{\eta^A}\,| \geq 2|\partial A|\right) \\
&\quad\leq \sum_{k=0}^{K-1} \mathbb{P}\left(\exists A \in \mathcal{C}_\ell \text{ such that } |\,\mathrm{GE}^{\eta^{A_{2^{k+1}}}} - \mathrm{GE}^{\eta^{A_{2^k}}}\,| \geq 2\alpha_k \ell\right) \\
&\quad\leq \sum_{k=0}^{K-1} \sum_{\substack{B \in \mathcal{C}_\ell^{2^k}, B' \in \mathcal{C}_\ell^{2^{k+1}} \\ \exists A \in \mathcal{C}_\ell \text{ with } B = A_{2^k}, B' = A_{2^{k+1}}}} \mathbb{P}\left(|\,\mathrm{GE}^{\eta^{B'}} - \mathrm{GE}^{\eta^B}\,| \geq 2\alpha_k \ell\right) \\
&\qquad\leq \sum_{k=0}^{K-1} \sum_{\substack{B \in \mathcal{C}_\ell^{2^k}, B' \in \mathcal{C}_\ell^{2^{k+1}} \\ \exists A \in \mathcal{C}_\ell \text{ with } B = A_{2^k}, B' = A_{2^{k+1}}}} C e^{-c \frac{\alpha_k^2 \ell^2}{\lambda^2 |B \Delta B'|}} \\
&\qquad\qquad\leq \sum_{k=0}^{K-1} C_d e^{C_d (k+1) \frac{\ell}{2^{k(d-1)}}} e^{-c_d \frac{\alpha_k^2 \ell}{\lambda^2 2^k}}
\end{aligned} \tag{31}$$

which one may check is less than the right-hand side of (19) when $0 \leq \lambda \leq \lambda_0$ with $\lambda_0 \leq c_d$ positive but sufficiently small, and letting $\alpha_k = \gamma 2^{-\frac{1}{4}\min\{k,K-k\}}$ with γ a normalizing constant ensuring that the α_k sum to 1.

*** Exercise: Extend argument to low, positive temperatures. Change ground energy to free energy.

Exercise: Extend argument to random-field Potts model. ***

2.1.2 Quantitative estimates on the absence of magnetization in low-dimensional systems

*** Here we will review results from Dario–Harel–Peled [5]. ***

*** Point to exercise (maybe in appendix?) on the absence of a magnetized phase for the two-dimensional random-field Ising model at zero temperature ***

*** Open problem: Uniformity of distribution of random-field Potts spin at the origin in dimension $d = 2$. ***

*** Mention also quantum version [1] and its accompanying papers in the physics literature ***

References

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