Advanced topics in Probability

Theme: Effects of disorder on statistical physics

Statistical physics:

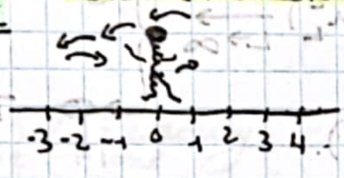
Idea: To determine the behavior of a large system from a description of its microscopic constituents and their interactions.

A focus on specific models.

Example Models:

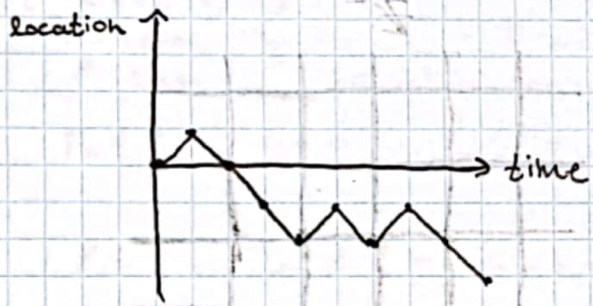
1) Random Walk

one dimensional

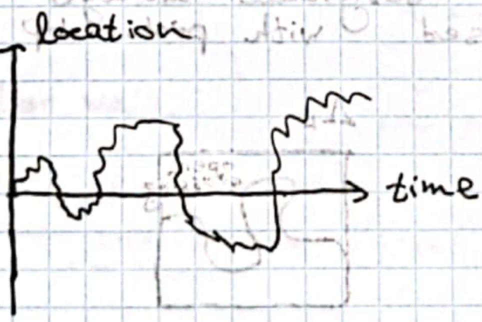


Two dimensions:

Simple RW: prob. $1/2$ to walk to each side.



Scaling limit: Brownian motion.



2) Ising Model: $\Lambda_L \subseteq \mathbb{Z}^2$

$\sigma : \Lambda_L \to \{-1, 1\}$

Probability measure:

Energy of σ:

$$H(\sigma) = -\sum_{u \sim v} \sigma_u \sigma_v$$

$u \sim v$: u adjacent to v

Temprature: $T > 0$

inverse Temp: $\beta = \frac{1}{T}$.

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$$\mathbb{P}(\sigma) = \frac{1}{Z} e^{-\beta H(\sigma)}$$

configurations with lower energy are favored, moreso if the temprature is low.

$$Z = \sum_{\sigma : \Lambda_L \to \{-1,1\}} e^{-\beta H(\sigma)}$$ is termed partition function.

Total magnetization: $m_L = \left(\sum_{v \in \Lambda_L} \sigma_v\right) \cdot \frac{1}{|\Lambda_L|}$ ← normalize

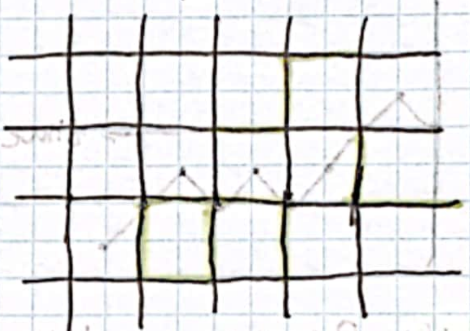
Phase transition: $\exists\, T_c \in (0,\infty)$ s.t.

$$\mathbb{E}(m_L^2) \xrightarrow[L\to\infty]{} \begin{cases} 0 & T > T_c \\ >0 & T < T_c \end{cases}$$

(In dimensions $d \geq 2$).

3) Percolation

$\mathbb{Z}^2$



Given $0<p<1$, designate an edge open with prob. p and closed with prob. $1-p$ indep. between edges.

Λ_L

Phase transition: $\exists\, p_c \in (0,1)$

$$\mathbb{P}\left(\left[\begin{matrix}\text{origin has an open} \\ \text{connection to } \infty\end{matrix}\right]\right) = \begin{cases} 0 & p < p_c \\ >0 & p > p_c \end{cases}$$

(in dimension $d \geq 2$)

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Disorder:

1) Random walk in random environment.

2) Random-Field models or spin glasses.
   - Giorgio Parisi - Nobel prize in physics 2021

3) Noise sensitivity and dynamical percolation.

4) Localization in quantom systems.

Books:
{ Ofer Zeitouny - Lectures on Probability Theory + St. Flour
{ Christophe Garban - Noise Sensitivity and percolation + Jeffrey Steif

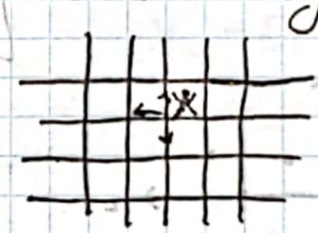
Random Operators - Disorder Effects
Michael Aizenman and Simone Warzel

Full tibles avilable on course website.

Random Walks in Random Environment

Brief discussion of random walks in a homogeneous environment:

Walk on $\mathbb{Z}^d$.



Nearest neighbor walk.

Transition Kernel = probability list on $\{\pm e_i\}_{i=1}^{d}$
where $e_i = (0,..,0,\underset{\uparrow\ i\text{-th coord}}{1},0,..,0)$

Write it as $\omega : \{\pm e_i\} \to [0,1]$, $\sum \omega(\cdot) = 1$

elliptic: $\exists \varepsilon > 0$ s.t $\omega(x) \in [\varepsilon, 1-\varepsilon]$.

Questions:
1) Recurrence / transience
2) Law of large numbers
3) Central limit theorem.

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1) Recurrence / transience

Starting at 0 (more generally, P^x for the walk measure starting at x).

Q: $\mathbb{P}^0\left(\begin{matrix}\text{Walker returns}\\ \text{to } 0\end{matrix}\right) = 1$?

$\left(\begin{matrix} = 1 : \text{recurrent} \\ < 1 : \text{transient}\end{matrix}\right)$

Investigate for simple random walk : $\omega(\pm e_i) = \frac{1}{2d}$

Polya's thm: Recurrent when $d = 1,2$
Transient when $d > 2$

Idea of proof: Let $p = \mathbb{P}^0(\text{Walker returns to } 0)$

$N = \#$ of visits to 0. N is distributed geometric$(1-p)$

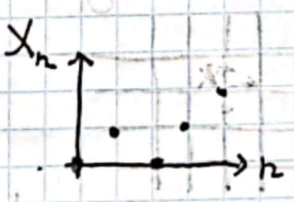
$\Rightarrow \mathbb{E}N = \frac{1}{1-p}$ infinite $\Longleftrightarrow p = 1$

However $N = \sum_{n=0}^{\infty} \mathbb{1}_{X_n = 0}$ (X_n) is the walk

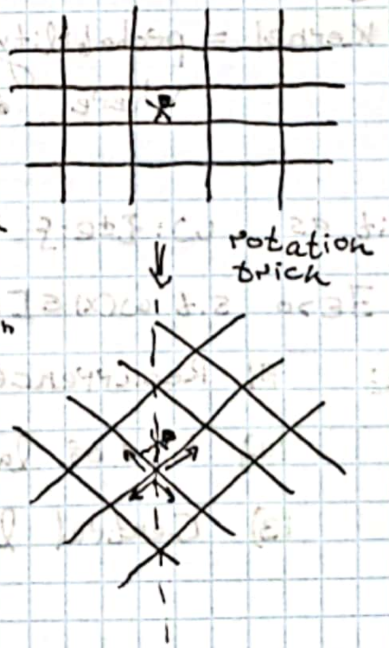
$$\mathbb{E}N = \sum_{n=0}^{\infty} \mathbb{P}(X_n = 0)$$

d=1: $\mathbb{P}(X_n = 0) = \begin{cases} 0 & n \text{ odd} \\ \frac{1}{2^n}\cdot\binom{n}{n/2} & n \text{ even} \end{cases}$ $\Rightarrow \mathbb{E}N = \infty$

$\frac{1}{2^n}\binom{n}{n/2} \approx c/\sqrt{n}$



d=2: After $45°$ rotation, the coords of X_n are indep. 1d simple random walk



$\Rightarrow \mathbb{P}(X_n = 0) = \begin{cases} 0 & n\text{-odd} \\ \approx c/n & n\text{-even} \end{cases}$

$\Rightarrow \mathbb{E}N = \infty.$

$d \geq 3$: $\mathbb{P}(X_n = 0) = \begin{cases} 0 & n\text{-odd} \\ \approx \underbrace{c/n^{d/2}}_{\text{requires a calculation}} & n\text{-even} \end{cases} \Rightarrow \mathbb{E}N < \infty$

2) **Law of large numbers**: Understand $\lim\limits_{n\to\infty} \frac{X_n}{n}$ for general transition kernel ω.

$$X_n = \sum_{k=1}^{n} \varepsilon_k \quad \text{where } (\varepsilon_k) \text{ i.i.d samples from } \omega.$$

$\Rightarrow$ Since the ε_k are bounded in every coord., can apply the usual law of large numbers in every coord. to get:

$$\lim_{n\to\infty} \frac{X_n}{n} = \mathbb{E}[\varepsilon_1] \quad - P^0\text{-a.s and in } L^1.$$

If $\lim\limits_{n\to\infty} \frac{X_n}{n} \neq 0$ we say that the walk is ballistic

In $d = 1,2$ we have that either the walk is either recurrent or ballistic.

3) **Central limit theorem**: Thm : $\dfrac{X_n - \mathbb{E}[\varepsilon_1]\cdot n}{\sqrt{n}} \xrightarrow{d} N(0,\Sigma)$

Probability measure on $\mathbb{R}^d$ with density $\dfrac{1}{(2\pi)^{d/2}\cdot\sqrt{\det\Sigma}} \cdot e^{-\frac{x^t\Sigma^{-1}x}{2}}$

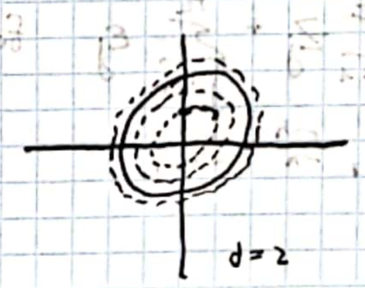
Σ = Covariance matrix.

$$\Sigma_{i,j} = \mathrm{Cov}(\varepsilon_{1,i}, \varepsilon_{1,j})$$

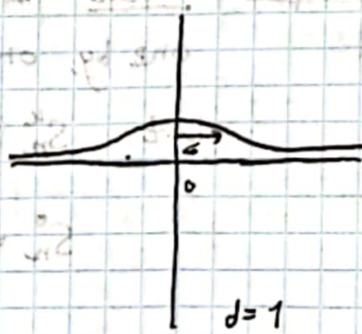
If $Z \sim N(0,\Sigma)$ then $\mathbb{E}[Z] = 0$

$$\Sigma_{i,j} = \mathrm{Cov}(Z_i, Z_j)$$

coordinates of the vector



$d=2$



$d=1$

Setup for central limit theorem:

(ε_j) IID random vectors in $\mathbb{R}^d$.

For simplicity, assume $\mu = \max_{1\le i\le d} \mathbb{E}(|\varepsilon_{1,i}|^3) < \infty$

$$S_n := \sum_{j=1}^{n} \varepsilon_j \quad , \vec{m} = \mathbb{E}[\varepsilon_1]$$

We want $\frac{S_n - n\mathbb{E}(\varepsilon_1)}{\sqrt{n}} \xrightarrow{d} N(0,\Sigma)$

$\Sigma_{ij} = \mathrm{Cov}(\varepsilon_{1,i}, \varepsilon_{1,j})$

This means that for every continuous and bounded $F: \mathbb{R}^d \to \mathbb{R}$ it holds:

$$\mathbb{E}\left[F\left(\frac{S_n - n\mathbb{E}[\varepsilon_1]}{\sqrt{n}}\right)\right] \xrightarrow{n\to\infty} \mathbb{E}(F(Z))$$

Where $Z \sim N(0,\Sigma)$

In this, it actually suffices to ~~show~~ consider only $F: \mathbb{R}^d \to \mathbb{R}$ smooth and compactly supported.

We prove using Lindeberg's method (1922).

Step 1: Let (Z_j) be IID $N(\vec{m},\Sigma)$

expectation m — covariance Σ

the same first and second moments as ε_1

Then $S_n' := \sum_{j=1}^{n} Z_j$

$S_n' \sim N(n\vec{m}, n\Sigma)$

since a sum of indep. Gaussians is a Gaussian

$\frac{S_n' - n\vec{m}}{\sqrt{n}} \sim N(0,\Sigma)$. **Note:** (Z_j) are indep. of the (ε_j).

Step 2: **Idea:** In S_n, replace the (ε_j) by the (Z_j) one by one.

Let $S_n^k := \sum_{j=1}^{k} Z_j + \sum_{j=k+1}^{n} \varepsilon_j$ for $0 \le k \le n$

$S_n^0 = S_n$, $S_n^n = S_n'$

Let $F:\mathbb{R}^d \to \mathbb{R}$ be smooth and comp. supp.

Write: $\mathbb{E}\left[F\left(\frac{S_n - n\cdot\vec{m}}{\sqrt{n}}\right)\right] - \mathbb{E}\left[F(N(0,\Sigma))\right]$

$$= \sum_{k=0}^{n-1} \underbrace{\mathbb{E}\left[F\left(\frac{S_n^k - n\cdot\vec{m}}{\sqrt{n}}\right)\right] - \mathbb{E}\left[F\left(\frac{S_n^{k+1} - n\cdot\vec{m}}{\sqrt{n}}\right)\right]}_{\text{Call this } d_k}$$

Goal: $\sum_{k=0}^{n-1} d_k \xrightarrow[n\to\infty]{} 0$

proof: Develop F in a Taylor expansion.

$$F(x+\delta) = F(x) + \underbrace{\sum_{i=1}^{d} \frac{\partial F}{\partial x_i}(x)\cdot\delta_i}_{\nabla F(x)\cdot\delta} + \sum_{i,j=1}^{d} c_{ij}\frac{\partial^2 F}{\partial x_i \partial x_j}\cdot\delta_i\delta_j$$

$$c_{ij} = \begin{cases} 1 & i\neq j \\ \frac{1}{2} & i=j \end{cases}$$

$$+ O\left(\max_i |\delta_i|^3\right)$$

F is smooth and compactly supported

Fix $0 \le k \le n-1$. Note that:

$$F\left(\frac{S_n^k - n\cdot\vec{m}}{\sqrt{n}}\right) = F\Bigg(\underbrace{\frac{1}{\sqrt{n}}\left(\sum_{j=1}^{k}(Z_j - \vec{m})\right) + \frac{1}{\sqrt{n}}\left(\sum_{j=k+2}^{n}(\varepsilon_j - \vec{m})\right)}_{X_k}$$

$$+ \underbrace{\frac{1}{\sqrt{n}}(\varepsilon_{k+1} - \vec{m})}_{\delta_k}\Bigg)$$

$$F\left(\frac{S_n^{k+1} - n\cdot\vec{m}}{\sqrt{n}}\right) = F\Big(X_k + \underbrace{\frac{1}{\sqrt{n}}(Z_{k+1} - \vec{m})}_{\delta_k'}\Big)$$

Note also, by the Taylor develop.

$$\mathbb{E}(F(X_k+\delta_k)) = \mathbb{E}F(X_k) + \sum_{i=1}^{d} \mathbb{E}(\partial_i F(X_k)\cdot\delta_{k,i}) \overset{\text{independent}}{=} \mathbb{E}(\partial_i F(X_k))\cdot\mathbb{E}[\delta_{k,i}]$$

$$+ \sum_{i,j=1}^{d} c_{ij}\,\mathbb{E}\left[\partial^2_{ij}F(X_k)\cdot\delta_{k,i}\,\delta_{k,j}\right] = \mathbb{E}[\partial^2_{ij}F(X_k)]\cdot\mathbb{E}[\delta_{k,i}\,\delta_{k,j}]$$

$$+ O\left(\mathbb{E}\max_i|\delta_{k,i}|^3\right)$$

Since ε_{k+1} and Z_{k+1} have the same first and second ~~order~~ moments we get

$$d_k = O\left(\mathbb{E}\max_i |\delta_{k,i}|^3 + \max_i |\delta_{k,j}'|^3\right) =$$

$$= O\left(\sum_{i=1}^{d} \mathbb{E}\left(|\delta_{k,i}|^3 + |\delta_{k,i}'|^3\right)\right) = O\left(1/n^{3/2}\right)$$

by def. of δ_k and δ_k' and the assumption $\max_i \mathbb{E}|\varepsilon_{1,i}|^3 < \infty$.

$$\Rightarrow \sum_{k=0}^{n-1} d_k = O\left(1/\sqrt{n}\right) \xrightarrow{n\to\infty} 0$$

Finishing the proof. ☒

Remarks: 1) The method is sometimes useful when Fourier methods are not applicable.

It has been extended to showing that:

$\mathbb{E}F(\varepsilon_1, \dots, \varepsilon_n) - \mathbb{E}F(Z_1, \dots, Z_n)$ is small when the ε_i share many moments with Z_j and F has a good Taylor expansion.

Tao-Vu use this to show Wigner matrix universality when first 4-moments match.

2) Also some extensions to dependent random variables.

Random Walk in random environment

Setup: at each vertex of $\mathbb{Z}^d$ we put a prob. distance over $\{\pm e_i\}_{i=1}^{d}$.

Then the walker walks in this environment.

There are two prob. meas.:

1) $\mathbb{P}$ - the distribution of the environment ω.

environment
$$\begin{cases} \omega: \mathbb{Z}^d \times \{\pm e_i\}_{i=1}^d \to [0,1] \\ \forall x \in \mathbb{Z}^d,\ \sum \omega(x,\cdot) = 1 \\ \text{Uniform ellipticity: } \omega: \mathbb{Z}^d \times \{\pm e_i\}_{i=1}^d \to [\varepsilon, 1-\varepsilon] \text{ for some } \varepsilon>0 \end{cases}$$

Many times we will take P to be a product measure, $(\omega(x,\cdot))_m$ IID

People also study the case that P is stationary and ergodic.

2) Given ω, P_ω^x is the prob. dist. of the walk started at x and walking in the environment ω.