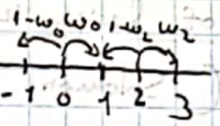
Lecture 3 — הרצאה 3

One-dimensional random walks in random environment

$\Omega = \{\omega : \mathbb{Z} \to [0,1]\}$



$\mathbb{P}$ is a prob. dist. on Ω (the environment measure)

1) $\mathbb{P}$ is stationary and ergodic

Stationary: $(\dots,\omega_{-1},\omega_0,\omega_1,\dots) \overset{d}{=} (\dots,\omega_0,\omega_1,\omega_2,\dots)$

at position 0 at position 0

Ergodic: For event $E \subseteq \Omega$ which is invariant:

$\omega \in E \iff$ all shifts of ω belong to E

it holds that $\mathbb{P}(E) \in \{0,1\}$

Usually (almost always) we take $\mathbb{P}$ to be IID.

2) Uniform ellipticity: $\exists \varepsilon > 0$ s.t. $\mathbb{P}(\omega_0 \in [\varepsilon, 1-\varepsilon]) = 1$

Two walk measures:

1) Given $\omega \in \Omega$ and $x \in \mathbb{Z}$, the quenched law of the walk is $\mathbb{P}_\omega^x$ which is the random walk measure starting at x and walking according to the prob. in ω.

2) Given $x \in \mathbb{Z}$, $\mathbb{P}^x$ is the (marginal dist. on the walk of the) joint dist. of ω and the walk $(X_n)_{n=0}^\infty$. $\mathbb{P}^x$ is called the annealed (or averaged law) of the walk.

Useful def.: $\rho_x = \dfrac{1-\omega_x}{\omega_x}$

Recurrence / transience:

Thm. (Solomon 1975):

1) $\mathbb{E}_P(\log \rho_0) < 0$: $\lim_{n\to\infty} X_n = \infty$ $\mathbb{P}^0$ a.s.

2) $\mathbb{E}_P(\log \rho_0) > 0$: $\lim_{n\to\infty} X_n = -\infty$ $\mathbb{P}^0$ a.s.

3) $\mathbb{E}_P(\log \rho_0) = 0$; $\limsup_{n\to\infty} X_n = \infty$, $\liminf_{n\to\infty} X_n = -\infty$ $\mathbb{P}^0$ a.s

Law of large numbers: $\left(\lim_{n\to\infty} \frac{X_n}{n}\right)$

Thm (Solomon 1975, Alili 1999 in ergodic case) Where formulation slightly differs

1) $\mathbb{E}_P(\rho_0) < 1$: $\lim_{n\to\infty} \frac{X_n}{n} = \frac{1-\mathbb{E}_P(\rho_0)}{1+\mathbb{E}_P(\rho_0)} = V_P$ (P^0-a.s)

2) $\mathbb{E}_P(\rho_0^{-1}) < 1$: $\lim_{n\to\infty} \frac{X_n}{n} = \frac{1-\mathbb{E}_P(\rho_0^{-1})}{1+\mathbb{E}_P(\rho_0^{-1})}$ (P^0-a.s)

3) $\mathbb{E}_P(\rho_0) \geq 1$, $\mathbb{E}_P(\rho_0^{-1}) \geq 1$: $\lim_{n\to\infty} \frac{X_n}{n} = 0$ (P^0-a.s)

Remark: There exists P for which $\mathbb{E}(\log \rho_0) < 0$ but $\mathbb{E}_P(\rho_0) \geq 1$, meaning that the walk is transient to ∞ with zero speed.

Back to the proof:

Instead of the law of large numbers, we rely on Birkhoff's ergodic thm.

Thm (Birkhoff): If $(Z_n)_{n\geq 0}^{\infty}$ are a stationary seq. of random variables taking values in some measurable space S, then for all $f: S \to \mathbb{R}$, $\mathbb{E} f(|Z_0|) < \infty$ it holds that

$$\frac{1}{n}\sum_{k=0}^{n-1} f(Z_k) \longrightarrow \mathbb{E}(f(Z_0)\,|\,\underbrace{\mathcal{I}}_{\text{invariant } \sigma\text{-algebra}}) \text{ a.s and in } L_1.$$

Instead of focusing on X_n, we focus on:

$$T_n := \min\{k \geq 0 : X_k = n\}, \quad n \geq 0$$

Also define:

$$\tau_n = T_n - T_{n-1} \text{ for } n \geq 1$$

Clearly, $\frac{1}{n}\sum_{k=1}^{n} \tau_k = \frac{T_n}{n}$ when $X_0 = 0$

We assume WLOG that $\mathbb{E}_P(\log \rho_0) < 0$ so that all $T_n < \infty$, $\mathbb{P}^0$-a.s.

Lemma (from last time): If $\frac{T_n}{n} \to (0,\infty]$ then also
$$\frac{1}{n} X_n \xrightarrow[n\to\infty]{} \frac{1}{\lim_{n\to\infty} \frac{T_n}{n}}$$

Lemma: Under P^0, $(\tau_n)_{n\geq 1}$ is a stationary and ergodic seq.

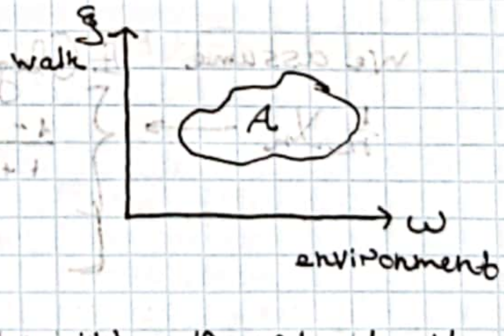
proof: The fact that τ_n is stationary is an immediate consequence of the stationarity of P.
e.g the event $\{\tau_1 \geq 100\}$ is a certain function of $(\dots,\omega_{-1},\omega_0)$ and the walk started at 0 and the event that $\{\tau_2 \geq 100\}$ is the same function of $(\dots,\omega_0,\omega_1)$ and the walk started at 1. increments

For the ergodicity, let $(\varepsilon_n)_{n\geq 0}$ be independent and uniformly dist. on $[0,1]$.
Represent the decisions of the walk using (ε_n) by writing: $X_{n+1} = X_n - 1_{\omega_{X_n} < \varepsilon_n} + 1_{\varepsilon_n \leq \omega_{X_n}}$

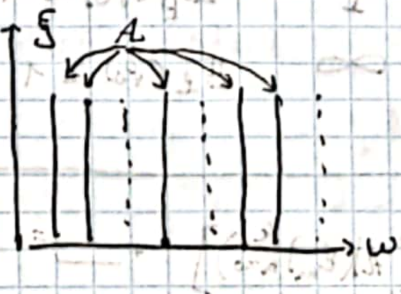
(i.e, walk right if $\varepsilon_n \leq \omega_{X_n}$ and otherwise, walk left).

Let A be an event, depending only on the $(\tau_n)_{n\geq 1}$ which is invariant.
(i.e $1_A(\tau_1,\tau_2,\dots) = 1_A(\tau_2,\tau_3,\dots)$)

Our goal is to show that $P^0(A) \in \{0,1\}$



Step 1: For a set of ~~each~~ $\omega \in \Omega$ with P prob. 1, it holds that $P^0_\omega(A) \in \{0,1\}$



Indeed, since A depends only on $(\tau_n)_{n\geq 0}$ and is invariant, then

$$\mathbb{P}^0_\omega(A \mid (\xi_k)_{k=0}^n) \quad \text{is the same for all}$$

$$(\xi_k)_{k=0}^n . \quad \text{i.e equal to } \mathbb{P}^0_\omega(A).$$

To see this, note that $\mathbb{1}_A(\tau_1, \tau_2, \ldots) = \mathbb{1}_A(\tau_{n+1}, \tau_{n+2}, \ldots)$

A fcn. of $(\xi_k)_{k=\tau_n}^\infty$

When $(\xi_k)_{k=0}^n$ are specified, they can affect τ_n, but cannot affect $(\xi_k)_{k=\tau_n}^\infty$

Since, given ω, A is a fcn. of the $(\xi_n)_{n=0}^\infty$ we conclude that $\mathbb{P}^0_\omega(A) \in \{0,1\}$.

(Since we can approximate A by an event which depends only on a finite number of ξ_k's.

$\forall \varepsilon > 0 \ \exists k$ and $B_k \in \sigma(\xi_0, \ldots, \xi_k)$ s.t $\mathbb{P}^0_\omega(A \triangle B_k) < \varepsilon$)

$$\mathbb{P}^0_\omega(A \triangle B_k) = \mathbb{P}^0_\omega(A \cap B_k^c) + \mathbb{P}^0_\omega(B_k \cap A^c) = \mathbb{E}^0_\omega[\mathbb{P}^0_\omega(A \cap B_k^c \mid (\xi_m)_{m=0}^k)] + \circledast$$

Step 2: $\mathbb{P}^0(A) \in \{0,1\}$.

By step 1, A is (up to prob. 0) a function of the environment ω. Since A is an invariant fcn. of (τ_n), it follows that it is also an invariant fcn of ω. ~~It follows~~

By ergodicity of P (the environment measure) it follows that $\mathbb{P}^0(A) \in \{0,1\}$

We assume $\mathbb{E}(\log \rho_0) \leq 0$ and need to show

$$\frac{1}{n} X_n \longrightarrow \begin{cases} \dfrac{1 - \mathbb{E}_P(\rho_0)}{1 + \mathbb{E}_P(\rho_0)} = V_P & \mathbb{E}_P(\rho_0) < 1 \\ 0 & \mathbb{E}_P(\rho_0) \geq 1 \end{cases} \quad \mathbb{P}^0\text{-a.s}$$

This is equivalent, by what we showed, to

$$\mathbb{E}^0_A(\tau_1) = \begin{cases} V_P^{-1} & \mathbb{E}_P(\rho_0) < 1 \\ \infty & \mathbb{E}_P(\rho_0) \geq 1 \end{cases}$$

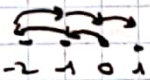
$$\circledast = \mathbb{E}^0_\omega[\mathbb{1}_{B_k^c} \underbrace{\mathbb{P}^0_\omega(A \mid (\xi_m)_{m=0}^k)}_{\mathbb{P}^0_\omega(A)}] + \underline{\quad} = \underbrace{\mathbb{P}^0_\omega(A) \cdot \mathbb{P}^0_\omega(B_k^c) + \mathbb{P}^0_\omega(A^c) \cdot \mathbb{P}^0_\omega(B_k)}_{< \varepsilon}$$

We now prove this:

The key is the following recursion:

is the time from the second visit to 0 to reach 1.

$$\tau_1 = \mathbb{1}_{X_1=1} + \mathbb{1}_{X_1=-1}\left(1 + \tau_{-1,0} + \tau_{0,1}\right)$$



$\tau_1 = 5$
$\tau_{-1,0} = 3$
$\tau_{0,1} = 1$

We take the P^0_ω expectation of this.

$$E^0_\omega(\tau_1) = \omega_0 + (1-\omega_0)\cdot\left(1 + E^0_{\theta^{-1}\omega}(\tau_1) + E^0_\omega(\tau_1)\right)$$

Where $\theta^{-1}(\dots,\omega_{-1},\omega_0,\omega_1\dots) = (\dots,\omega_{-2},\omega_{-1},\omega_0,\dots)$

$$\Longleftrightarrow \omega_0 E^0_\omega(\tau_1) = 1 + (1-\omega_0) E^0_{\theta^{-1}\omega}(\tau_1)$$

$$\Longleftrightarrow E^0_\omega(\tau_1) = \frac{1}{\omega_0} + \rho_0 E_{\theta^{-1}\omega}(\tau_1)$$

$$\Longleftrightarrow E^0_\omega(\tau_1) = 1 + \rho_0 + \rho_0 E_{\theta^{-1}\omega}(\tau_1)$$

Assume for now that $E^0(\tau_1) < \infty$

Now take E^0 (assume P is IID).

$$E^0(\tau_1) = 1 + \underset{E^0(\rho_0)}{\underset{\shortparallel}{E_P(\rho_0)}} + E^0(\underbrace{\rho_0}_{\text{a fcn of } \omega_0} \underbrace{E^0_{\theta^{-1}\omega}(\tau_1)}_{\text{a fcn of } (\dots,\omega_{-2},\omega_{-1})})$$

$$\overset{\text{independence}}{=} 1 + E_P(\rho_0) + E_P(\rho_0) E^0\left(E^0_{\theta^{-1}\omega}(\tau_1)\right) =$$

$$\overset{\text{stationarity}}{=} 1 + E_P(\rho_0) + E_P(\rho_0) E^0(\tau_1)$$

$$\Longleftrightarrow \left(1 - E_P(\rho_0)\right) E^0(\tau_1) = 1 + E_P(\rho_0)$$

$$\Longrightarrow \begin{cases} E_P(\rho_0) < 1 \\ E^0(\tau_1) = \dfrac{1 + E_P(\rho_0)}{1 - E_P(\rho_0)} = V_P^{-1} \end{cases}$$

In conclusion: $E^0(\tau_1) < \infty \Longrightarrow E_P(\rho_0) < 1$ and $E^0(\tau_1) = V_P^{-1}$.

To get the full result start with a truncated recurrence:

$$\tau_1 \mathbb{1}_{\{\tau_1 \le M\}} \le \mathbb{1}_{\{X_1=1\}} + \mathbb{1}_{\{X_1=-1\}}\left(1 + \tau_{-1,0}\mathbb{1}_{\{\tau_{-1,0}\le M\}} + \tau_{0,1}\mathbb{1}_{\{\tau_{0,1}\le M\}}\right)$$

Repeat the argument to conclude that

$$(1-\mathbb{E}_P(\rho_0))\cdot \mathbb{E}^0(\tau_1 \mathbb{1}_{\tau_1\le M}) \le 1+\mathbb{E}_P(\rho_0)$$

which implies that if $\mathbb{E}_P(\rho_0)<1$, then

$$\mathbb{E}^0(\tau_1\cdot \mathbb{1}_{\tau_1\le M}) \le V_P^{-1}$$

$\Downarrow$ taking $M\to\infty$

$$\mathbb{E}^0(\tau_1)\le V_P^{-1}<\infty$$

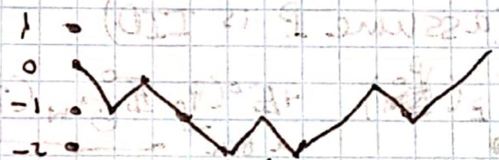
$\blacksquare$

Limit theorems:

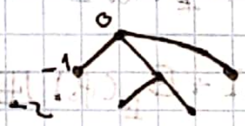
We consider first the transient case, i.e

$\mathbb{E}_P(\log\rho_0)<0$ (transient to $+\infty$).

The following branching process perspective is useful:



$\Updownarrow$ tree encoding



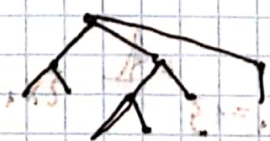
Three excursions from 0 to -1 before hitting 1 and in the second excursion there were 2 excursions to -2

$N_0=3$

$N_{-1}=2$

$N_{-2}=0$

Write N_x for the number of excursions from x to $x-1$ before the walk hits 1.



$N_0=3$

$N_1=5$

$N_{-2}=2$

$N_{-3}=0$

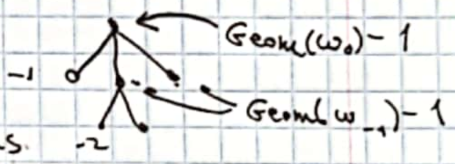
Then: $\tau_1 = \underset{\substack{\uparrow \\ \text{step from} \\ 0 \text{ to } 1}}{1} + 2 \sum_{x \le 0} N_x$.

(Note $N_x < \infty$ P_0 a.s for all $x \le 0$ since walk is transient to $+\infty$).

This allows another proof for the formula for $E^0(\tau_1)$:

$$E^0(\tau_1) = 1 + 2 \sum_{x \le 0} E^0[N_x] \quad (*)$$

Each time the walk goes to $x-1$ from x it makes a geometric number of $x-1 \to x-2$ before returning to x. (geometric (ω_{x-1}) -1)
these geometrics are all independent under P^0_ω.



I.e, the tree has an inhomogeneous Galton-Watson dist. under P^0_ω.

From this we see that for each $x \le 0$,

$$E^0(N_x) = E^0(E^0_\omega(N_x)) \underset{\substack{\uparrow \\ \text{with } N_{+1} := 1}}{=} E^0\Big(\sum_{j=1}^{N_{x+1}} \underbrace{E(\mathrm{Geom}(\omega_x) - 1)}_{= \frac{1}{\omega_x} - 1 = \frac{1-\omega_x}{\omega_x} = \rho_x} \Big)$$

$$= E^0(\rho_x \cdot N_{x+1})$$

$$\underset{\uparrow}{=} E^0(\rho_x) \cdot E^0(N_{x+1}) = E_0(\rho_x)^{|x|+1}$$

Assume
ρ IID
and use indep.
N_{x+1} is a fcn. of $(\omega_{x+1}, \omega_{x+2}, \ldots)$

$\to$ We get $E^0(\tau_1) = 1 + 2 \sum_{x \le 0} [E^0(\rho_x)]^{|x|+1} =$

$$\underset{\substack{\uparrow \\ \text{or } \infty \\ \text{if } E^0(\rho_x) \ge 1}}{=} 1 + \frac{2 E^0(\rho_x)}{1 - E^0(\rho_x)} = V_\rho^{-1}$$

The tree representation allows to prove the following

Lemma: Assume $\mathbb{P}$ IID and $\mathbb{E}_{\mathbb{P}}(\log \rho_0) < 0$.

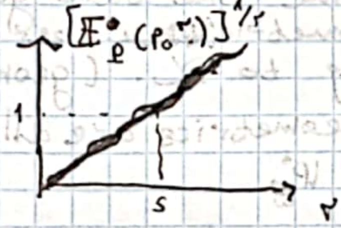
Let $S = \sup\{ r > 0 : \mathbb{E}_{\mathbb{P}}(\rho_0^r) < 1 \}$

Then $\mathbb{E}^0(\tau_1^t) < \infty$ if and only if $t < S$.

Remarks: 1) This generalizes the relation:

$$\mathbb{E}^0(\tau_1) < \infty \iff \mathbb{E}_{\mathbb{P}}(\rho_0) < 1$$

2) Uniform ellipticity shows ρ_0 is bounded, whence $r \mapsto \mathbb{E}_{\mathbb{P}}^0(\rho_0^r)$ is cont. and by Jensen, $[\mathbb{E}_{\mathbb{P}}^0(\rho_0^r)]^{1/r}$ increases with r



In addition, $\mathbb{E}_{\mathbb{P}}(\log \rho_0) = \lim_{r \downarrow 0} \mathbb{E}_{\mathbb{P}}\left(\frac{\rho_0^r - 1}{r}\right)$

(since $\frac{x^r - 1}{r} \to \log x$ as $r \downarrow 0$, as $x^r = e^{r \log x}$).