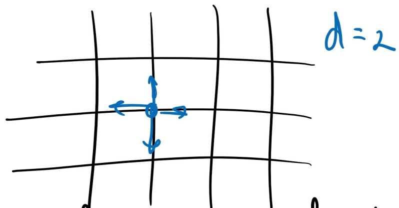
Lecture 2

Random walk in random environment

General definitions:

Nearest-neighbour walks on $\mathbb{Z}^d$



Today: $d=1$ but the following def. are for general d

ENVIRONMENT

- M^d - The set of probability measures on $\{\pm e_i\}_{i=1}^d$
  where $e_i = (0,\dots,1,0\dots0) \in \mathbb{Z}^d$
  (1 at the i^{th} location)
- $\Omega = \{\omega : \mathbb{Z}^d \to M^d\}$ where $\ast \sum \omega(x,\cdot) = 1$
  $\ast\ \omega(x,\cdot) \geq 0$
- On Ω we have a prob. measure $\mathbb{P}$ (the dist. of the environment), assume to be:

  ① stationary and ergodic

  stationary: for $\tau \in \mathbb{Z}^d$, the dist. of $\omega(\cdot,\cdot)$ equals the dist. of $\omega(\cdot+\tau,\cdot)$

  ergodic: means that if E is an event on Ω which is invariant to translations ($\omega \in E \leftrightarrow \forall \tau \in \mathbb{Z}^d\ \omega(\cdot+\tau,\cdot) \in E$) then $\mathbb{P}(E) \in \{0,1\}$

In fact we will mostly take $\mathbb{P}$ to be I.I.D.

The $(\omega(x,\cdot))_{x\in\mathbb{Z}^d}$ all have the same dist on M^d and they are indep.

② Uniform elliptiaty:

$\exists \varepsilon>0$ s.t. $\mathbb{P}(\forall x\in \mathbb{Z}^d, \forall e\in\{\pm e_i\}\ \omega(x,e)\geq\varepsilon)=1$

WALK

Given $\omega\in\Omega$, the walk $(X_n)_{n=0}^{\infty}$ is the markov chain with transition prob. $\mathbb{P}(X_{n+1}=x+e \mid X_n=x)=\omega(x,e)$

$\forall x\in\mathbb{Z}^d, \forall e\in\{\pm e_i\}_{i=1}^{d}$

The prob dist P_ω^x is the dist. over the walk in the environment ω with $P_\omega^x(X_0=x)=1$

We say that P_ω^x is the quenched dist. of the walk.
frozen ω

Joint law: In addition, we give a name to the joint law of $(\omega,(X_n)_{n=0}^{\infty})$ when $X_n\sim P_\omega^x$ and call their dist $\mathbb{P}^x$.

The marginal of $\mathbb{P}^x$ on $(X_n)_{n=0}^{\infty}$, also denoted $\mathbb{P}^x$, is called the annealed (or averaged) dist. of the walk.

I.e., To sample from the quenched dist. we are given ω and x and sample from P_ω^x.

To sample from the annealed measure, we are given x, we sample from $\mathbb{P}$ and then we sample $(X_n)_{n=0}^{\infty}$ from P_ω^x

To illustrate the difference:

$$P^x_\omega(X_0=x,\ X_1=x+e_1,\ X_2=x,\ X_3=x-e_2) = \omega(x,e_1)\omega(x+e_1,e_1)\omega(x,-e_2)$$

$$\mathbb{P}^x(X_0=x, X_1=x+e_1, X_2=x,\ X_3=X-e_2) = \mathbb{E}\big[\omega(x,e_1)\omega(x+e_1,-e_1)\,\omega(x,-e_2)\big)$$

under P

The quenched law is a Markov chain. The annealed law is not, but is invariant in dist. to changing x.

Questions to ask

① Recurrence/Transience: does the walk return to its starting point inf. often

② Law of large numbers: Does $\frac{X_n}{n}$ have a limit? what is the limit?

③ Central limit theorem.

One-dimensional case

From now on, $d=1$.

Instead of $\omega(x,x+1)$ we write ω_x (so that ω_x is the prob. to go right from x)

$\omega(x,x+1) = \omega(x, x+e_1)$

Set: $\rho_x := \frac{1-\omega_x}{\omega_x} \in \left[\frac{\varepsilon}{1-\varepsilon}, \frac{1-\varepsilon}{\varepsilon}\right]$

Recurrence / transience

It is clear that $P^0(-\infty < \limsup_n X_n < \infty) = 0$

($\{\limsup X_n = k\}$ is the same as saying that $X_n = k$ for infintely many n, but X_n only equals $k+1$ finitely many times, so the claim follows from the strong markov property and uniform ellipticity).

Thus we are left with three options

① $\lim_{n\to\infty} X_n = \infty$ - transience to ∞

② $\lim_{n\to\infty} X_n = -\infty$ - transience to $-\infty$

③ $\limsup_{n\to\infty} X_n = \infty$, $\liminf_{n\to\infty} X_n = -\infty$ - recurrence

Thm (Solomom 1975)

P is stationary and ergodic:

a) $\mathbb{E}_P(\log \rho_n) < 0 \implies \lim_{n\to\infty} X_n = +\infty$ $\mathbb{P}^0$ a.s.

b) $\mathbb{E}_P(\log \rho_n) > 0 \implies \lim_{n\to\infty} X_n = -\infty$ $\mathbb{P}^0$ a.s.

c) $\mathbb{E}_P(\log \rho_n) = 0 \implies \limsup_{n\to\infty} X_n = \infty$ $\mathbb{P}^0$ a.s.

$\liminf_{n\to\infty} X_n = -\infty$ $\mathbb{P}^0$ a.s.

proof:

For R, L positive integers we study

$V_{R,L}(x) = \mathbb{P}_W^x$(the walk hits $+R$ before $-L$)

depends also on ω

We calculate $V_{R,L}$ using recurrence relations.

$$V_{R,L}(x) = \omega_x V_{R,L}(x+1) + (1-\omega_x) V_{R,L}(x-1) \quad \forall -L < x < R \qquad (*)$$

and $V_{R,L}(R) = 1 \quad V_{R,L}(-L) = 0$

we get:

$$V_{R,L}(x) = \frac{\sum_{j=-L}^{x-1} \prod_{y=-L+1}^{j} \rho_y}{\sum_{j=-L}^{R-1} \prod_{y=-L+1}^{j} \rho_y}$$

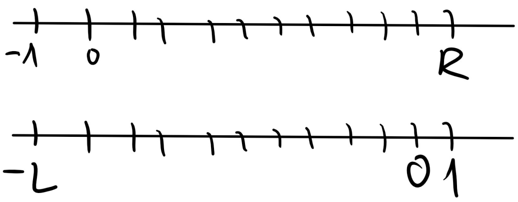
interpreting $\prod_{y=-L+1}^{-L} \rho_y$ as 1

<u>Derivation</u> let $g(x) = V_{R,L}(x+1) - V_{R,L}(x)$

Then, $(*) \iff 0 = \omega_x g(x) - (1-\omega_x) g(x-1) \iff g(x) = \rho_x g(x-1)$

$$\Rightarrow g(x) = \rho_x \cdot \rho_{x-1} \cdots \rho_{-L+1} \underbrace{g(-L)}_{= V_{R,L}(-L+1)} \quad \text{and} \quad \sum_{-L \le y \le R-1} g(y) = 1$$

Recurrence means that



$$\begin{cases} \lim_{R\to\infty} V_{R,1}(0) = 0 = \lim_{R\to\infty} \dfrac{1}{\sum_{j=-1}^{R-1} \prod_{y=0}^{j} \rho_y} = \dfrac{1}{\lim_{R\to\infty} \sum_{j=-1}^{R-1} e^{\sum_{y=0}^{j} \log \rho_y}} \\ \lim_{L\to\infty} V_{1,L}(0) = 1 \end{cases}$$

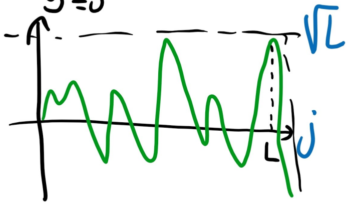
For simplicity, focus on ρ IID (get same results for ergodic ρ by Birkhoff's ergodic thm and related arguments)

If $\mathbb{E}(\log \rho_0) \geq 0$ then $\lim_{R\to\infty} V_{R,1}(0) = 0$

If $\mathbb{E}(\log \rho_0) \leq 0$ then $\lim_{L\to\infty} V_{1,L}(0) = 1$

$\mathbb{E} \log \rho_0 = 0$

$\sum_{y=0}^{j} \log \rho_y$

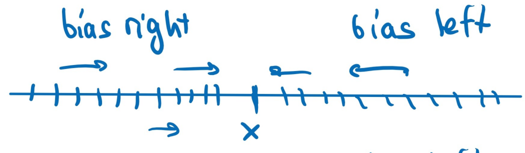


This already gives case (c) and using such ideas and the formula for $V_{R,L}(x)$ one gets also (a) and (b)

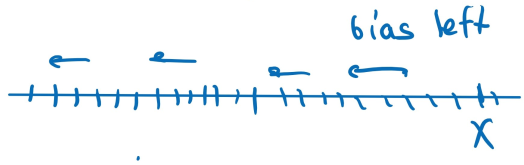
Law of large numbers

Why the behavior can differ from a homogeneous random walk? traps!

two sided trap:



one sided trap:



What determines a trap is the product of the ρ_i by the formula for $V_{R,L}(x)$

Theorem (Solomon 1975, Alili 1999, in ergodic case where statement slightly differs)

a) $\mathbb{E}_P(\rho_0) < 1 \Rightarrow \lim\limits_{n\to\infty} \frac{X_n}{n} = \frac{1-\mathbb{E}_P(\rho_0)}{1+\mathbb{E}_P(\rho_0)} =: V_P \quad P^0\text{-a.s.}$

b) $\mathbb{E}_P(\rho_0^{-1}) < 1 \Rightarrow \lim\limits_{n\to\infty} \frac{X_n}{n} = -\frac{1-\mathbb{E}_P(\rho_0^{-1})}{1+\mathbb{E}_P(\rho_0^{-1})} \quad P^0\text{-a.s.}$

c) $\mathbb{E}_P(\rho_0) \geq 1$
$\mathbb{E}_P(\rho_0^{-1}) \geq 1$ $\Rightarrow \lim\limits_{n\to\infty} \frac{X_n}{n} = 0 \quad P^0\text{-a.s.}$

Remark: Comparing the two theorems we see that it is possible for a walk to be transient to $+\infty$ but non-ballistic.

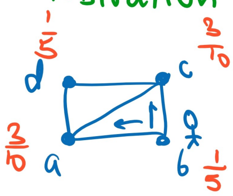
I.e., $\lim\limits_{n\to\infty} X_n = +\infty$, $\lim\limits_{n\to\infty} \frac{X_n}{n} = 0$

proof: prelude (Birkhoff ergodic thm – a generalization of the law of large numbers)

Suppose $(Y_1, Y_2, \ldots)$ is a stationary seq. taking values in some measurable space $(S, \mathcal{S})$, that is, $(Y_1, Y_2, \ldots) \overset{d}{=} (Y_2, Y_3, \ldots)$

Example: E.g., $(Y_n)_{n=1}^{\infty}$ are sampled from a (time-homogeneous) Markov chain with Y_1 dist. as the stationary dist.

illustration



Thm: (Birkhoff)

For every meas. $f: S \to \mathbb{R}$ s.t. $E|f(Y_1)| < \infty$

$$\frac{1}{n}\sum_{k=1}^{n} f(Y_k) \xrightarrow[n\to\infty]{} E[f(Y_1) \mid \mathcal{I}] \text{ a.s. and in } L^1$$

$\mathcal{I}$ is the sigma algebra of invariant events that is, $\mathcal{I}$ contains all the measurable $E \subseteq S^{\{1,2,\ldots\}}$ such that

$$(y_1, y_2, \ldots) \in E \iff (y_2, y_3, \ldots) \in E$$

The seq. $(Y_n)_{n=1}^{\infty}$ is called ergodic if: $\underbrace{\mathbb{P}}_{\text{under the dist of } (Y_n)_{n=1}^{\infty}}(E) \in \{0,1\} \; \forall E \in \mathcal{I}$

In this case the right-hand side of Birkhoff's thm is just $E[f(Y_1)]$.

Moving to the proof of Solomon's thm with Birkhoff's thm in mind, we can write $\frac{1}{n}(X_n - X_0) = \frac{1}{n}\sum_{k=1}^{n}(X_k - X_{k-1})$ but unfortunately $(X_k - X_{k-1})_{k=1}^{n}$ is <u>not</u> stationary, neither under P_ω^x nor under $\mathbb{P}^x$. Instead proceed as follows:

Without loss of generality, assume $\limsup_{n\to\infty} X_n = +\infty$ P^0-a.s. ($\mathbb{E}_P[\log \rho_0] \le 0$)

Define: $T_n := \min\{k \ge 0 : X_k = n\}$ for $n \ge 0$

$\tau_0 = 0$, $\tau_n = T_n - T_{n-1}$ for $n \ge 1$

<u>Claim</u>: $(\tau_n)_{n=1}^{\infty}$ is a stationary and ergodic seq. under P^0

taking the claim for granted let's proceed.

<u>Birkhoff</u>: $\dfrac{T_n}{n} \xrightarrow[n\to\infty]{} \mathbb{E}^0[\tau_1]$ P^0-a.s. (and in L^1 if $\mathbb{E}^0[\tau_1] < \infty$)

even if $\mathbb{E}^0[\tau_1] = \infty$

Indeed $\frac{1}{n} T_n = \frac{1}{n} \sum_{k=1}^{n} \tau_k$ and if $\mathbb{E}^0[\tau_1] = \infty$ we can use that $\frac{1}{n} T_n \ge \frac{1}{n} \sum_{k=1}^{n} \tau_k \cdot \mathbb{1}_{\{\tau_k \le M\}} \xrightarrow[n\to\infty]{} \mathbb{E}[\tau_1 \mathbb{1}_{\{\tau_1 \le M\}}] \xrightarrow[M\to\infty]{} \mathbb{E}[\tau_1]$

Birkhoff

<u>Lemma</u>

$$\lim_{n\to\infty} \frac{X_n}{n} = \frac{1}{\mathbb{E}^0[\tau_1]} = \frac{1}{\lim_{n\to\infty} \frac{T_n}{n}}$$

<u>proof</u>:

let k_n be such that $T_{k_n} \le n \le T_{k_n+1}$

Since $\dfrac{T_n}{n} \longrightarrow \alpha = \mathbb{E}^0(\tau_1)$

We also have $\underbrace{\dfrac{k_n}{n}}_{\to \frac{1}{\alpha}} - \dfrac{1}{n}\underbrace{(n - T_{k_n})}_{} \le \dfrac{X_n}{n} \le \dfrac{k_n}{n} \longrightarrow \dfrac{1}{\alpha}$

$= 1 - \dfrac{T_{k_n}}{n} \longrightarrow 0$

exercise

exercise