Final Exam

PROBLEM 1. (1, 13 points) The DE $\dddot{y} + a_1\ddot{y} + a_2\dot{y} + \alpha y = 0$ with constant real coefficients has the particular solution $\cos t - 2\sin t$. Find its general solution and the coefficients a_1 and a_2 as functions of α.

(2, 12 points) Determine whether the following constant solutions of the first-order DE's are stable or unstable:

(a) $\dot{y} = -y\sqrt{|y|}$, $y(0) = 0$; **(b)** $\dot{y} = |\sin y|$, $y(0) = 0$; **(c)** $\dot{y} = \cos(y^2)$, $y(0) = \sqrt{\pi/2}$.

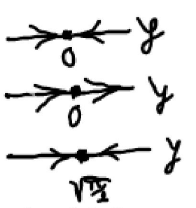
Answer. (1) General solution: $y =$ $c_1 \cos t + c_2 \sin t + c_3 e^{-\alpha t}$, $c_1, c_2, c_3 \in \mathbb{R}$. Coefficients: $a_1 = \alpha$, $a_2 = 1$. **(2)** Stable (asymptotically), unstable, stable (asymptotically).

Solution. (1) Let χ be the characteristic polynomial of the equation, that is $\chi(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + \alpha = 0$. Considering the form of the given solution and the fact that all the coefficients are real, we deduce that $\pm i$ are roots of χ, thus, we have that

$$\chi(\lambda) = (\lambda - i)(\lambda + i)(\lambda - \lambda_3) = \lambda^3 - \lambda_3\lambda^2 + \lambda - \lambda_3, \tag{א׳}$$

where λ_3 is the remaining root of the polynomial. Thus, $\alpha = -\lambda_3$, so $a_1 = \alpha$, $a_2 = 1$. Consecutively, the general solution has the form $c_1 \cos t + c_2 \sin t + c_3 e^{-\alpha t}$, where $c_1, c_2, c_3 \in \mathbb{R}$.

(2) In each item we take the function $v(y) = (y - y(0))^2$ as Lyapunov's function and differentiate it with respect to the equation (we denote the derivative by v^*), that is $v^* = 2(y - y(0))\dot{y}$, so we have for the three cases (the inequalities are true in some vicinity of the initial condition)



$$\begin{aligned} v^*_{(a)} &= -2y^2\sqrt{|y|} < 0, \quad y \neq 0 \\ v^*_{(b)} &= 2y|\sin y| > 0, \quad 0 < y < \tfrac{\pi}{2} \\ v^*_{(c)} &= 2\left(y - \sqrt{\pi/2}\right)\cos(y^2) < 0, \quad y \neq \sqrt{\tfrac{\pi}{2}} \text{ (small vicinity)} \end{aligned} \tag{ב׳}$$

Thus, in the first and in the third cases the constant solution is stable, while in the second case it is unstable. ❦

PROBLEM 2. (1, 14 points) Find the general solution and the solution of the Cauchy problem for the DE

$$y' - \frac{1}{2} y \tan x = y^3 \cos x \tag{ג׳}$$

with the initial condition $y(0) = 1$ (maybe, as an implicit function).

(2, 11 points) Check the stability of the equilibrium at the origin for $\alpha = 0$ and $\alpha = 2$ for the system

$$\begin{cases} \dot{x} = \sin(x - 3y) + x\cos y \\ \dot{y} = \alpha e^{x-2y} + \log(1 + x) - \alpha \end{cases} \tag{ד׳}$$

Answer. (1) $y = \pm 1/\sqrt{-2x\cos x + C\cos x}$ and $y \equiv 0$. **(2)** Unstable for $\alpha = 0$ and asymptotically stable for $\alpha = 2$.

Solution. (1) This is a Bernoulli equation. Divide both parts by y^3 (note that $y \equiv 0$ is a solution) to obtain

$$\frac{y'}{y^3} - \frac{1}{2}\frac{1}{y^2}\tan x = \cos x \tag{ה׳}$$

and use the substitution $z = 1/y^2$, then $z' = -2y'/y^3$, so Equation (ה׳) becomes

$$-\frac{z'}{2} - \frac{z}{2}\tan x = \cos x \Longrightarrow z' + z\tan x = -2\cos x. \quad x \neq \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z} \tag{ו'}$$

A function μ, upon satisfying

$$\mu = \mu' \tan x \Longrightarrow \frac{\mu'}{\mu} = \tan x \longrightarrow \mu = \frac{1}{\cos x}, \tag{ז'}$$

serves as an integrating factor. Therefore,

$$(z\mu)' = z'\mu + z\mu' = z'\mu + 2z\mu\tan x \Longrightarrow (z\mu)' = -2\mu\cos x = -2 \Longrightarrow$$
$$\frac{z}{\cos x} = -2x + C \Longrightarrow z = -2x\cos x + C\cos x. \quad C \in \mathbb{R} \tag{ח'}$$

Thus,

The general solution: $$y = \pm\frac{1}{\sqrt{z}} = \pm\frac{1}{\sqrt{-2x\cos x + C\cos x}}, \quad C \in \mathbb{R}, \quad y = 0. \tag{ט'}$$

(2) First, note that $x \equiv 0, y \equiv 0$ is a solution, so the origin is indeed an equilibrium point. Computing the Jacobian at the origin yields

$$J = \begin{pmatrix} \cos(x-3y) + \cos y & -3\cos(x-3y) - x\sin y \\ \alpha e^{x-2y} + \frac{1}{1+x} & -2\alpha e^{x-2y} \end{pmatrix} \Longrightarrow J(0,0) = \begin{pmatrix} 2 & -3 \\ \alpha+1 & -2\alpha \end{pmatrix} \tag{י'}$$

Since the original equation has the form $\binom{\dot x}{\dot y} = F(x,y)$, where the RHS is differentiable, we get that $F(x,y) = J(0,0)\cdot\binom{x}{y} + o\left(\binom{x}{y}\right)$ as $\binom{x}{y} \to 0$.

The eigenvalues of the Jacobian can be easily found and equal $1 \pm i\sqrt{2}$ for $\alpha = 0$ (so the point is unstable because the real parts are positive) and -1 (of multiplicity two) for $\alpha = 2$ (so the point is asymptotically stable).

Problem 3. (1, 14 points) Find the general solution of the DE

$$(x^2 - xy - y^2)\,dx + x^2\,dy = 0 \tag{י״א}$$

(maybe, as an implicit function).

(2, 11 points) The function $f : \mathbb{R}^2 \to \mathbb{R}$ is known to be locally Lipschitz. Can the function $\cos t - 1$ satisfy the DE $\ddot y = f(y, \dot y)y$ for $t \in [-1,1]$ or for $t \in [5,7]$?

Answer. (1) $x \equiv 0$; $(y-x)/(y+x) = Cx$, $C \in \mathbb{R}$; $y = x$; $y = -x$; (2) No.

$x(y) \equiv 0$ is a solution (for $x \neq 0$)

Solution. (1) The equation is homogeneous, so we use the substitution $y = zx$ to get

$$(x^2 - xy - y^2)\,dx + x^2\,dy = (x^2 - zx^2 - z^2x^2)\,dx + x^2(z\,dx + x\,dz) = x^2((1-z^2)\,dx + x\,dz). \tag{י״ב}$$

Considering the potential solutions $x \equiv \text{const}$, we deduce

$$1 - z^2 + xz' = 0 \Longrightarrow \frac{z'}{z^2-1} = \frac{1}{x} \Longrightarrow \frac{1}{2}\log\left|\frac{z-1}{z+1}\right| = \log|x| + c \Longrightarrow \sqrt{\left|\frac{z-1}{z+1}\right|} = e^c|x|, \tag{י״ג}$$

so, returning to y, we get

$z = \pm 1$ is a solution, but $z^2 - 1 \neq 0$ $(y = \pm x)$

$$\frac{y-x}{y+x} = Cx^2, \; C \in \mathbb{R}, \; C \neq 0; \quad y = x; \; y = -x \qquad \text{(י"ד)}$$

Note that if x is a constant function (of y), then only $x \equiv 0$ fits the original equation.

(2) Denote $\varphi \equiv 0$, $\psi \equiv \cos t - 1$. Note that at the points $2\pi\mathbb{Z}$ both functions vanish (in particular, they are equal there). Also, their derivatives at those points are equal (again, both vanish), since

$$\psi' = -\sin t|_{2\pi\mathbb{Z}} = 0. \qquad \text{(ט"ו)}$$

Now, note that the equation has a unique solution for any initial condition $y(t_0) = y_0$, $y'(t_0) = y_1$, since $f(\dot{y}, y) \cdot y$ is Lipschitz-continuous as a function of y and $\dot{y}$ in some neighbourhood U of an arbitrary point (y_0, y_1). ~~Herefrom,~~ Therefore, the theorem of existence and uniqueness ~~must~~ hold~~s~~.

However, since both φ and ψ both vanish at zero and their derivatives there are equal, the theorem of existence and uniqueness is violated for the initial condition $y(0) = 0$, $y'(0) = 0$. Also, since $3 < \pi < 3.2$, we have $2\pi \in [5, 7]$, so, for the initial condition $y(2\pi) = 0$, $y'(2\pi) = 0$, and we again have a contradiction. ❦

Problem 4. Consider the matrix $A = \begin{pmatrix} 0 & 1 \\ -1 & 2 \end{pmatrix}$. Solve the DE $\dot{y} = Ay$, $y \in \mathbb{R}^2$ for an arbitrary initial condition $y(0) = C \in \mathbb{R}^2$ and compute e^{tA}.

Answer. $y = e^{tA} C$, $e^{tA} = \begin{pmatrix} e^t(1-t) & e^t t \\ -e^t t & e^t(1+t) \end{pmatrix}$

Solution. ◆ *Method 1 (for higher-order equations fans).* Note that the system is equivalent to

$$\begin{cases} \dot{\alpha} = \beta \\ \dot{\beta} = -\alpha + 2\beta \end{cases} \Longrightarrow \ddot{\alpha} = \dot{\beta} = -\alpha + 2\dot{\alpha}. \qquad \text{(ט"ז)}$$

The last equation can be solved by constructing the characteristic polynomial as follows

$$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2, \qquad \text{(י"ז)}$$

so the solution has the form

$$\alpha = e^t(a_1 + a_2 t) \Longrightarrow \beta = e^t(a_1 + a_2 + a_2 t). \qquad \text{(י"ח)}$$

If the initial condition is $C = \binom{c_1}{c_2}$, then $a_1 = c_1$, $a_2 = c_2 - c_1$. The matrix exponential can be found from the fact that if v_1 and v_2 are the columns of the e^{tA}, then $v_1(0) = \binom{1}{0}$, $v_2(0) = \binom{0}{1}$. Considering the forms of α and β we find

$$e^{At} = \begin{pmatrix} e^t(1-t) & e^t t \\ -e^t t & e^t(1+t) \end{pmatrix} \qquad \text{(י"ט)}$$

◆ *Method 2 (for matrix enjoyers).* Find the Jordan form

$$A = SJS^{-1}, \text{ where } S = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \quad J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad S^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} \qquad \text{(כ')}$$

Since

$$e^{tJ} = \begin{pmatrix} e^t & te^t \\ 0 & e^t \end{pmatrix} \Longrightarrow e^{tA} = Se^{tJ}S^{-1} = \begin{pmatrix} e^t(1-t) & e^t t \\ -e^t t & e^t(t+1) \end{pmatrix} \qquad \text{(כ"א)}$$

The linear combinations of the columns constitute the set of general solutions, just as described in the previous method.

Problem 5. Solve the DE

$$\dot{y} = Ay + \begin{pmatrix} 0 \\ 0 \\ e^t + 1 \end{pmatrix}, \quad y \in \mathbb{R}^3 \quad \text{for} \quad A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} \tag{כ״ב}$$

It is known that one of the eigenvalues of A is equal to 2.

Answer. See Equation (כ״ז) below.

Solution. Denote $y = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$ and write the system as

$$\begin{cases} \dot{\alpha} = \beta \\ \dot{\beta} = \gamma \\ \dot{\gamma} = -2\alpha + \beta + 2\gamma + e^t + 1 \end{cases} \tag{כ״ג}$$

Then, $\ddot{\alpha} = \gamma$ and

$$\dddot{\alpha} = \dot{\gamma} = -2\alpha + \dot{\alpha} + 2\ddot{\alpha} + e^t + 1 \Longrightarrow \dddot{\alpha} - 2\ddot{\alpha} - \dot{\alpha} + 2\alpha = e^t + 1. \tag{כ״ד}$$

First we solve the corresponding homogeneous equation. The characteristic polynomial is

$$\chi(\lambda) = \lambda^3 - 2\lambda^2 - \lambda + 2 = (\lambda - 2)(\lambda - 1)(\lambda + 1). \tag{כ״ה}$$

Thus, the general solution has the form $c_1 e^{2t} + c_2 e^{-t} + c_3 e^t$. For the RHS e^t there is resonance, and a particular ~~the~~ solution can be found in the form ate^t. Plugging it into the equation, we can find that $a = -1/2$. For the RHS equal to 1 we can find a particular solution 1/2, since there is no resonance.

Thus, the general solution can be written as

$$\begin{aligned} \alpha &= c_1 e^{2t} + c_2 e^{-t} + c_3 e^t + \frac{1}{2} - \frac{te^t}{2} \Longrightarrow \\ \beta &= 2c_1 e^{2t} - c_2 e^{-t} + c_3 e^t - \frac{e^t}{2} - \frac{te^t}{2} \Longrightarrow \\ \gamma &= 4c_1 e^{2t} + c_2 e^{-t} + c_3 e^t - e^t - \frac{te^t}{2} \end{aligned} \tag{כ״ו}$$

Finally, we have

$$y = c_1 e^{2t} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + c_3 e^t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} - \frac{te^t}{2} \\ -\frac{e^t}{2} - \frac{te^t}{2} \\ -e^t - \frac{te^t}{2} \end{pmatrix} \tag{כ״ז}$$

Problem 6 (Bonus for the Group 1+, 10 points). The function $f : \mathbb{R}^2 \to \mathbb{R}$ is locally Lipschitz. Can both functions $\sin(2t)$ and $2\arctan t$ simultaneously be solutions of the DE $\ddot{y} = f(y, \dot{y})$ for all $t \in [-1, 1]$?

Answer. No.

Solution. Applying logic similar to the one seen in the solution of item (2) in Problem 3, we see that the uniqueness and existence theorem must hold for any initial condition. However, again, for $\varphi = \sin 2t$ and $\psi = 2\arctan t$ we have

$$\varphi(0) = \psi(0) = 0, \quad \varphi'(0) = 2\cos 2t\big|_0 = 2, \quad \psi'(0) = \frac{2}{1+t^2}\bigg|_0 = 2, \tag{כ״ח}$$

so the existence and uniqueness theorem is violated for the initial condition $y(0) = 0$, $y'(0) = 2$.

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