Lecture 11 The method of complex magnitude

Example $\ddot{y}+y=\cos t$ once more

$\cos t = Re\, e^{it}$ $y \in \mathbb{R}$ 1

consider $\ddot{y}+y=e^{it}$ resonance $y \in \mathbb{C}$

$y_{pc} = d\,t e^{it}$, $d \in \mathbb{C}$

$\dot{y}_{pc} = d(e^{it} + i t e^{it})$

$\ddot{y}_{pc} = d(ie^{it} + ie^{it} - te^{it}) = d(2ie^{it} - te^{it})$

$d(2ie^{it} - te^{it}) + d\,te^{it} = e^{it}$

$2id\,\cancel{e^{it}} = \cancel{e^{it}}$ $d = \frac{1}{2i} = -\frac{1}{2}i$

$y_{pc} = -\frac{1}{2} i t e^{it} = -\frac{1}{2} t i(\cos t + i \sin t)$

$y_{pc} = -\frac{1}{2} t(-\sin t + i\cos t)$

$y_p = Re\, y_{pc} = \frac{1}{2} t \sin t$

Then $Re\, p(\frac{d}{dt})\, y_{pc} = Re\, e^{it}$

$p(\frac{d}{dt})\, y_p = \cos t$

The answer is the same

The matrix case, matrix quasi polynomials

$\dot{y} = Ay + B(t)$ is put in Moodle

Once more on Wronskian

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$$P(\lambda) = (\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$$

$$P\left(\frac{d}{dt}\right)y = 0 \Rightarrow y = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t} + \dots + C_n e^{\lambda_n t}$$

$$W\left[e^{\lambda_1 t}, \dots, e^{\lambda_n t}\right] = \begin{pmatrix} e^{\lambda_1 t}, & e^{\lambda_2 t}, & \dots & e^{\lambda_n t} \\ \lambda_1 e^{\lambda_1 t}, & \lambda_2 e^{\lambda_2 t}, & & \lambda_n e^{\lambda_n t} \\ \vdots & & & \\ \lambda_1^{n-1} e^{\lambda_1 t}, & \lambda_2^{n-1} e^{\lambda_2 t}, & \dots, & \lambda_n^{n-1} e^{\lambda_n t} \end{pmatrix}$$

$$\begin{array}{c} \| \\ W(t) \end{array} \qquad W(0) = \det \underbrace{\begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \vdots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{pmatrix}}_{}$$

Vandermond

Theorem

$$W[\dots]\Big|_{t=0} = 0 \Leftrightarrow \exists i,j: \ \lambda_i = \lambda_j$$

Trivial ...

Question How does $W(t)$ change in time?

Theorem Abel – Liouville

Let $\varphi_1(t), \dots, \varphi_n(t)$ be solutions of the linear equation

$$\dot{y} = A(t)y, \quad y \in \mathbb{R}^n, \quad A(t) \in \mathbb{R}^{n\times n}$$

$$W(t) = W[\varphi_1, \dots, \varphi_n](t) = \det \underbrace{(\varphi_1(t) \dots \varphi_n(t))}_{\Phi(t)}$$

Then $\dot{W}(t) = \text{Trace}\, A(t) \cdot W(t)$

$$A(t) = \begin{pmatrix} a_{11}(t) & \dots & a_{1n}(t) \\ a_{21}(t) & & a_{2n}(t) \\ \vdots & \ddots & \\ a_{n1}(t) & & a_{nn}(t) \end{pmatrix}, \quad \text{Trace } A = a_{11} + a_{22} + \dots + a_{nn}$$

Proof Let A_i, Φ_i be the rows of A, Φ, $i = 1, \dots, n$

$$\Phi = \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_n \end{pmatrix}, \quad A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}, \quad \frac{d}{dt}\det C = \frac{d}{dt}\sum (-1)^{\dots} c_{1j_1} \cdots c_{nj_n}$$

$$= \sum (-1) \left[\dot c_{1j_1} \cdots + c_{1j_1} \dot c_{2j_2} \cdots + \dots + c_{1j_1} c_{2j_2} \cdots \dot c_{nj_n} \right]$$

Then $\dot W = \frac{d}{dt} \det \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix} =$

$$= \det \begin{pmatrix} \dot\Phi_1 \\ \Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix} + \det \begin{pmatrix} \Phi_1 \\ \dot\Phi_2 \\ \vdots \\ \Phi_n \end{pmatrix} + \dots + \det \begin{pmatrix} \Phi_1 \\ \Phi_2 \\ \vdots \\ \dot\Phi_n \end{pmatrix}$$

$$\dot\Phi = A\Phi \Rightarrow \dot\Phi_i = A_i \Phi = a_{i1}\Phi_1 + a_{i2}\Phi_2 + \dots + a_{in}\Phi_n$$

$$\det \begin{pmatrix} \Phi \\ \vdots \\ \dot\Phi_i \\ \vdots \\ \Phi_n \end{pmatrix} = 0 + a_{ii} \det \begin{pmatrix} \Phi_1 \\ \vdots \\ \Phi_i \\ \vdots \\ \Phi_n \end{pmatrix} + 0 = a_{ii} \det \Phi = a_{ii} W$$

$$\Rightarrow \dot W = (a_{11} + a_{22} + \dots + a_{nn}) W$$

$\boxed{\text{QED}}$

Scalar case. **Theorem** $\dot W = -a_1(t) W$

$$y^{(n)} + a_1(t) y^{(n-1)} + \dots + y_n(t) y = 0$$

$$\dot{\vec y} = A(t) \vec y, \quad A(t) = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ -a_n & \dots & -a_2 & -a_1 \end{pmatrix}$$

$$\text{Trace } A(t) = -a_1(t)$$

$$\Rightarrow \boxed{\dot W = -a_1(t) W}$$

Example Equation order reduction

$$\ddot{y} + a_1(t)\dot{y} + a_2(t)y = 0$$

A solution $y_p = f(t)$ is given

Find the general solution

$$W = \begin{vmatrix} f & y \\ \dot{f} & \dot{y} \end{vmatrix}, \quad \dot{W} = -a_1(t)W, \quad W \neq 0$$

$$\Rightarrow \int \frac{\dot{W}}{W}dt = -\int a_1(t)\,dt, \quad \ln|W| = \dots$$

$$W(t) = C_1 e^{-\int a_1(s)ds} = C_1 \underset{\text{choose}}{W_0(t)}$$

Trick ~~Example~~ $\begin{vmatrix} f & y \\ \dot{f} & \dot{y} \end{vmatrix} = C_1 W_0(t)$

$$\frac{f\dot{y} - \dot{f}y}{f^2} = C_1 W_0(t)/f^2$$

$$\frac{d}{dt}\left(\frac{y}{f}\right) = C_1 \frac{1}{f(t)^2} W_0(t), \quad y = f(t)\int C_1 \frac{1}{f(t)^2} W_0(t)\,dt$$

Example $(t^2+1)\ddot{y} - 2t\dot{y} + 2y = 0$

$f = t$ is a solution: $-2t + 2t = 0$

$$a_1 = -\frac{2t}{t^2+1} \;!!$$

$$\dot{W} = +\frac{2t}{t^2+1}W, \quad \frac{d\ln|W|}{dt} \quad \ln|W| = \ln(t^2+1) + c$$

$$W = C_1(t^2+1)$$

$$t\dot{y} - y \cdot 1 = \begin{vmatrix} t & y \\ 1 & \dot{y} \end{vmatrix} = C_1(t^2+1)$$

$$\frac{t\dot{y} - y \cdot 1}{t^2} = C_1 \frac{t^2+1}{t^2}$$

$$\frac{d}{dt}\left(\frac{y}{t}\right) = C_1\left(1 + \frac{1}{t^2}\right) \qquad \frac{y}{t} = C_1 t - \frac{C_1}{t} + C_2$$

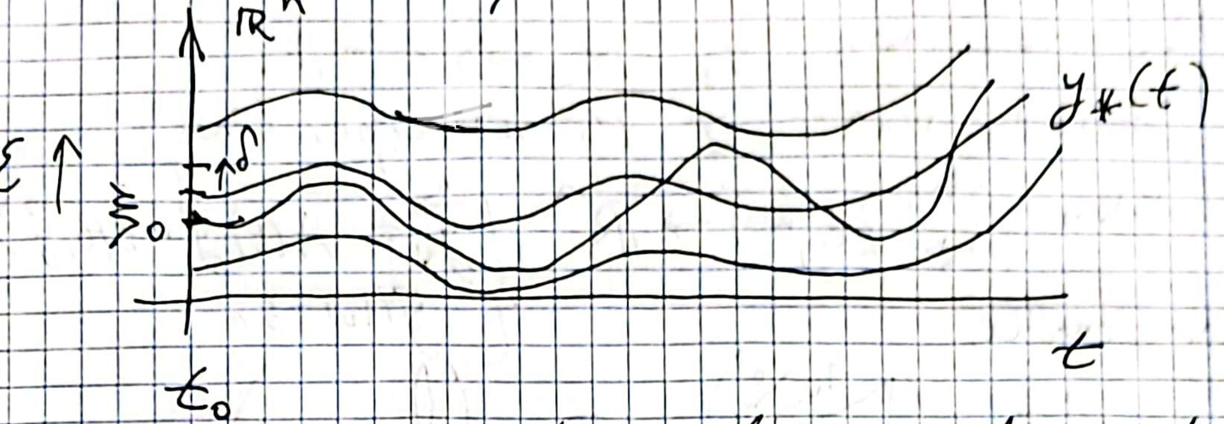
$$\boxed{y = C_1(t^2 - 1) + C_2 t}$$

Stability Theory. Introduction

Cauchy problem:

$$\begin{cases} \dot{y} = f(t, y) \\ y(t_0) = \xi_0 \end{cases} \qquad y \in \mathbb{R}^n,\ f \in C$$

solution uniqueness is not assured



The solution continuously depends on time and initial condition over any <u>closed</u> finite time interval $t_0 \le t \le t_1$.
Does it continuously depend on the initial condition over $[t_0, \infty)$?
If it does $\Rightarrow$ stability (according to Lyapunov)
otherwise: instability

$\boxed{\text{Depends on } t_0!!}$

<u>Definition</u> Let the solution $y_*: [t_0, \infty) \to \mathbb{R}^n$ satisfy the equation $\dot{y} = f(t, y)$ with the initial condition $y(t_0) = \xi_0$. It's called Lyapunov stable at the init. time t_0 if

1. $\exists\, \delta_* > 0$: solution $y(t)$ exists for any initial condition $\|y(t_0) - \xi_0\| < \delta_*$ for $\forall t \ge t_0$
2. $\forall \varepsilon > 0\ \exists\, \delta > 0 \quad \|y(t_0) - \xi_0\| < \delta \Rightarrow$
   $\forall t \geqslant t_0 \quad \|y(t) - y_*(t)\| < \varepsilon$

The case of the scalar equation

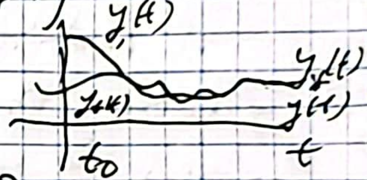
$$y^{(n)} = f(t, y, \dot y, \dots, y^{(n-1)}), \quad y \in \mathbb{R}, \; f \in C$$

$$f : \mathbb{R}^{n+1} \to \mathbb{R}$$

Init. condition: $\vec y(t_0) = \begin{pmatrix} y(t_0) \\ \dot y(t_0) \\ \vdots \\ y^{(n-1)}(t_0) \end{pmatrix} = \xi_0 \in \mathbb{R}^n$

The solution $y_* : [t_0, \infty) \to \mathbb{R}$ is called stable if the corresponding solution $\vec y_* : [t_0, \infty) \to \mathbb{R}^n$, $\begin{cases} \dot{\vec y} = \Delta \vec y + B(t, \vec y) \\ \vec y(t_0) = \xi_0 \end{cases}$ is stable

$$\Delta = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ f(t, \vec y) \end{pmatrix}$$



Asymptotic stability:

Lyapunov stability 1., 2. + 3.

3. $\exists \delta_1 > 0 : \|y(t_0) - \xi_0\| < \delta_1 \Rightarrow \lim_{t \to \infty} \|y(t) - y_*(t)\| = 0$

$\delta \leq$

Global asymptotic stability: $\delta_*, \delta_1 = \infty$

Example $\dot y = A y$, $A \in \mathbb{R}^{n \times n}$, $y \in \mathbb{R}^n$

Constant solution $y_*(t) \equiv 0$ is asymptotically stable if all eigenvalues of A lie in $\mathbb{C}_-$

$$\operatorname{Spec} A \subset \mathbb{C}_- = \{ \operatorname{Re} \lambda < 0 \mid \lambda \in \mathbb{C} \}$$

Proof: $y(t) = e^{At} y(0)$, $e^{At} \to 0$ (see Jordan form)

Def Stable equilibrium (critical point, singular point)

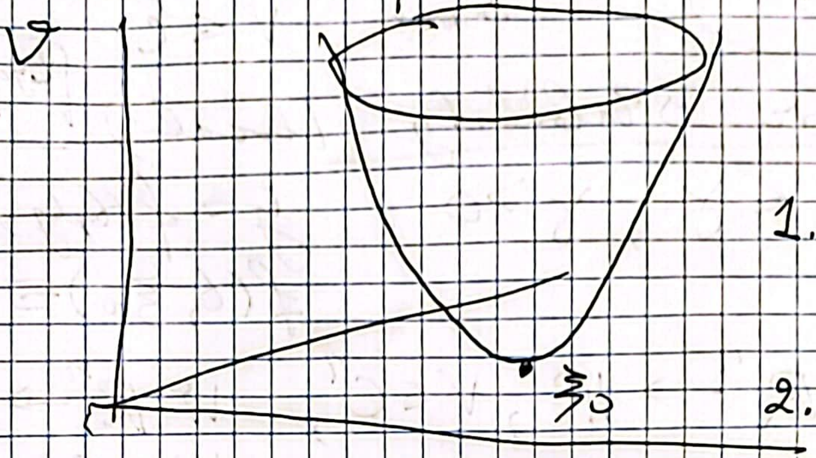
$t \geq t_0$: $\dot y = f(t, y)$, $f(t, \xi_0) \equiv 0$, $y(t) \equiv \xi_0$

Lyapunov function

Consider $\dot{y} = f(t,y)$, $f(t,\xi_0) \equiv 0$ $\forall t \geq t_0$

How to check the (asymptotic) stability of the equilibrium ξ_0?

The Lyapunov function is a new "distance"



$v : \mathbb{R}^n \to \mathbb{R}$

$v \in C^1$, $\|y - \xi_0\| < \delta$

1. v is positive definite
   $v(y) \geq 0$, $v(y) = 0 \Leftrightarrow y = \xi_0$
2. $\dot{v} \leq 0$

$\Longrightarrow$

Theorem (Lyapunov) Stability

Let $y(t)$ be a solution, $\dot{y} = f(t,y)$

$$\begin{pmatrix} \dot{y}_1 \\ \vdots \\ \dot{y}_n \end{pmatrix} = \dot{y} = f(t,y) = \begin{pmatrix} f_1(t,y) \\ \vdots \\ f_n(t,y) \end{pmatrix},$$

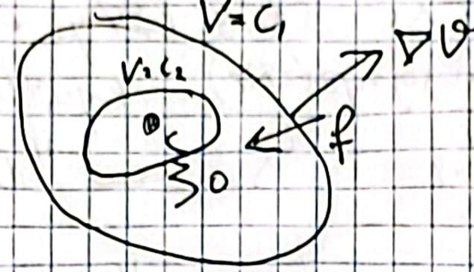
consider $v(y(t))$

gradient $\nabla v(y)$

$$\frac{d}{dt} v(y(t)) = \frac{\partial v}{\partial y}(y(t)) \cdot \dot{y}(t) = \left(\frac{\partial v}{\partial y_1}(y(t)), \dots, \frac{\partial v}{\partial y_n}(y(t))\right) \cdot \begin{pmatrix} f_1(t, y(t)) \\ \vdots \\ f_n(t, y(t)) \end{pmatrix}$$

$$= \nabla v \cdot f = L_f v = \left(\frac{\partial v}{\partial y_1} f_1 + \frac{\partial v}{\partial y_2} f_2 + \dots + \frac{\partial v}{\partial y_n} f_n\right)$$

$L_f V$ is called the Lie derivative of the function V with respect to the vector (field) f (directional derivative)



$\dot{V} = \nabla V \cdot f \le 0$

$\Rightarrow$ the velocity "looks" inside $V \le C$

Lyapunov 1892

<u>Asymptotic stability</u> (local)

$\|y - \xi_0\| \le \gamma, \ \gamma > 0.$ $\quad \dot{y} = f(t,y)$

$f(t, \xi_0) \equiv 0$

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1. $V : \mathbb{R}^n \to \mathbb{R}, \ V \in C^1, \quad W : \mathbb{R}^n \to \mathbb{R}$

$W \in C$

2. V, W are positive definite in $\|y - \xi_0\| \le \gamma$: $\quad V(y), W(y) \ge 0$

$V(y) = 0 \Leftrightarrow y = \xi_0$

$W(y) = 0 \Leftrightarrow y = \xi_0$

3. $\dot{V} = \nabla V(y) \cdot f(t,y) \le -W(y) \le 0$

$\Rightarrow$ asymptotic stability.

<u>Example</u> $\quad \dot{y} = Ay, \ y \in \mathbb{R}^n, \ \xi_0 = 0$

$\operatorname{Spec} A \subset \mathbb{C}_- \Rightarrow$ global as. stability

<u>Theorem (Lyapunov)</u> (corollary)

$V(y) = y^T H y, \quad H = \int_0^\infty e^{A^T t} e^{At} \, dt$

converges

A3. Stability in the first approximation (Linearization)

$$\dot{y} = A(y-\xi_0) + R(t,y) = f(t,y)$$

$f \in C$, $A \in \mathbb{R}^{n\times n}$

uniformly!

$R(t,y) = o(y-\xi_0)$:

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall t \geq t_0 : \|y-\xi_0\| < \delta \Rightarrow \|R(t,y)\| < \varepsilon \|y-\xi_0\|$

Linearization.

particular case $\dot{y} = f(y) = A(y-\xi_0) + o(y-\xi_0)$

Let $\mathbb{C}_- = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda < 0\}$

$\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$

Theorem $y(t) \equiv \xi_0$ is (locally)

1. asymptotically stable if $\operatorname{Spec} A \subset \mathbb{C}_-$
2. unstable if $\operatorname{Spec} A \cap \mathbb{C}_+ \neq \emptyset$

Example
$$\begin{cases} \dot{x} = \sin(x-2y-1) \\ \dot{y} = 3x-5y+xy-3 \end{cases} \qquad \xi_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\left.\frac{\partial f}{\partial(x,y)}\right|_{(1,0)} = \left.\begin{pmatrix} \nabla f_1 \\ \nabla f_2 \end{pmatrix}\right|_{(1,0)} = \left.\begin{pmatrix} \cos(x-2y-1), & -2\cos(x-2y-1) \\ 3+y, & -5+x \end{pmatrix}\right|_{(1,0)}$$

$$= \begin{pmatrix} 1, & -2 \\ 3, & -4 \end{pmatrix} = A, \quad p(\lambda) = \lambda^2 + 3\lambda + 2 = (\lambda+1)(\lambda+2)$$

Asymptotic stability

2. $\begin{cases} \dot{x} = \sin(x-2y-1) \\ \dot{y} = 3x-5y+7xy-3 \end{cases}$ $\xi_0 = (1,0)$

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$$A = \frac{\partial f}{\partial(x,y)}(1,0) = \begin{pmatrix} \cos(x-2y-1), & -2\cos(x-2y-1) \\ 3+7y, & -5+7x \end{pmatrix}\Bigg|_{(1,0)}$$

$$= \begin{pmatrix} 1, & -2 \\ 3, & 2 \end{pmatrix}, \quad p(\lambda) = \lambda^2 - 3\lambda + 8$$

$9-32=-29$ $\quad \lambda_{1,2} = \frac{3}{2} \pm \frac{1}{2}\sqrt{29}\, i \in \mathbb{C}$

unstable

3. $\boxed{\xi_0 = 0}$ $\begin{cases} \dot{x} = -4y - x^3 \\ \dot{y} = 3x - y^3 \end{cases}$ $A = \begin{pmatrix} 0 & -4 \\ 3 & 0 \end{pmatrix}$, $p(\lambda) = \lambda^2 + 12$

Theorem doesn't help!

Lyapunov function $V(x,y) = 3x^2 + 4y^2$

$$\dot{V} = 6x(-4y - x^3) + 8y(3x - y^3)$$

$$= -6x^4 - 8y^4 \le 0 \quad \text{negative definite}$$

A.S. stability (global)

4. $\begin{cases} \dot{x} = -4y + x^3 \\ \dot{y} = 3x + y^3 \end{cases}$

still $p(\lambda) = \lambda^2 + 12$

Theorem doesn't help

$$\dot{V} = 6x(-4y + x^3) + 8y(3x + y^3) = 6x^4 + 8y^4 \ge 0$$

positive definite

instability!

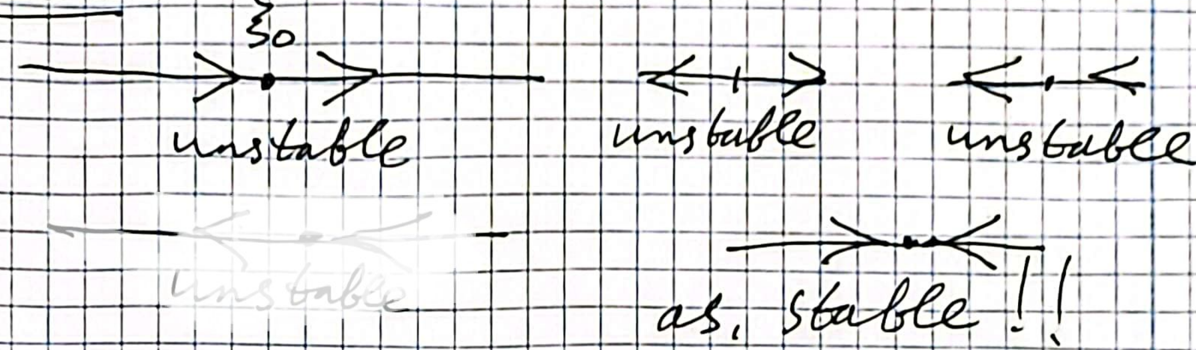
(Fin. time escape to infinity)

Scalar case

$$\dot{y} = f(y), \quad y \in \mathbb{R}, \quad f(\xi_0) = 0, \quad f \in C$$

Is the solution $y(t) \equiv \xi_0$ stable?

Theorem



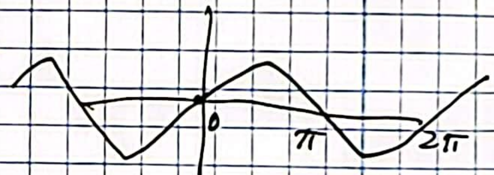
Proof. Let $\xi_0 = 0$, $V = (y - \xi_0)^2 = y^2$ pos. definite !!

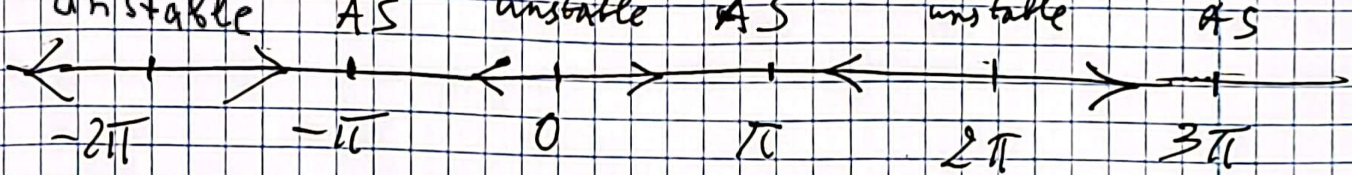
$\dot{V} = 2y \cdot f(y)$ if $\dot{V}$ negative definite $\Rightarrow$ as. st.

In all other cases there are escaping solutions!

QED.

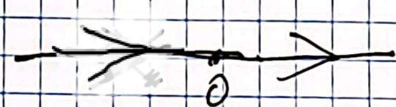
Example 1. $\dot{y} = \sin y$





AS: Asymptotically stable

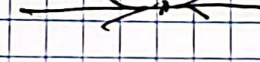
2. $\dot{y} = |y| \sin y^2, \quad \xi_0 = 0$



unstable

3. $\dot{y} = -y^4 \operatorname{tg} y, \quad \xi_0 = 0$

AS



Kurzweil 1956 Existence of a Lyapunov function

Lyapunov Alexanddr Michailovich 1857-1918 Odessa