Lecture 10

$\lambda = \alpha \pm \beta i, \ \alpha, \beta \in \mathbb{R}$

$$P_{\alpha+\beta i, k} = e^{\alpha t}\left[\cos(\beta t)(d_0 + d_1 t + \dots + d_{k-1} t^{k-1}) + \sin(\beta t)(f_0 + f_1 t + \dots + f_{k-1} t^{k-1})\right]$$

$d_0, d_1, \dots, d_{k-1}, f_0, f_1, \dots, f_{k-1} \in \mathbb{R}$ (or $\mathbb{C}$!)

These are the solutions of the DE $p(\frac{d}{dt})y = 0$ 1

$$p(\lambda) = (\lambda^2 - 2\alpha\lambda + \alpha^2 + \beta^2)^k$$

Example $\ddot{y} + y = 0 \quad \alpha = 0, \beta = 1$

$$y = \underset{1}{\underset{\parallel}{e^{0t}}}\left[\cos t \cdot c_1 + \sin t \cdot c_2\right]$$

All the results are true over $\mathbb{C}$

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0 \qquad a_1 \dots a_n \in \mathbb{C}$$

$$y: \mathbb{R} \to \mathbb{C}$$

$$L^{\mathbb{C}}_{\lambda, k} = \{e^{\lambda t}(d_0 + d_1 t + \dots + d_{k-1} t^{k-1}) \mid d_0 \dots d_{k-1} \in \mathbb{C}\}$$

solutions of $\left(\frac{d}{dt} - \lambda I_{\mathbb{C}}\right)^k y = 0, \ \lambda \in \mathbb{C}$

$\dim L^{\mathbb{C}}_{\lambda,k} = k$

Real numbers, complex eigenvalues (roots)

$a_1 \dots a_n \in \mathbb{R}$, $\alpha \pm \beta i$ conjugate pair of roots

The minimal polynomial

$$s^2 - 2\alpha s + \alpha^2 + \beta^2$$
$$= (s - \alpha)^2 - (\beta i)^2 = (s - \alpha - \beta i)(s - \alpha + \beta i)$$

The basis of $L^{\mathbb{C}}_{\alpha+\beta i, k} \oplus L^{\mathbb{C}}_{\alpha-\beta i, k}$ over $\mathbb{C}$ consists of

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$$e^{(\alpha+\beta i)t} \left\{1, \frac{t}{1!}, \dots, \frac{t^{k-1}}{(k-1)!}\right\} \cup e^{(\alpha-\beta i)t} \left\{1, \frac{t}{1!}, \dots, \frac{t^{k}}{k!}\right\}$$

$$e^{(\alpha+\beta i)t} = e^{\alpha t}(\cos(\beta t) + i\sin(\beta t))$$

$$e^{(\alpha-\beta i)t} = e^{\alpha t}(\cos(\beta t) - i\sin(\beta t))$$

$$e^{\alpha t}\cos\beta t = \frac{1}{2}\left(e^{(\alpha+\beta i)t} + e^{(\alpha-\beta i)t}\right)$$

$$e^{\alpha t}\sin\beta t = \frac{1}{2i}\left(e^{(\alpha+\beta i)t} - e^{(\alpha-\beta i)t}\right)$$

Thus $L^{\mathbb{C}}_{\alpha+\beta i, k} \oplus L^{\mathbb{C}}_{\alpha-\beta i, k} =$

$$= \left\{ e^{\alpha t}\left[\cos(\beta t)(d_0 + d_1 t + \dots + d_{k-1}t^{k-1}) + \sin(\beta t)(f_0 + f_1 t + \dots + f_{k-1}t^{k-1})\right] \;\middle|\; \begin{matrix} d_0 \dots d_{k-1} \\ f_0 \dots f_{k-1} \end{matrix} \in \mathbb{C} \right\}$$

__Definition__ Generalised (real) quasi polynomial $\alpha \pm \beta i$, $\alpha, \beta \in \mathbb{R}$

$$L^{\mathbb{R}}_{\alpha\pm\beta i, k} = \left\{ e^{\alpha t}\left[\cos\beta t\,(d_0 + \dots + d_{k-1}t^{k-1}) + \sin\beta t\,(f_0 + \dots + f_{k-1}t^{k-1})\right] \right\}$$

$$d_0 \dots d_{k-1}, f_0, \dots, f_{k-1} \in \mathbb{R}$$

They constitute the general solution of the DE

$$\left(\frac{d^2}{dt^2} - 2\alpha\frac{d}{dt} + (\alpha^2+\beta^2) I_G\right)^k y = 0 \qquad \beta \neq 0$$

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In the case $\beta = 0$ get

$$\left(\frac{d}{dt} - \alpha I_G\right)^k y = 0 \quad \left(, \sin\beta t = 0\right)$$

over $\mathbb{R}$ (and over $\mathbb{C}$ for $\alpha_0 - \alpha_{k-1}, f_0, \dots, f_{k-1} \in \mathbb{C}$)

<u>Example</u> $(\lambda^2 - 1)(\lambda^2 + \lambda + 1)^2$

$$= \lambda^6 + 2\lambda^5 + 2\lambda^4 - 2\lambda^2 - 1$$

$\lambda = \pm 1, \; -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ roots

$$y^{(6)} + 2y^{(5)} + 2y^{(4)} - 2y'' - y = 0$$

Solution:

$$y = c_1 e^{t} + c_2 e^{-t}$$
$$+ e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{3}}{2} t\right)(c_3 + c_4 t)$$
$$+ e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{3}}{2} t\right)(c_5 + c_6 t)$$

$c_1, c_2, \dots, c_6 \in \mathbb{R} \Rightarrow$ solution over $\mathbb{R}$

$\in \mathbb{C} \Rightarrow$ over $\mathbb{C}$!

Non homogeneous high-order scalar DE with constant coefficients

Example $\ddot{y} - 2\dot{y} + y = \frac{e^t}{t} + 1$ $(\in L_{0,1})$, $t > 0$

$\begin{cases} y(1) = 2 \\ \dot{y}(1) = 0 \end{cases}$, $t_0 = 1$

$\lambda^2 - 2\lambda + 1 = (\lambda - 1)^2$

$y_h = c_1 e^t + c_2 t e^t \in L_{1,2}$, $y = y_h + y_{p_1} + y_{p_2}$

$y_{p_2} = \text{const}$ check $\Rightarrow$ $y_{p_2} = 1$, $y_{p_1} = ?$

$\Phi(t) = \begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix}$, $\vec{y}_{p_1} = \Phi(t)\tilde{c}(t)$

$\dot{\vec{y}} = A\vec{y} + B$, $B = \begin{pmatrix} 0 \\ \frac{e^t}{t} \end{pmatrix}$

$\begin{cases} \dot{\tilde{c}}_1 e^t + \dot{\tilde{c}}_2 t e^t = 0 \\ \dot{\tilde{c}}_1 e^t + \dot{\tilde{c}}_2 (e^t + te^t) = \frac{e^t}{t} \end{cases}$ $\det \Phi = W$
$= e^{2t} + te^{2t} - te^{2t} = e^{2t}$

Kramer

$\dot{\tilde{c}}_1 = -\frac{te^t \frac{e^t}{t}}{e^{2t}} = -1$, $\tilde{c}_1 = -\int_1^t dt = 1 - t$

$\dot{\tilde{c}}_2 = \frac{e^t \frac{e^t}{t}}{e^{2t}} = \frac{1}{t}$, $\tilde{c}_2 = \int_1^t \frac{1}{t} dt = \ln t \Big|_1^t = \ln t$

$y_{p_2}(1) = 1$, $\dot{y}_{p_2}(1) = 0 \Rightarrow y_{p_1}(1) = \dot{y}_{p_1}(1) = 0$ | $\tilde{c}_1(1) = \tilde{c}_2(1) = 0$

$y = c_1 e^t + c_2 t e^t + \underbrace{(1-t)e^t + t\ln t\, e^t}_{y_{p_1}} + \underbrace{1}_{y_{p_2}}$

$\begin{pmatrix} \cancel{2}^{\,1} \\ 0 \end{pmatrix} = \begin{pmatrix} y(1) \\ \dot{y}(1) \end{pmatrix} = \begin{pmatrix} e^t & te^t \\ e^t & e^t + te^t \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + 0 + \begin{pmatrix} \cancel{1}^{\,0} \\ 0 \end{pmatrix} \Bigg|_{t=1}$

$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} e$

$c_1 = \frac{2}{e}$

$c_2 = -\frac{1}{e}$

Method of undetermined coefficients, Resonance

$$y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = e^{\mu t} \; b_1(t) + b_2(t)$$

$$P_n\left(\frac{d}{dt}\right) y = b_1(t) + b_2(t) \qquad a_1 \ldots a_n \in \mathbb{R} \text{ (or } \mathbb{C}\text{)}$$

$$\Rightarrow y = \underbrace{y_h}_{\text{known}} + \boxed{y_{p_1} + y_{p_2}} \longrightarrow \text{are searched separately one by one}$$

Consider the case

1. $P_n\left(\frac{d}{dt}\right) y = b(t)$, $b(t) \in L_{\mu, k}$, $\mu \in \mathbb{R}$ (or $\mathbb{C}$)

2. or $b(t) \in L_{\mu, \bar{\mu}, k}$

$\mu = \alpha \pm \beta i$

Consider the first case
Search the particular solution in the form $y_p \in L_{\mu, \ell}$ will be simplified

Theorem $\mu \in \mathbb{R}$ or $\mu \in \mathbb{C}$

Let ~~$P_n\left(\frac{d}{dt}\right)$~~ $p(\lambda) = q(\lambda)(\lambda - \mu)^m$

where $q(\mu) \neq 0$, $m \geq 0$ is the multiplicity of the root μ

Then there exists a particular solution

$$y_p = t^m e^{\mu t}\left[d_0 + d_1 t + \ldots + d_{k-1} t^{k-1}\right] \in t^m L_{\mu, k}$$

Proof: $e^{\mu t}, \frac{t}{1!} e^{\mu t}, \ldots, \frac{t^{k+m-1}}{(k+m-1)!} e^{\mu t}$ – basis of $L_{\mu, k+m}$

$e_1 \quad e_2 \quad \ldots \quad e_{k+m}$

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We have seen that

$$\frac{d}{dt} - \mu I_{\mathcal{O}} : e_{k+m} \mapsto e_{k+m-1} \mapsto \dots \mapsto e_1 \mapsto 0 \qquad \begin{pmatrix} 0 & 1 & & 0 \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

$$\frac{d}{dt} - \gamma I_{\mathcal{O}} = \frac{d}{dt} - \mu I_{\mathcal{O}} - (\gamma - \mu) I_{\mathcal{O}}$$

$\gamma \neq \mu$

$$j > 1 : e_j \longmapsto e_{j-1} + (\mu - \gamma) e_j \qquad \begin{pmatrix} \mu-\gamma & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \mu-\gamma \end{pmatrix}$$

$$j = 1 : e_1 \mapsto (\mu - \gamma) e_1 \qquad \underline{\text{nonsingular}}$$

$$q\left(\frac{d}{dt}\right)\left(\frac{d}{dt} - \mu I_{\mathcal{O}}\right)^m y = b(t) \in L_{\mu,k}, \quad y \in L_{\mu,k+m}$$

$$\left(\frac{d}{dt} - \mu I_{\mathcal{O}}\right)^m \underbrace{q\left(\frac{d}{dt}\right) y}_{\tilde{y} \in L_{\mu,k+m}} = b = \beta_1 e_1 + \dots + \beta_k e_k$$

$$\tilde{y} = \beta_1 e_{1+m} + \dots + \beta_k e_{k+m} \in L_{\mu,k+m}$$

$$\hat{y} = q\left(\frac{d}{dt}\right)^{-1} \tilde{y} \in L_{\mu,k+m}$$

$$\left(\frac{d}{dt} - \mu I_{\mathcal{O}}\right)^k q\left(\frac{d}{dt}\right) \underbrace{q\left(\frac{d}{dt}\right)^{-1} \tilde{y}}_{\hat{y}} = \left(\frac{d}{dt} - \mu I_{\mathcal{O}}\right)^k \tilde{y} = b$$

$$p\left(\frac{d}{dt}\right) \hat{y} = b \in L_{\mu,k+m}$$

$$\text{Let } \hat{y} = \underbrace{f_0 \underset{e^{\mu t}}{\overset{\parallel}{e_1}} + f_1 e_2 + \dots + f_m \underset{\frac{t^{m-1}}{(m-1)!} e^{\mu t}}{\overset{\parallel}{e_m}}}_{\Delta y} + \underbrace{d_0 e_{m+1} + \dots + d_{m+k-1} e_{m+k}}_{y_p = e^{\mu t}\left[\frac{d_0 t^m}{m!} + \dots + \frac{d_{m+k-1} t^{m+k-1}}{(m+k-1)!}\right]}$$

$$b = p\left(\frac{d}{dt}\right)\left[\Delta y + y_p\right] = p\left(\frac{d}{dt}\right) y_p \qquad \underline{QED}$$

One says that there is resonance if $m > 0$. (swing, pendulum)

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Example

$$b_1 \in L_{2,2} \quad b_2 \in L_{0,1} \quad b_3 \in L_{-1,1}$$

$$\ddot{y} + 3\dot{y} + 2y = te^{2t} + 1 + e^{-t}$$

$$\lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1)$$

$$y = y_h + y_p = y_h + y_{p1} + y_{p2} + y_{p3}$$

$$y = c_1 e^{-2t} + c_2 e^{-t} + \underbrace{y_{p1} + y_{p2}}_{\text{no resonance}} + \underbrace{y_{p3}}_{\substack{\text{resonance} \\ m=1}}$$

1. $$y_{p1} = e^{2t}(d_1 + d_2 t)$$

$$\dot{y}_{p1} = e^{2t} d_2 + 2e^{2t}(d_1 + d_2 t) = e^{2t}(2d_1 + d_2 + 2d_2 t)$$

$$\ddot{y}_{p1} = e^{2t} 2d_2 + 2e^{2t}(2d_1 + d_2 + 2d_2 t)$$
$$= e^{2t}(4d_1 + 4d_2 + 4d_2 t)$$

substitute:

$$e^{2t}(4d_1 + 4d_2 + 4d_2 t + 3(2d_1 + d_2 + 2d_2 t) + 2(d_1 + d_2 t)) = te^{2t}$$

$$\begin{array}{l|l} t^0 & 4d_1 + 4d_2 + 6d_1 + 3d_2 + 2d_1 = 0 \\ t^1 & 4d_2 + 6d_2 + 2d_2 = 1 \end{array}$$

$$\begin{cases} 12d_1 + 7d_2 = 0 \\ 12d_2 = 1 \end{cases} \quad d_2 = \frac{1}{12}, \quad d_1 = -\frac{7}{12} \cdot \underset{\overset{\shortparallel}{d_2}}{\frac{1}{12}} = -\frac{7}{144}$$

$$\boxed{y_{p1} = e^{2t}\left(-\frac{7}{144} + \frac{1}{12}t\right)}$$

2. $y_{p2} = d \Rightarrow 2d = 1 \quad d = \frac{1}{2} \quad \boxed{y_{p2} = \frac{1}{2}}$

3. $b_3 = e^{-t} \in L_{-1,1}$ resonance $m=1$

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$y_{p3} = f t e^{-t} \in t^m L_{-1,1}$

$\dot{y}_{p3} = f(e^{-t} - te^{-t}) = fe^{-t}(1-t)$

$\ddot{y}_{p3} = -fe^{-t} - fe^{-t}(1-t) = fe^{-t}(t-2)$

substitute in $\ddot{y}_{p3} + 3\dot{y}_{p3} + 2y_{p3} = e^{-t}$

$fe^{-t}[t-2+3(1-t)+2t] = 1 \cdot e^{-t}$

$f[1+0] = 1 \qquad f = 1 \qquad \boxed{y_{p3} = te^{-t}}$

$$y = \underbrace{c_1 e^{-2t} + c_2 e^{-t}}_{y_h} + \underbrace{e^{2t}\left[-\frac{7}{144} + \frac{1}{12}t\right] + \frac{1}{2} + te^{-t}}_{y_p = y_{p1} + y_{p2} + y_{p3}}$$

The case of a real DE with complex roots

$$p\left(\frac{d}{dt}\right) y = b \in L^{\mathbb{R}}_{\alpha \pm \beta i, k}$$

$$b(t) = e^{\alpha t}\left[\cos(\beta t)(\hat{d}_1 + \hat{d}_2 t + \dots + \hat{d}_k t^{k-1}) + \sin(\beta t)(\hat{f}_1 + \hat{f}_2 t + \dots + \hat{f}_k t^{k-1})\right].$$

$p(\lambda)$: roots complex and real

Theorem $\quad y_p = t^m e^{\alpha t}\left[\cos(\beta t)(d_1 + d_2 t + \dots + d_k t^{k-1}) + \sin(\beta t)(f_1 + f_2 t + \dots + f_k t^{k-1})\right]$

$\boxed{y_p \in t^m L^{\mathbb{R}}_{\alpha \pm i\beta, k}}$

It is enough that $\hat{d}_k$ or $\hat{f}_k \neq 0$ for $\beta \neq 0$ and then all $d_1, \dots, d_k, f_1, \dots, f_k$ are to appear

Proof replace $\cos(\beta t) = \frac{1}{2}(e^{i\beta t} + e^{-i\beta t})$

$\sin(\beta t) = \frac{1}{2i}(e^{i\beta t} - e^{-i\beta t})$

Apply the previous theorem over $\mathbb{C}$, find a solution

$$p\left(\frac{d}{dt}\right) y_{\mathbb{C}p} = b$$

real coefficients — real-valued function

$$y_{\mathbb{C}p} \in t^m L^{\mathbb{C}}_{\alpha+i\beta,k} + t^m L^{\mathbb{C}}_{\alpha-i\beta,k}$$

$= t^m L^{\mathbb{R}}_{\alpha\pm i\beta,k}$ over $\mathbb{C}$

$$\Rightarrow p\left(\frac{d}{dt}\right) \underbrace{\operatorname{Re} y_{\mathbb{C}p}}_{y_p} = b$$

Q.E.D.

Example $\ddot{y} + y = \cos t + 2 + t \in L^{\mathbb{R}}_{i,1}$, $2 + t \in L_{0,2}$

$\lambda^2 + 1 = 0$, $\lambda = \pm i$ resonance $m = 1$

$$y_h = C_1 \cos t + C_2 \sin t$$

1. $b_1 = t \in L_{0,2}$, $y_{p1} = d_1 + d_2 t$

$\dot{y}_{p1} = d_2$

$\ddot{y}_{p1} = 0$

$d_1 + d_2 t = t + 2$

$\Rightarrow d_1 = 2,\ d_2 = 1$

$y_{p1} = t + 2$

2. $b_2 = \cos t$ $\quad y_{p2} = t[\cos t \cdot d_1 + \sin t \cdot d_2]$

$$\dot{y}_{p2} = d_1 \cos t + d_2 \sin t + t(d_2 \cos t - d_1 \sin t)$$

$$\ddot{y}_{p2} = 2d_2 \cos t - 2d_1 \sin t + t(-d_1 \cos t - d_2 \sin t)$$

$$\ddot{y}_{p2} + y_{p2} = 2d_2 \cos t - 2d_1 \sin t = \cos t$$

$\Rightarrow d_1 = 0,\ d_2 = \frac{1}{2}$, $y_{p2} = \frac{1}{2} t \sin t$

$$y = C_1 \cos t + C_2 \sin t + t + 2 + \frac{1}{2} t \sin t$$