RANDOM MATRICES HOMEWORK SHEET 1

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To hand in by December 26 to the instructor in class.
The solutions should be written in English if possible.

The numbering of exercises is from "An Introduction to Random Matrices" by Anderson, Guionnet, Zeitouni which is available at `http://cims.nyu.edu/~zeitouni/cupbook.pdf`.

(i) Exercise 2.1.5: Recall the semicircle distribution whose density is $\sigma(x) := \frac{1}{2\pi}\sqrt{4-x^2}1_{|x|\leqslant 2}$. Define its Stieltjes transform by

$$S(z) := \int_{\mathbb{R}} \frac{\sigma(x)}{x-z}dx, \quad z \in \mathbb{C}\setminus[-2,2].$$

Prove that

$$S(z) = \frac{z}{2}\left(\sqrt{1-\frac{4}{z^2}}-1\right), \quad z \in \mathbb{C}\setminus[-2,2].$$

Hint: You may rely on the generating function of the Catalan numbers (Lemma 2.1.3).

(ii) Solve Exercise 2.1.30 from the book.

Clarification: The assumptions on X_N is that it is a real symmetric $N\times N$ matrix whose entries are independent except for the symmetry restriction (that is, on and above diagonal entries are independent), though not necessarily identically distributed, have zero mean and satisfy the bound $\sup_{N,i,j}\mathbb{E}e^{\lambda N X_N(i,j)^2} \leqslant C$ for some $\lambda, C > 0$.
In part (a) one needs to add the assumption that $\|z\|_2 = 1$.
In part (b) the term $z^T X_N z_i$ should be replaced by $(z-z_i)^T X_N z_i$. One may also prove instead the related inequality that $(1-\delta)^2 \sup_{z:\,\|z\|_2=1} z^T X_N z \leqslant \sup_{z_i\in\mathcal{N}_\delta} z_i^T X_N z_i$.

Hints to part (a): It may be of use to prove that if $W_1,\dots,W_N$ are independent zero mean random variables satisfying $\sup_i \mathbb{E}e^{\lambda W_i^2} \leqslant C$ for some $\lambda, C > 0$ then there exist $\lambda', C' > 0$ (depending only on λ and C) such that $\mathbb{E}e^{\lambda'(a_1W_1+\cdots+a_NW_N)^2} \leqslant C'$ for all $a_1,\dots,a_N\in\mathbb{R}$ satisfying $a_1^2+\cdots+a_N^2 = 1$. One way to approach this is to first prove that there exists $c>0$ such that $\sup_i \mathbb{E}e^{sW_i} \leqslant e^{cs^2}$ for all $s\in\mathbb{R}$.

It may also be helpful to note that if $X = Y + Z$ for random matrices X, Y, Z then the event $\|Xz\|_2 > C$ implies that either $\|Yz\|_2 > \frac{C}{2}$ or $\|Zz\|_2 > \frac{c}{2}$. This can be used to avoid dealing with the lack of independence stemming from the symmetry of X_N.

(iii) Recall that a sequence of probability measures (μ_n) on $\mathbb{R}$ converges weakly to a probability measure μ on $\mathbb{R}$ if

$$\int f d\mu_n \to \int f d\mu \quad \text{as } n\to\infty$$

for every bounded, continuous $f:\mathbb{R}\to\mathbb{R}$.

(a) Prove that μ_n converges weakly to μ if and only if

$$\sup_f \left|\int f d\mu_n - \int f d\mu\right| \to 0 \quad \text{as } n\to\infty$$

where the supremum is taken over all bounded, Lipschitz functions with constant 1, that is, all f in

$$\text{BLip} := \{f:\mathbb{R}\to\mathbb{R} :\ |f(x)|\leqslant 1 \text{ for all } x \text{ and } |f(x)-f(y)|\leqslant|x-y| \text{ for all } x,y\}.$$

Remark: The same is true for probability measures over any Polish space.
Hint: For each $\varepsilon > 0$ there is an M with $\mu([-M,M]) \geqslant 1-\varepsilon$. Approximate with piecewise linear functions.

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(b) Let d be a metric on probability measures on $\mathbb{R}$ satisfying that $d(\mu_n, \mu) \to 0$ as $n \to \infty$ if and only if μ_n converges weakly to μ. Let (μ_n) be a sequence of *random* probability measures and μ be a *deterministic* probability measure. Prove that μ_n converges to μ in the metric d in probability, in the sense that

$$\text{for every } \varepsilon > 0, \ \lim_{n\to\infty} \mathbb{P}(d(\mu_n, \mu) > \varepsilon) = 0,$$

if and only if μ_n converges to μ weakly in probability, in the sense that

$$\text{for every bounded, continuous } f : \mathbb{R} \to \mathbb{R} \text{ and every } \varepsilon > 0, \ \lim_{n\to\infty} \mathbb{P}\left(\left|\int f d\mu_n - \int f d\mu\right| > \varepsilon\right) = 0.$$

Remark: Part (a) of the exercise gives a metric satisfying the condition. In class we applied this to the case that μ_n is the empirical measure of eigenvalues of an $n \times n$ Wigner matrix and μ is the semicircle law.

Hint: Starting with convergence in d in probability, one may use an argument of the form "every subsequence has a further subsubsequence . . . ". In the other direction, one may develop the ideas in part (a) of the exercise.

(iv) Exercise 2.3.4.

(a) Prove that for any $u \geqslant 0$, $v \in \mathbb{R}$,

$$uv \leqslant u \log u - u + e^v.$$

Remark: This is a consequence of Young's inequality (but may also be proved directly). As usual, we set $0 \log 0 := 0$.

(b) Let P be a probability measure on $\mathbb{R}^d$. Let $f : \mathbb{R}^d \to [0, \infty)$ be in $L^1(P)$. Prove that

$$\int f \log\left(\frac{f}{\int f dP}\right) dP = \sup\left\{\int fg dP \ : \ g : \mathbb{R}^d \to \mathbb{R} \text{ satisfies } \int e^g dP \leqslant 1\right\}.$$

(c) Let $Q_1, \ldots, Q_d$ be probability measures on $\mathbb{R}$ and $P := Q_1 \times Q_2 \times \cdots \times Q_d$. Let $g : \mathbb{R}^d \to R$ satisfy $\int e^g dP \leqslant 1$. Define

$$g^i(x_1, \ldots, x_d) := \log\left(\frac{\int e^{g(x_1,\ldots,x_d)} dQ_1(x_1) \ldots dQ_{i-1}(x_{i-1})}{\int e^{g(x_1,\ldots,x_d)} dQ_1(x_1) \ldots dQ_i(x_i)}\right), \quad 1 \leqslant i \leqslant d.$$

Prove that for any $f : \mathbb{R}^d \to [0, \infty)$ in $L^1(P)$,

$$\int fg dP \leqslant \sum_{i=1}^{d} \int\int f_i \cdot (g^i)_i dQ_i dP, \tag{1}$$

where for $h : \mathbb{R}^d \to \mathbb{R}$ and fixed $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_d$ we let $h_i : \mathbb{R} \to \mathbb{R}$ be defined by $h_i(x_i) := h(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_d)$ (thus, a more detailed form of the integral on the right-hand side of (1) is $\int\int f_i(x_i) \cdot (g^i)_i(x_i) dQ_i(x_i) dP(x_1, \ldots, x_d)$.

(d) Deduce that if $Q_1, \ldots, Q_d$ satisfy the log-Sobolev inequality with constant $c > 0$ then the same is true for their product measure P.