Lecture 8 - 8 הרצאה

Random Walks in random environments in $d\geq 2$

We discussed the general theory, where many questions are still open.

We mention now special cases which were easier to treat:

1) **Balanced environment:** IID, uniformly elliptic envir. satisfying that $\omega(x,e)=\omega(x,-e)$ P-a.s at every x and $e\in\{\pm e_i\}$.

Here, under $\mathbb{P}^x_\omega$, the walk (X_n) is a martingale.

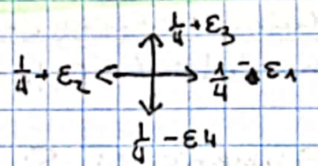
Lawler 1982 - law of large numbers and quenched central limit thm.

Later, recurrence in $d=2$ and transience in $d\geq 3$.

Tool: Existance of an invariant abs. cont. prob. meas. for environment viewed from the point of view of the particle.

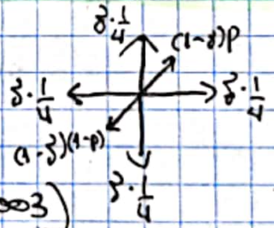
2) **Small perturbations of simple random walk.**

(Bricmont - Kupiainen 1991, Sznitman - Zeitouni 2006)



3) **IID, uniformly elliptic, with the first d_0 coordinates doing simple random walk.**

Law of large numbers, central limit thm



(Bolthausen - Sznitman - Zeitouni 2003)

When d_0 sufficiently large.

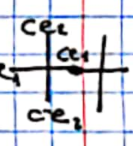
Tool: Cut ~~points~~ times - times where the future of the walk doesn't intersect it's past. Have infinitely many cut times for simple random walk in dimensions $d\geq 3$.

Then use independence of the environment.

(see survey of Biskup 2011)

4) **Random Conductance model:** Each edge is equipped with a non-negative real - the conductance.

The prob. to go on an edge is $\omega(x,e)=\dfrac{c_{x,e}}{\sum_{f\in\{\pm e_i\}} c_{x,f}}$



The conductances may be IID, but the resulting ω are not IID.

Example: Simple random walk on a percolation cluster.

Tool: Such walks are reversible and this gives access to their stationary measure.

5) Random walk in IID Dirichlet random environment (Sabot et al.)

The Dirichlet distribution is a dist. on a vector $(p_1, \dots, p_k)$ with $p_i \geq 0$, $\sum_{i=1}^{k} p_i = 1$.

Its density wrt. to the natural Lebesgue measure is

$\frac{1}{Z} \prod_{i=1}^{k} p_i^{\alpha_i - 1}$ where $\alpha_1, \dots, \alpha_k > 0$ are the parameters of the distribution.

normalization

Law of large numbers, central limit thm, recurrence/transience

Tool: Underlying integrable structure.

Lattice Spin systems

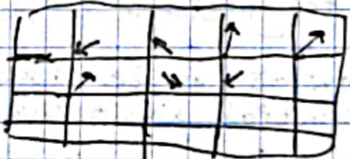
Main reference: Lectures on the spin and loop $O(n)$ models / Peled - Spinka

$\to$ Ising

Λ

Framework: Single-site space: Measure space $(S, \mathcal{S}, \mu)$

set, σ-algebra, measure on σ-algebra



$\to$ XY

Λ

Examples: 1) Ising: $S = \{-1, 1\}$, μ = counting measure.

$$H_\Lambda^\tau(\sigma) = -\sum_{\substack{u \sim v \\ u, v \in \Lambda}} \sigma_u \sigma_v - \sum_{\substack{u \sim v \\ u \in \Lambda \\ v \notin \Lambda}} \sigma_u \tau_v$$

measures the total number of agreements of adjacent spins, between σ and σ and τ

2) XY model: $S = S^1$ - unit circle

$\mathcal{S}$ - Borel σ-alg.

μ = Lebesgue measure.

- For each finite $\Lambda \subseteq \mathbb{Z}^d$ and boundary values $\tau: \mathbb{Z}^d \to S$ have an energy fcn, called Hamiltonian.

(space of config.: $\Omega = \{\sigma : \mathbb{Z}^d \to S\}$, $\tau \in \Omega$)

which is $H_\Lambda^\tau : \Omega \to \mathbb{R}$.

Prob. measure at temprature T, has density

$\frac{1}{Z} e^{-\beta H_\Lambda^{\tau}(\sigma)}$ with respect to $\prod_{v\in\Lambda} \delta(\sigma(v))$

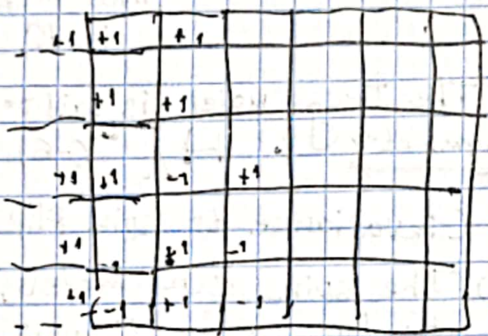
$\beta = \frac{1}{T}$

In the models we will discuss (Ising, XY, ...) We only care about σ in Λ and τ outside Λ, and moreover only about τ on the external vertex bdy of Λ. We sometimes let $\sigma : \Lambda \to S'$ and $\tau : \partial_{ext}\Lambda \to S$.

Unwrapping def. the prob. of σ in the ising model is: $\mathbb{P}_\Lambda^\tau(\sigma) = \frac{1}{Z} \exp\Big(\beta\Big(\sum_{\substack{u\sim v\\ u,v\in\Lambda}} \sigma_u\sigma_v + \sum_{\substack{u\sim v\\ u\in\Lambda\\ v\notin\Lambda}} \sigma_u\tau_v\Big)\Big)$

also depends on β

normalization



$\tau = +1$

$\beta = \frac{1}{T}$ tunes how important is the energy.

$T=\infty \Rightarrow \beta = 0$ means every configurations in Λ is equally likely.

$T=0$ or $\beta=\infty$ means that we sample uniformly among configurations with the lowest possible energy. ← ground states.

Remarks: Two additional bdy cond. 1) Free - τ is not specified,

$$H_\Lambda^{free}(\sigma) = -\sum_{\substack{u\sim v\\ u,v\in\Lambda}} \sigma_u\sigma_v$$

2) periodic - regard Λ as a discrete torus.

Extension: Potts model with q states:

$S = \{1, 2, \dots, q\}$ (q-possible spin values at every vertex)

$$H_\Lambda^\tau(\sigma) = -\sum_{\substack{u\sim v\\ u,v\in\Lambda}} \mathbb{1}_{\sigma_u=\sigma_v} - \sum_{\substack{u\sim v\\ u\in\Lambda}} \mathbb{1}_{\sigma_u=\tau_v}$$
$v\notin\Lambda$

2) Spin $O(n)$ model with $n \geq 2$:

$S = S^{n-1}$ ($(n-1)$-dim. sphere in $\mathbb{R}^n$)

$n=2$: XY model 

$n=3$: Heisenberg model 

($n=1$: $S^0 = \{-1,1\}$ – so we get the Ising model).

$\mathcal{S}$ – borel σ-algebra, γ – Lebesgue-measure

$$H_\Lambda^\tau(\sigma) = - \sum_{\substack{u \sim v \\ u,v \in \Lambda}} \sigma_u \cdot \sigma_v - \sum_{\substack{u \sim v \\ u \in \Lambda \\ v \notin \Lambda}} \sigma_u \cdot \tau_v$$

inner product in $\mathbb{R}^n$.

Remarks: 1) The Ising model is said to have a discrete symmetry: $H_\Lambda^{-\tau}(-\sigma) = H_\Lambda^\tau(\sigma)$

(invariance to spin flip)

2) The spin $O(n)$ models, $n \geq 2$, are said to have a continuous symmetry:

$H_\Lambda^{R\tau}(R\sigma) = H_\Lambda^\tau$

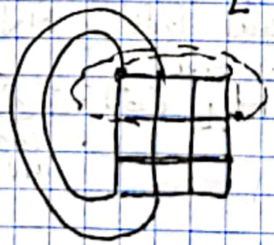
where R is a rotation in $\mathbb{R}^n$.

This is the reason for the name $O(n)$.

Properties of the models

We consider the above models with periodic bdy cond., on the discrete Torus:

$$\mathbb{T}_L^d = \{-L+1, -L+2, \dots, L\}^d$$



We study the correlation $\rho_{x,y} := \mathbb{E}^{per}_{\mathbb{T}_L^d}(\sigma_x \cdot \sigma_y)$

usual product for Ising.
Inner product for $n \geq 2$

For fixed distant $x, y \in \mathbb{T}_L^d$

1) Non-negativity and monotonicity:

$\rho_{x,y} \geq 0$ always.

We also expect $\rho_{x,y}$ to increase as the temperature lowers.

This holds for the Ising and XY models

(Ginibre / Griffiths ineq.)

but is open for $n \geq 3$.

2) High temperature regime:

$\exists \beta_0(d,n)$ s.t $\forall \beta < \beta_0(d,n)$:

$$\rho_{x,y} \leq C_{d,n,\beta}\, e^{-c_{d,n,\beta}\|x-y\|_1}$$ ← on the torus

uniformly in L. In $d=1$, $\beta_0 = \infty$

3) Ising model Phase transition in $d \geq 2$!

$\exists \beta_c(d) \in \mathbb{R}^+$ s.t $\forall \beta < \beta_c$:

$$\rho_{x,y} \leq C_{d,\beta}\, e^{-c_{d,\beta}\|x-y\|_1}$$

$\forall \beta > \beta_c$ (and $d \geq 2$):

$$\rho_{x,y} \geq c_{d,\beta} > 0$$ Long-range order

uniformly in x,y and L.

Criticality: $\beta = \beta_c$

$d=2$: $\mathbb{E}_{\mathbb{Z}^2}(\sigma_x \sigma_y) \sim C \|x-y\|_2^{-1/4}$

$\exists d_0$ s.t $d \geq d_0$:

$$\mathbb{E}_{\mathbb{Z}^d}(\sigma_x \sigma_y) \sim C_d \|x-y\|_2^{-(d-2)}$$

For other d, only known that $\mathbb{E}_{\mathbb{Z}^d}(\sigma_x\sigma_y) \xrightarrow[\|x-y\| \to \infty]{} 0$

(i.e, the phase transition is continuous)

4) XY model $(n=2)$: $(d=2)$

Two dimensions:

Mermin-Wagner thm: No continuous-symmetry breaking in two dimensions.

At every $\beta<\infty$, $\rho_{x,y} \leq \dfrac{C(\beta)}{\|x-y\|_1^{c(\beta)}}$ No long-range order.

Berezinskii-Kosterlitz-Thouless phase transition:

$\exists \beta_0$ s.t $\forall \beta>\beta_0$,

$$\rho_{x,y} \geq \frac{C(\beta)}{\|x-y\|_1^{C(\beta)}} \quad (d=2)$$

(Fröhlich - Spencer (1982))

5) Spin $O(n)$ models with $n \geq 3$ $(d=2)$

Mermin-Wagner: $\rho_{x,y} \leq \dfrac{C(n,\beta)}{\|x-y\|_1^{c(n,\beta)}}$

Polyakov prediction: exp. decay at all $\beta<\infty$.
(No BKT phase transition).

6) Spin $O(n)$ model, $n \geq 2$, in dimension $d \geq 3$:

Long-range order: $\exists \beta_1(d,n)$ $\forall \beta>\beta_1(d,n)$,

$$\frac{1}{|\mathbb{T}_L^d|^2} \sum_{x,y \in \mathbb{T}_L^d} \rho_{x,y} \geq C_{d,n,\beta}.$$

(proved by the famous infra-red bound of Fröhlich - Simon - Spencer - 1976).

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