***Ordinary Differential Equations – 1*, 0366-2103,** Tel-Aviv University

Problem Collection

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1. Find the general solution and solve the Cauchy problem, where it is relevant

   a. $(1+y^2\sin 2x)dx - 2y\cos^2 x\, dy = 0$

   b. $y' + 2x = \cos(y - x + x^2)$

   c. $xy' = \dfrac{x^2+y^2}{x+y}$

   d. $y' - x^2 y = x^2 / y^2$

   e. $\begin{cases} y_1' = 4y_1 - 5y_2 \\ y_2' = y_1 - 2y_2 + 2x \end{cases}$

   f. $y'' - 5\,y' = 3x^2 + \sin x$, $y(0)=0, y'(0)=0$

   g. $y^{(4)} + \ddot{y} = e^t + t$

   h. $\begin{cases} \dot{x} = y \\ \dot{y} = z \\ \dot{z} = -y + 2z + 1 + e^t \end{cases}$, also calculate the exponent e^{At} of the system matrix A.

2. Find a solution bounded as $t \to \frac{\pi}{2}$, $0 < t < \pi/2$, of the DE $\dot{x}\sin 2t = 2(x + \cos t)$

3. In what form (only provide the formula corresponding to the method of undetermined coefficients) one searches for a particular solution of the following DE?

   a. $y^{(7)} + 5y^{(6)} + 6y^{(5)} = 1 + te^t + te^{-2t} - t^3 + te^{-3t}\sin t - e^{-3t}$

   b. $p(\frac{d}{dt})y = f(t)$, where
   $p(\lambda) = (\lambda^2+4\lambda+5)(\lambda^2+4\lambda-5)(\lambda-1)^2$, $f(t) = t + te^{-2t}\cos t - t\cos t - e^{-5t} + e^t + t^2e^{2t} + t^3$

   c. $P(\frac{d}{dt})x = 1 + 2t - e^t - \cos t + t\sin 2t$, where $P(\lambda) = \lambda(\lambda^2-1)^3(\lambda^2+1)^2 Q(\lambda)$. The roots of the polynomial Q have real parts larger than 5. Does the DE have stable solutions?

4. Linearize the system of nonlinear DEs $\begin{cases} \dot{x} = -\cos(x + \sin y) \\ \dot{y} = \tan(3x) + 2\cos x \end{cases}$ at its equilibrium $(\frac{\pi}{2}, 0)$ and solve the obtained linear system.

5. Which pair of vector functions $\begin{pmatrix} t \\ t-1 \end{pmatrix}, \begin{pmatrix} t-1 \\ t \end{pmatrix}$ or $\begin{pmatrix} t \\ t+1 \end{pmatrix}, \begin{pmatrix} t+1 \\ t \end{pmatrix}$ can constitute a pair of solutions for the DE $\dot{y} = A(t)y,\ y \in \mathbb{R}^2$ with the matrix $A(t)$ continuously depending on $t \in [0,1]$? Explain.

   a. Find the corresponding matrix $A(t)$ using the DE for the fundamental matrix.

   b. Solve the Cauchy problem $\dot{y} = A(t)y$, $y_1(0) = 1, y_2(0) = 2$.

6. Which pair $(f_1(t), f_2(t))$ of two functions $(1, \tan t)$ or $(t, \tan t)$ can satisfy the DE $\ddot{y} + a_1(t)\dot{y} + a_2(t)y = 0$ with continuous real-valued coefficients in the interval $t \in (-\pi/2, \pi/2)$. Calculate the Wronskian $W[y, f_1, f_2](t)$ for some other solution $y(t)$ of this DE.

7. Solve the DE $(t^2 - 1)\ddot{y} - 6y = 1$ if it is known that the corresponding homogeneous problem has a polynomial particular solution.

8. Solve the DE $(t^2 - 1)\ddot{y} + 4t\dot{y} + 2y = 6t$ if its two particular solutions are $y_1 = t$ and $y_2 = \dfrac{t^2 + t + 1}{t+1}$.

9. Solve the DE and the Couchy problem $\ddot{y} + \dot{y}\tan t = \sin(2t), |t| < \frac{1}{2}\pi$, $y(0) = 1, \dot{y}(0) = 1$, if $y = \sin t$ is a particular solution of the corresponding homogeneous DE.

10. Find the linear approximation with respect to $\varepsilon \in \mathbb{R}$, $\varepsilon \approx 0$, of the Cauchy-problem solution over the time interval $t \in [-T, T]$ for some $T > 0$.

   a. $\ddot{y} = -2\sin y + y$, $y(0) = 0, \dot{y}(0) = \varepsilon$

   b. $\ddot{y} = -\varepsilon \sin t \cos y - y$, $y(0) = 2\varepsilon^3 - \varepsilon^2, \dot{y}(0) = \varepsilon^2$

   c. $\ddot{y} + 2\dot{y}\cos\varepsilon + y - (e^t y + te^{-t} + \dot{y})\sin\varepsilon = 0$, $y(0) = \tan\varepsilon, \dot{y}(0) = (1+\varepsilon)^2$

11. Calculate the matrix exponent e^{At} for the matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 3 \end{pmatrix}$, and solve the Cauchy problem $\dot{y} = Ay, y(0) = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}$. It is acceptable to not calculate matrix products in the final answer.

12. What are the possible DE orders n such that for some smooth real-valued function $f(t, y, \dot{y}, ..., y^{(n-1)})$ functions $y = t^3 + t^7$ and $y = t^3$ both satisfy $y^{(n)} = f(t, y, \dot{y}, ..., y^{(n-1)})$ for any $t \in \mathbb{R}$? Give an example for the found minimal order. Explain.

13. What is the minimal order n of the linear time-invariant DE $y^{(n)}+a_1 y^{(n-1)}+...+a_n y=0$ which has both solutions $y=t^4$, $y=t^2\cos t+t^4$? Explain and find the DE in the form $p(\frac{d}{dt})y=0$. There is no need to multiply polynomials in the final form of $p(\lambda)$.

14. Which of the functions t^2 or t^2+1 can satisfy the DE $\ddot{y}+a(t)\dot{y}+b(t)y=0$ for any $t\in\mathbb{R}$ and continuous $a,b:\mathbb{R}\to\mathbb{R}$? Explain and provide an example.

15. What is the minimal order n of the DE $y^{(n)}+a_1 y^{(n-1)}+...+a_n y=0$ with constant real coefficients, if it has the solution $y=t\cos t+t^3$? Explain and find the DE. Does it have stable solutions?

16. It is known that the DE $y^{(5)}+a_1 y^{(4)}+a_2\dddot{y}+a_3\ddot{y}+a_4\dot{y}=0$ with constant real coefficients has the solution $y=\cos t+\sin(2t)$. Find the coefficients and the general solutions.

17. Check the stability of the following equilibriums of scalar differential equations:

$$\dot{y}=-y^3,\, y(0)=0; \qquad \dot{y}=-y^3+y^2,\, y(0)=0; \qquad \dot{y}=-|y|+\sin^2 y,\, y(0)=0;$$

$$\dot{y}=\cos(y^3),\, y(0)=\left(\tfrac{1}{2}\pi\right)^{1/3} \text{ and } y(0)=\left(\tfrac{3}{2}\pi\right)^{1/3}; \qquad \dot{y}=-y^3\cos y,\, y(0)=\tfrac{1}{2}\pi \text{ and } y(0)=0;$$