Advanced Topics in Probability (Fall 2021)
Homework Problems

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1 Random walks in a homogeneous environment and one-dimensional random walks in random environment

1. (Recurrence/transience for homogeneous random walks on $\mathbb{Z}^d$)
Let X be a random variable taking values in $\mathbb{Z}^d$ and satisfying the following assumptions:

- (Finite second moment) $\mathbb{E}(\|X\|^2) < \infty$
(where $\|x\| = \sqrt{\sum_{j=1}^d |x_j|^2}$ is the Euclidean norm).
- (Full linear span) $\mathbb{P}(\theta \cdot X \neq 0) > 0$ for all $\theta \in \mathbb{R}^d \setminus \{0\}$
(where $x \cdot y = \sum_{j=1}^d x_j y_j$ is the standard inner product).

Define the characteristic function $\phi : \mathbb{R}^d \to \mathbb{C}$ by

$$\phi_X(\theta) := \mathbb{E}\exp(i\theta \cdot X). \tag{1}$$

(a) Prove that ϕ_X is continuous.
Prove that if $\phi_X(\theta_0) = 1$ for some θ_0 then $\phi_X(\theta_0 + \theta) = \phi_X(\theta)$ for all θ.

(b) Prove that

$$\phi_X(\theta) = 1 + i\theta^t\,\mathbb{E}(X) - \frac{1}{2}\theta^t\,\mathbb{E}(XX^t)\theta + o(\|\theta\|^2) \text{ as } \|\theta\| \downarrow 0 \tag{2}$$

where A^t denotes the transpose of the matrix (or vector) A and where we regard θ and X as column vectors.
Hint: It may help to use the fact that $|e^{ix} - (1 + ix - \frac{1}{2}x^2)| \leqslant \min\left\{|x|^2, \frac{|x|^3}{6}\right\}$ for $x \in \mathbb{R}$.

(c) Define the random walk

$$S_0 := 0 \quad \text{and} \quad S_n := X^1 + \cdots + X^n \quad \text{for } n \geqslant 1, \tag{3}$$

where the $(X^j)_{j\geqslant 1}$ are independent and distributed as X.
Prove that S_n is transient in dimensions $d \geqslant 3$.

(d) Assume additionally that X is symmetric, i.e., $X \overset{d}{=} -X$.
In dimensions $d = 1, 2$, prove that S_n is recurrent.

Remark 1: The Fourier inversion formula may be helpful. For any random variable Z taking values in $\mathbb{Z}^d$ it holds that

$$\mathbb{P}(Z = z) = (2\pi)^{-d} \int_{[-\pi,\pi]^d} \exp(-i\theta \cdot z)\phi_Z(\theta)d\theta, \quad z \in \mathbb{Z}^d. \tag{4}$$

Remark 2: The assumptions of finite second moment, full linear span and symmetry (in dimensions $d = 1, 2$) are not necessary for the conclusion of the exercise.
In dimensions $d \geqslant 3$, any random walk for which the support of X has at least a three-dimensional linear span is transient.
In dimension $d = 2$ recurrence follows from the assumption that $\frac{S_n}{\sqrt{n}}$ converges to a Gaussian distribution.
In dimension $d = 1$, recurrence follows from the assumption that $\frac{S_n}{n}$ converges to 0 in probability (Chung-Fuchs theorem).

These facts, and their extensions to non-lattice random walks, can be found, e.g., in the book "Probability: Theory and Examples" by Rick Durrett.

For the next two exercises we remind Birkhoff's ergodic theorem.

Let $(A, \mathcal{A})$ be a measurable space. Let $\Omega := \{(y_n)_{n\geqslant 1}\colon y_n \in A \text{ for all } n\}$. Define the shift map $T : \Omega \to \Omega$ by $T(y_1, y_2, \ldots) = (y_2, y_3, \ldots)$. The invariant sigma algebra $\mathcal{I}$ is the set of all measurable $E \subset \Omega$ satisfying $TE = E$ (where $TE = \{Ty\colon y \in E\}$). A random sequence $Y = (Y_1, Y_2, \ldots) \in \Omega$ is called stationary if $Y \overset{d}{=} T(Y)$.

The ergodic theorem: Let Y be a stationary sequence. For each $f : \Omega \to \mathbb{R}$ satisfying $\mathbb{E}\,|f(Y)| < \infty$ it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f(T^k(Y)) = \mathbb{E}(f \mid \mathcal{I}) \quad \text{a.s. and in } L^1. \tag{5}$$

2. (Partial sums of a stationary sequence)
Let Y be a real-valued stationary and ergodic sequence (i.e., $A = \mathbb{R}$). Define $S_n := \sum_{k=1}^{n} Y_k$ for $n \geqslant 1$. Prove that

$$\mathbb{P}\left(\left\{\lim_{n\to\infty} S_n = \infty\right\} \setminus \left\{\liminf_{n\to\infty} \frac{1}{n} S_n > 0\right\}\right) = 0 \tag{6}$$

(that is, the sums of a stationary sequence cannot increase to infinity at a slower than linear rate).

Hint: For fixed $k \geqslant 1$, consider the event $\{S_n > 1 \text{ for all } n \geqslant k\}$.

Remark 1: The ergodicity assumption is not needed as follows from the statement above by decomposing into ergodic components (i.e., conditioning on $\mathcal{I}$).

Remark 2: This result is called Kesten's lemma.

3. (Transience/recurrence in a stationary and ergodic environment)
Recall Solomon's recurrence/transience criterion for one-dimensional random walk in random environment:
Let P be a stationary and ergodic measure on $\omega = (\omega_n)_{n\in\mathbb{Z}}$ (where ω_n is the

transition probability from n to $n+1$ in the random walk). Define $\rho_n := \frac{1-\omega_n}{\omega_n}$. Assume that for some $\varepsilon > 0$, $P(\omega_0 \in [\varepsilon, 1-\varepsilon]) = 1$.
Then the random walk is (i) transient to $+\infty$ if $\mathbb{E}\log(\rho_0) < 0$, (ii) recurrent if $\mathbb{E}\log(\rho_0) = 0$ and (iii) transient to $-\infty$ if $\mathbb{E}(\log(\rho_0)) > 0$.

We proved this in class when P is IID. Extend the proof to the stationary and ergodic case.

Hint: The result (6) may be helpful. Also note that our derivation in class of the exit probability $v_{R,L}$ is valid for any value of the transition probabilities ω.

4. (Second moment of the hitting time)
In the notation of one-dimensional random walk in random environment: Let P be IID and uniformly elliptic. Prove that $\mathbb{E}(\tau_1^2) < \infty$ if and only if $\mathbb{E}_P(\rho_0^2) < 1$.

5. (Annealed central limit theorem)
In the notation of one-dimensional random walk in random environment:
Let P be IID and uniformly elliptic. Assume that $\mathbb{E}_P(\rho_0^2) < 1$.
Prove that as n tends to infinity, $\frac{T_n - nv_P^{-1}}{\sqrt{n}}$ converges in distribution to $N(0, \sigma^2)$ for some $\sigma > 0$.

Hint: Use the central limit theorem for the partial sums of a stationary sequence presented in class.

Remark: You may use the fact stated in class that the assumption $\mathbb{E}_P(\rho_0^2) < 1$ implies that $\mathbb{E}(\tau_1^{2+\delta}) < \infty$ for some $\delta > 0$.

2 Random walks in a random environment in general dimensions

We consider nearest-neighbor random walks in random environment on $\mathbb{Z}^d$. We use the usual notation from class:

- The environment measure P is a measure on $(\omega_x)_{x\in\mathbb{Z}^d}$, where each ω_x is a probability distribution on the $2d$ unit vectors $e_1, -e_1, e_2, -e_2, \ldots$ ($e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ with the 1 in the ith position). We often make the assumption that the (ω_x) are independent and identically distributed and the assumption of uniform ellipticity (i.e., there exists $\varepsilon > 0$ such that $\omega_x(e) > \varepsilon$ for all $x \in \mathbb{Z}^d$ and $e \in \{\pm e_i\}_{i=1}^d$, P-a.s.).
- The quenched measures $\mathbb{P}_\omega^x$ are the measures on the random walk $(X_n)_{n\geqslant 0}$ in the environment ω when $X_0 = x$.
- The annealed measures $\mathbb{P}^x$ are the joint distribution of the environment ω sampled from P and the walk $(X_n)_{n\geqslant 0}$ sampled from $\mathbb{P}_\omega^x$ (sometimes $\mathbb{P}^x$ refers to the marginal of this distribution on the walk).

1. (Walking in uncharted territory)
Let the environment measure P be independent (P need not be IID or elliptic). Let $x, y \in Z^d$. In this problem we formalize the idea that two random walks

starting at x and y evolve independently if their trajectories do not intersect, even under the annealed measure.

Denote by $\mathbb{P}^{x,y}$ the joint distribution on ω sampled from P and on two random walks $(X_n^1)_{n\geqslant 0}$ and $(X_n^2)_{n\geqslant 0}$ sampled independently from $\mathbb{P}_\omega^x$ and $\mathbb{P}_\omega^y$, respectively.

(a) Let $k,\ell \geqslant 1$ be integers and let $p=(p_0,p_1,\dots,p_k), q=(q_0,q_1,\dots,q_\ell)$ be nearest-neighbor paths in $\mathbb{Z}^d$ of lengths k and ℓ respectively. Assume that $\mathbb{P}^y((X_n^2)_{n=0}^\ell = q) > 0$. Prove that

$$\begin{aligned}\mathbb{P}^{x,y}((X_n^1)_{n=0}^k = p, (X_n^1)_{n=0}^{k-1}\cap (X_n^2)_{n=0}^{\ell-1} = \emptyset \,|\, (X_n^2)_{n=0}^\ell = q)\\ = \mathbb{P}^x((X_n^1)_{n=0}^k = p)1_{(p_n)_{n=0}^{k-1}\cap(q_n)_{n=0}^{\ell-1}=\emptyset}.\end{aligned} \tag{7}$$

(b) Let $k,\ell \geqslant 1$ be integers. Deduce that for any $A\subset (\mathbb{Z}^d)^{\{0,1,\dots,k\}}$ it holds that

$$\begin{aligned}&\mathbb{P}^{x,y}((X_n^1)_{n=0}^k \in A, (X_n^1)_{n=0}^{k-1}\cap (X_n^2)_{n=0}^{\ell-1} = \emptyset \,|\, (X_n^2)_{n=0}^\ell)\\ &= \mathbb{P}^x((X_n^1)_{n=0}^k \in A, (X_n^1)_{n=0}^{k-1}\cap (X_n^2)_{n=0}^{\ell-1} = \emptyset), \quad \mathbb{P}^{x,y}\text{-almost surely.}\end{aligned} \tag{8}$$

(In this notation we mean that the the right-hand side is a function of $(X_n^2)_{n=0}^\ell$. Precisely, if we define for each $q=(q_0,\dots,q_\ell)$ the function $f(q) := \mathbb{P}^x((X_n^1)_{n=0}^k \in A, (X_n^1)_{n=0}^{k-1}\cap (q_n)_{n=0}^{\ell-1} = \emptyset)$ then the right-hand side of (8) should be understood as $f((X_n^2)_{n=0}^\ell)$.)

(c) Deduce that for any measurable $A\subset (\mathbb{Z}^d)^{\{0,1,\dots\}}$ it holds that

$$\begin{aligned}&\mathbb{P}^{x,y}((X_n^1)_{n\geqslant 0} \in A, (X_n^1)_{n\geqslant 0}\cap (X_n^2)_{n\geqslant 0} = \emptyset \,|\, (X_n^2)_{n\geqslant 0})\\ &= \mathbb{P}^x((X_n^1)_{n\geqslant 0} \in A, (X_n^1)_{n\geqslant 0}\cap (X_n^2)_{n\geqslant 0} = \emptyset), \quad \mathbb{P}^{x,y}\text{-almost surely.}\end{aligned} \tag{9}$$

(again, the right-hand side equals $f((X_n^2)_{n\geqslant 0})$ where now
$f(q) := \mathbb{P}^x((X_n^1)_{n\geqslant 0} \in A, (X_n^1)_{n\geqslant 0}\cap (q_n)_{n\geqslant 0} = \emptyset)$ with $q\in(\mathbb{Z}^d)^{\{0,1\dots\}}$).

Remark: Note in particular that the right-hand side of (8) is deterministically at most $\mathbb{P}^x((X_n^1)_{n=0}^k \in A)$. An analogous fact holds for (9).

Remark: A version of (8) can be devised when the length of the paths involved is itself random. Another version can be devised involving a single random walk whose trajectory has two disjoint parts (e.g., as with "regeneration times").

2. (Making trajectories intersect - part of the Merkl-Zerner proof of the $0-1$ law in two dimensions)
Fix $d=2$. Let P be independent (P need not be IID or elliptic). Let $L\geqslant 1$ integer.

Prove that there exists $y_L\in\mathbb{Z}$ such that the following holds. Set $x=(-L,0)$ and $y=(L,y_L)$. Let $\mathbb{P}^{x,y}$, $(X_n^1)_{n\geqslant 0}$ and $(X_n^2)_{n\geqslant 0}$ be as in the previous problem. Then

$$\begin{aligned}\mathbb{P}^{x,y}(X_n^1\cdot(1,0)\to\infty, X_n^2\cdot(1,0)\to-\infty, (X_n^1)_{n\geqslant 0}\cap(X_n^2)_{n\geqslant 0}=\emptyset)\\ \leqslant \frac{1}{2}\mathbb{P}^x(X_n^1\cdot(1,0)\to\infty)\mathbb{P}^y(X_n^2\cdot(1,0)\to-\infty).\end{aligned} \tag{10}$$

3 Lattice spin systems

1. (Localization of 1-Lipschitz integer-valued height functions at low temperature)

   Let $d \geqslant 2$ and $L \geqslant 1$ be integers. Let $x > 0$. Let $G = (V, E)$ be the d-dimensional discrete cube of side length $2L + 1$: the graph with

   $$V = \{-L, -L+1, \ldots, L\}^d \tag{11}$$

   with two vertices adjacent if they differ in exactly one coordinate, and by exactly one in that coordinate. The configuration space of 1-Lipschitz integer-valued height functions with zero boundary conditions is

   $$\Omega := \left\{ \phi : V \to \mathbb{Z} : \begin{array}{c} \phi(u) - \phi(v) \in \{-1, 0, 1\} \text{ for } \{u, v\} \in E, \\ \phi(v) = 0 \text{ for } v \text{ with } \|v\|_\infty = L \end{array} \right\}. \tag{12}$$

   We place a probability measure μ_x on Ω by setting the probability of each $\phi \in \Omega$ to be proportional to $x^{N(\phi)}$ where

   $$N(\phi) := \{\{u, v\} \in E \colon \phi(u) \neq \phi(v)\} \tag{13}$$

   is the number of nearest-neighbor pairs where the values of ϕ differ.

   Prove that there exists some $x_0 > 0$, which may depend on d but not on L, such that whenever $0 < x < x_0$ then

   $$\mu_x(\phi(v) \neq 0) < 0.01 \text{ for all } v \in V. \tag{14}$$

   Hint: Find a way to adapt the Peierls argument to this setting.

2. (Delocalization of two-dimensional real-valued height functions)

   Let $U : \mathbb{R} \to \mathbb{R}$ be a twice-continuously differentiable function satisfying

   (a) (even function) $U(x) = U(-x)$ for all $x \in \mathbb{R}$.

   (b) (growth at infinity) $\frac{U(x)}{\log x} \to \infty$ as $x \to \infty$.

   (c) (bounded second derivative) $\sup_{x \in \mathbb{R}} U''(x) < \infty$.

   (one example to have in mind is the function $U(x) = x^2$).

   Let $L \geqslant 1$ be an integer. Let $G = (V, E)$ be the two-dimensional discrete square of side length $2L + 1$, as in (11) but now with $d = 2$. The configuration space of real-valued height functions with zero boundary conditions is

   $$\Omega := \{\phi : V \to \mathbb{R} \colon \phi(v) = 0 \text{ for } v \text{ with } \|v\|_\infty = L\}. \tag{15}$$

   We place a probability measure μ on Ω by setting the density of each $\phi \in \Omega$ to be proportional to

   $$\exp\left(- \sum_{\{u,v\} \in E} U(\phi(u) - \phi(v)) \right). \tag{16}$$

Here, the density is with respect to the natural product Lebesgue measure ($\prod_v d\phi(v)$ over all $v \in V$ with $\|v\|_\infty \neq L$). It is not obvious that (16) can indeed be normalized to be a probability measure but this is ensured by the growth at infinity assumption on U.

Let ϕ be sampled from μ. Prove that there exists some $c > 0$, which may depend on U but not on L, such that

$$\mathrm{Var}(\phi(0,0)) \geqslant c \log L. \tag{17}$$

Hint: Adapt the proof of the Mermin–Wagner theorem (no continuous-symmetry breaking in two dimensions) to this setting.

3. (Decay of correlations in the two-dimensional XY model: the Dobrushin–Shlosman method)

   Let $L \geqslant 1$ be an integer. Let $G = (V, E)$ be the *two-dimensional* discrete square of side length $2L + 1$ as in the previous problem. The configuration space of the XY model with zero-angle boundary conditions is

$$\Omega := \{\theta : V \to \mathbb{R}/2\pi\mathbb{Z} \colon \theta(v) = 0 \text{ for } v \text{ with } \|v\|_\infty = L\}. \tag{18}$$

   Let $\beta > 0$. The probability measure μ of the XY model at inverse temperature β is the measure on Ω assigning density proportional to

$$\exp\left(\beta \sum_{\{u,v\}\in E} \cos(\theta(u) - \theta(v))\right) \tag{19}$$

   to each $\theta \in \Omega$. The density is with respect to the natural product Lebesgue measure (product of the Lebesgue measures on $\mathbb{R}/2\pi\mathbb{Z}$ at each $v \in V$ with $\|v\|_\infty \neq L$).

   Let θ be sampled from μ. Consider the sigma algebra generated by all the differences in angles $\theta(u) - \theta(v)$ when $\|u\|_\infty = \|v\|_\infty$. Precisely,

$$\mathcal{F} := \sigma\big(\theta(u) - \theta(v) \colon u, v \in V,\ \|u\|_\infty = \|v\|_\infty\big). \tag{20}$$

   (given the information in this sigma algebra, the only degrees of freedom left in θ is to perform a uniform rotation to the spins in each 'layer' (by layer we mean the set of $v \in V$ with fixed $\|v\|_\infty$).

   (a) For $0 \leqslant j \leqslant L-1$, let $X_j := \theta(je_1) - \theta((j+1)e_1)$, where $e_1 = (1, 0)$. Prove that the (X_j) are conditionally independent given $\mathcal{F}$.
   Deduce that

$$\mathbb{E}\left(e^{i\theta(0,0)}\right) = \mathbb{E}\left(\prod_{j=0}^{L-1} \mathbb{E}\left(e^{iX_j} \mid \mathcal{F}\right)\right). \tag{21}$$

   (b) Prove that there exists some $c > 0$, which does not depend on β or L, such that for each $0 \leqslant j \leqslant L-1$,

$$\mathbb{E}\left(e^{iX_j} \mid \mathcal{F}\right) \leqslant 1 - \frac{c}{\max\{\beta, 1\}(j+1)}. \tag{22}$$

Deduce power-law decay of correlations - that is, deduce that there exists $\tilde{c} > 0$, which does not depend on β or L, such that

$$\mathbb{E}\left(e^{i\theta(0,0)}\right) \leqslant \frac{1}{L^{\tilde{c}/\max\{\beta,1\}}}. \tag{23}$$

Hint: One way to prove (22) is to use an argument of Mermin–Wagner type, by introducing a suitable 'spin wave'.

Remark: The proof may be easily adapted to the spin $O(n)$ model for all $n \geqslant 2$ (the XY model is the case $n = 2$), by conditioning on all but the first two components of each spin.

Remark: The idea to condition on the sigma algebra $\mathcal{F}$ is based on the 1975 work of Dobrushin–Shlosman.

4. (The XY model at strong external magnetic field)

Let $G = (V, E)$ be a finite connected graph of maximal degree Δ. Let $V_0 \subset V$ be a non-empty subset (the 'boundary subset'). For each $\tau : V_0 \to \mathbb{R}/2\pi\mathbb{Z}$ define the configuration space of the XY model with boundary condition τ as

$$\Omega^\tau := \{\theta : V \to \mathbb{R}/2\pi\mathbb{Z} \colon \theta(v) = \tau(v) \text{ for } v \in V_0\}. \tag{24}$$

At *external field strength* $h > 0$, the energy of a configuration θ is defined to be

$$H^h(\theta) := -\sum_{\substack{u\sim v\\ u,v\in V}} \cos(\theta(u) - \theta(v)) - h\sum_{v\in V} \cos(\theta(v)). \tag{25}$$

At inverse temperature $\beta > 0$, external field strength $h > 0$ and boundary condition $\tau : V_0 \to \mathbb{R}/2\pi\mathbb{Z}$, define the probability measure $\mathbb{P}^{\beta,h,\tau}$ of the XY model as the measure on Ω^τ assigning density proportional to $\exp(-\beta H^h(\theta))$ to each $\theta \in \Omega^\tau$ (with respect to the Lebesgue measure $\prod_{v\in V\setminus V_0} d\theta(v)$). The associated expectation is denoted $\mathbb{E}^{\beta,h,\tau}$.

Denote by ρ the metric on $\mathbb{R}/2\pi\mathbb{Z}$ given by $\rho(\alpha_1, \alpha_2) = |e^{\alpha_1} - e^{i\alpha_2}| = \sqrt{2 - 2\cos(\alpha_1 - \alpha_2)}$. A function $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ is called Lipschitz if $|f(\alpha_1) - f(\alpha_2)| \leqslant \rho(\alpha_1, \alpha_2)$ for all $\alpha_1, \alpha_2 \in \mathbb{R}/2\pi\mathbb{Z}$.

Prove that there exists $h_0, C, c > 0$, depending only on the maximal degree Δ, such that if $h > h_0$ then for all $v \in V$ and all Lipschitz $f : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$,

$$\sup_{\tau_1,\tau_2:V_0\to\mathbb{R}/2\pi\mathbb{Z}} |\,\mathbb{E}^{\beta,h,\tau_1} f(\theta(v)) - \mathbb{E}^{\beta,h,\tau_2} f(\theta(v))| \leqslant Ce^{-c\, d_G(v,V_0)}, \tag{26}$$

with $d_G(v, V_0)$ the graph distance in G between v and the boundary subset V_0.

Hint: Use the Dobrushin uniqueness theorem.

Remark: One standard choice for the graph G is a cube in $\mathbb{Z}^d$, as in (11), with V_0 chosen as its vertex boundary in $\mathbb{Z}^d$.

Remark: A similar result holds for the Ising model and for the spin $O(n)$ models with $n \geqslant 3$. Additionally, as the result is uniform in the temperature, it holds also at zero temperature (i.e., in the limit $\beta \to \infty$).

Remark: The decay estimate (26) holds, in fact, for all $h > 0$ (without restricting to large h, but with C, c depending on h) but this is harder to establish.

4 Disordered lattice spin systems

The random-field Ising model is defined as follows: In a finite domain $\Lambda \subset \mathbb{Z}^d$ with boundary conditions $\tau : \partial_\circ\Lambda \to \{-1, 1\}$ (here, $\partial_\circ\Lambda := \{v \in \mathbb{Z}^d \setminus \Lambda \colon \exists u \in \Lambda, u \sim v\}$ is the external vertex boundary of Λ), the energy of a configuration $\sigma : \Lambda \to \{-1, 1\}$ is given by

$$H^{\Lambda,\tau,\eta}(\sigma) = -\sum_{\substack{u\sim v\\ u,v\in\Lambda}} \sigma_u\sigma_v - \sum_{\substack{u\sim v\\ u\in\Lambda, v\in\Lambda^c}} \sigma_u\tau_v - \lambda\sum_{u\in\Lambda} \eta_u\sigma_u, \tag{27}$$

where $\lambda > 0$ is the disorder strength and $(\eta_v)_{v\in\mathbb{Z}^d}$ is the disorder, given by independent and identically distributed standard Gaussian random variables (more general disorder distributions are possible but we restrict to the Gaussian case for simplicity).

The model is considered here only at zero temperature ($\beta = \infty$), in which case it is supported on the (almost surely) unique configuration $\sigma^{\Lambda,\tau,\eta}$ minimizing the energy (27). Such a configuration is called a *finite-volume ground state.*

We use $\mathbb{P}$ and $\mathbb{E}$ for probability and expectation over η.

1. (Infinite-volume ground states in the random-field Ising model)

   A configuration $\sigma : \mathbb{Z}^d \to \{-1, 1\}$ is called a ground state (in infinite volume) if whenever $\sigma' : \mathbb{Z}^d \to \{-1, 1\}$ differs from σ in finitely many vertices we have $H^\eta(\sigma) \leqslant H^\eta(\sigma')$.

   One should note that the energies $H^\eta(\sigma)$ above are not well defined themselves, as the formula (27) involves an infinite non-convergent sum in infinite volume. However, the difference $H^\eta(\sigma) - H^\eta(\sigma')$ is well defined: It equals $H^{\Lambda,\tau,\eta}(\sigma) - H^{\Lambda,\tau,\eta}(\sigma')$ for any finite domain Λ such that $\sigma = \sigma'$ outside Λ, with the boundary conditions τ taken to be the restriction of σ (or σ') to $\partial_\circ\Lambda$.

   (a) Let $\Lambda \subset \mathbb{Z}^d$ be finite. Let $\tau_1, \tau_2 : \partial_\circ\Lambda \to \{-1, 1\}$ satisfy $\tau_1 \geqslant \tau_2$ pointwise. Prove that for each fixed $(\eta_v)_{v\in\mathbb{Z}^d}$ we have $\sigma^{\Lambda,\tau_1,\eta} \geqslant \sigma^{\Lambda,\tau_2,\eta}$ pointwise.

   Remark: Related facts hold true at positive temperatures, by the famous Fortuin-Kasteleyn-Ginibre (FKG) inequality. However, the zero-temperature case is more elementary.

   (b) Let $d = 2$. Prove that almost surely (in η), there *exists* a *unique* ground state $\sigma : \mathbb{Z}^d \to \{-1, 1\}$ (in infinite volume).

   Hint: Rely on the theorem from class which proves that the ground state of the two-dimensional random-field Ising model is disordered at all positive disorder strengths.

   (c) Let $d \geqslant 3$ and suppose $\lambda > 0$ is small (as small as is needed for the proof, but still positive). Prove that almost surely (in η) there are at least two different ground states (in infinite volume) for the d-dimensional random-field Ising model.

   Hint: Rely on the theorem from class which proves that the $d \geqslant 3$ random-field Ising model exhibits long-range order at weak disorder strength.

2. (Strong disorder regime of the random-field Ising model)

   For each dimension $d \geqslant 1$, prove that there exists $\lambda_0(d) > 0$ (depending only on d) such that the following holds for all $\lambda > \lambda_0(d)$: There exists $C, c > 0$ such that for all $L \geqslant 1$ we have

$$\mathbb{E}\left(\sigma_0^{\Lambda_L,+,\eta} - \sigma_0^{\Lambda_L,-,\eta}\right) \leqslant Ce^{-cL}, \tag{28}$$

where $\Lambda_L = \{-L, -L+1, \ldots, L\}^d$ is the lattice cube of side length $2L+1$, and where $+$ stands for the constant $+1$ boundary condition while $-$ stands for the constant -1 boundary condition.

Remark: We focus here on the ground state for simplicity but the same result continues to hold at every positive temperature (even if the temperature depends on the disorder). If the temperature is sufficiently high then a similar result holds at all disorder strengths due to the Dobrushin Uniqueness theorem.

Remark: In dimension $d = 1$, one may take $\lambda_0(d) = 0$ meaning that (28) holds at all $\lambda > 0$.

3. (A fact about convex Lipschitz functions - an exercise from the last lecture)

   There exists $C > 0$ such that the following holds. Let $g : \mathbb{R} \to \mathbb{R}$ be convex, differentiable and λ-Lipschitz (the Lipschitz assumption means that $|g(x) - g(y)| \leqslant \lambda |x - y|$ for all $x, y \in \mathbb{R}$). Define, for $r > 0$,

$$N_r(g) := \{h : \mathbb{R} \to \mathbb{R} \colon h \text{ is convex, differentiable and } \lambda\text{-Lipschitz} \colon \sup_{x\in\mathbb{R}} |h(x) - g(x)| \leqslant r\}. \tag{29}$$

   Prove that, for each $r, \delta > 0$,

$$\mathrm{Leb}(\{x \in \mathbb{R} \colon \exists h \in N_r(g), |h'(x) - g'(x)| \geqslant \delta\} \leqslant \frac{C\lambda r}{\delta^2}, \tag{30}$$

   where Leb stands for Lebesgue measure on $\mathbb{R}$.

   Remark: The differentiability assumptions are made for simplicity. A version of this fact holds with left and right derivatives.