Lecture 9

Nonhomogeneous systems
Method of parameter variation

$$\begin{cases} \dot{y} = A(t)y + B(t), & y \in \mathbb{R}^n,\ A, B \in C \\ y(t_0) = \xi \end{cases}$$

$$y = y_h + y_p \qquad y_p = ?$$

$A = \text{const}$

$y_h = e^{At}c$

Suppose that $\varphi_1(t), \dots, \varphi_n(t)$ – fundamental solutions for $\dot{y} = A(t)y$

$$y_h = c_1\varphi_1 + \dots + c_n\varphi_n = \begin{pmatrix} \varphi_1 & \varphi_2 & \dots & \varphi_n \end{pmatrix}\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \Phi(t)c$$

fundamental matrix $\dot{\Phi} = A\Phi$

Let us search for the particular solution in the form

$$y_p = \Phi(t)\tilde{c}(t), \quad \tilde{c}(t) = ?$$

substitute $y = \Phi(t)\tilde{c}(t)$

$$\cancel{\dot{\Phi}\tilde{c}} + \Phi\dot{\tilde{c}} = \cancel{A\Phi\tilde{c}} + B$$

$$\dot{\tilde{c}} = \Phi^{-1}(t)B(t)$$

$$\tilde{c}(t) = \int_{t_0}^{t} \Phi^{-1}(s)B(s)\,ds + \cancel{D} \xrightarrow{} 0$$

take for convenience

$$\Rightarrow \tilde{c}(t_0) = 0$$

$$y = \Phi(t)c + \underbrace{\Phi(t)\tilde{c}(t)}_{y_p}, \quad y_p(t_0) = 0$$

Initial condition $y(t_0) = \Phi(t_0)c + 0$

$$c = \Phi^{-1}(t_0)\underbrace{y(t_0)}_{\xi} = \Phi^{-1}(t_0)\xi$$

Example:
$$\begin{pmatrix}\dot y_1\\ \dot y_2\end{pmatrix} = \begin{pmatrix}2 & -1\\ 3 & -2\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} + \begin{pmatrix}e^t\\ e^t\end{pmatrix}$$

$$\begin{pmatrix}y_1(0)\\ y_2(0)\end{pmatrix} = \begin{pmatrix}0\\ 2\end{pmatrix} \qquad \begin{vmatrix}2-\lambda & -1\\ 3 & -2-\lambda\end{vmatrix} = \lambda^2 - 1$$

$$y = y_h + y_p$$

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$\lambda_1 = 1 \qquad y_1 - y_2 = 0 \qquad e_1 = \begin{pmatrix}1\\1\end{pmatrix}$

$\lambda_2 = -1 \qquad 3y_1 - y_2 = 0 \qquad e_2 = \begin{pmatrix}1\\3\end{pmatrix}$

$$y_h = c_1\begin{pmatrix}1\\1\end{pmatrix}e^t + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t}$$

$$y_p = \tilde c_1(t)\begin{pmatrix}1\\1\end{pmatrix}e^t + \tilde c_2(t)\begin{pmatrix}1\\3\end{pmatrix}e^{-t} = \underbrace{\begin{pmatrix}e^t & e^{-t}\\ e^t & 3e^{-t}\end{pmatrix}}_{\Phi(t)}\tilde c(t)$$

$$\cancel{\dot\Phi\tilde c} + \Phi\tilde c' = \dot y_p = \cancel{A\Phi\tilde c} + B$$

$$\tilde c' = \Phi^{-1}B = \frac{1}{2}\begin{pmatrix}3e^t & -e^{-t}\\ -e^t & e^t\end{pmatrix}\begin{pmatrix}e^t\\ e^t\end{pmatrix}$$

$$\tilde c' = \frac{1}{2}\begin{pmatrix}3 - te^{-t}\\ -e^{2t} + te^t\end{pmatrix}$$

$$\tilde c(t) = \frac{1}{2}\int_0^t \cancel{3e^{-s}}\begin{pmatrix}3 - se^{-s}\\ -e^{2s} + se^{s}\end{pmatrix} ds$$

$$\tilde c_1 = \int_0^t\left(\frac{3}{2} - \frac{s}{2}e^{-s}\right)ds = \frac{3}{2}t + \frac{1}{2}\left[se^{-s}\Big|_0^t - \int_0^t e^{-s}\,ds\right]$$

$$= \frac{3}{2}t - \frac{1}{2} + \frac{1}{2}e^{-t} + \frac{1}{2}te^{-t}$$

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$$\tilde{c}_2 = \frac{1}{2}\int_0^t \left(se^{s} - e^{2s}\right) ds$$

$$= -\frac{1}{4}e^{2t} + \frac{1}{4} + \frac{1}{2}se^{s}\Big|_0^t - \frac{1}{2}\int_0^t e^{s}\,ds$$

$$= -\frac{1}{4}e^{2t} + \frac{1}{4} + \frac{1}{2}te^{t} - \frac{1}{2}e^{t} + \frac{1}{2}$$

$$\tilde{c}_2 = -\frac{1}{4}e^{2t} - \frac{1}{2}e^{t} + \frac{1}{2}te^{t} + \frac{3}{4}$$

$$y = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{t} + c_2\begin{pmatrix}1\\3\end{pmatrix}e^{-t} + \begin{pmatrix}e^{t} & e^{-t}\\ e^{t} & 3e^{-t}\end{pmatrix}\cdot\begin{pmatrix}-\frac{1}{2} + \frac{3}{2}t + \frac{1}{2}te^{-t} + \frac{1}{2}e^{-t}\\ \frac{3}{4} - \frac{1}{4}e^{2t} + \frac{1}{2}te^{t} - \frac{1}{2}e^{t}\end{pmatrix}$$

$$\begin{pmatrix}0\\2\end{pmatrix} = y(0) = c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}1\\3\end{pmatrix} + 0 \qquad \begin{pmatrix}1 & 1\\1 & 3\end{pmatrix}\begin{pmatrix}c_1\\c_2\end{pmatrix} = \begin{pmatrix}0\\2\end{pmatrix}$$

$$\begin{pmatrix}c_1\\c_2\end{pmatrix} = \frac{1}{2}\begin{pmatrix}3 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}0\\2\end{pmatrix} = \begin{pmatrix}-1\\1\end{pmatrix}$$

The case of a scalar DE

$$y^{(n)} + a_1(t)y^{(n-1)} + a_2(t)y^{(n-2)} + \dots + a_{n-1}(t)\dot{y} + a_n(t)y = b(t)$$

$a_1, a_2, \dots, a_n, b \in C$ continuous functions

$$\vec{y} = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix} \in \mathbb{R}^n, \quad A = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b(t) \end{pmatrix}$$

$$\dot{\vec{y}} = A(t)\vec{y} + B(t), \quad \vec{y} \in \mathbb{R}^n$$

Let $f_1(t), f_2(t), \dots, f_n(t)$ be fundamental solutions of the homogeneous DE $y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0$

Then $\begin{pmatrix} f_1 \\ \dot{f}_1 \\ \vdots \\ f_1^{(n-1)} \end{pmatrix}, \dots, \begin{pmatrix} f_n \\ \dot{f}_n \\ \vdots \\ f_n^{(n-1)} \end{pmatrix}$ are n fundamental solutions of $\dot{\vec{y}} = A\vec{y} + B$

$$\vec{y} = \vec{y}_h + \vec{y}_p, \quad \vec{y}_h = \Phi(t)C, \quad \Phi(t) = \begin{pmatrix} f_1 & \dots & f_n \\ \dot{f}_1 & \dots & \dot{f}_n \\ \vdots & & \vdots \\ f_1^{(n-1)} & \dots & f_n^{(n-1)} \end{pmatrix}$$

$\vec{y}_p = ?$ Search it in the form

$$\dot{\Phi} = A\Phi, \quad \vec{y}_p = \Phi(t)\tilde{C}(t), \quad \cancel{\dot{\Phi}\tilde{C}} + \Phi\dot{\tilde{C}} = \cancel{A\Phi\tilde{C}} + B$$

$$\begin{cases} \dot{\tilde{C}}_1 f_1 + \dot{\tilde{C}}_2 f_2 + \dots + \dot{\tilde{C}}_n f_n = 0 \\ \dot{\tilde{C}}_1 \dot{f}_1 + \dot{\tilde{C}}_2 \dot{f}_2 + \dots + \dot{\tilde{C}}_n \dot{f}_n = 0 \\ \quad \vdots \\ \dot{\tilde{C}}_1 f_1^{(n-2)} + \dot{\tilde{C}}_2 f_2^{(n-2)} + \dots + \dot{\tilde{C}}_n f_n^{(n-2)} = 0 \\ \dot{\tilde{C}}_1 f_1^{(n-1)} + \dot{\tilde{C}}_2 f_2^{(n-1)} + \dots + \dot{\tilde{C}}_n f_n^{(n-1)} = b \end{cases}$$

$$\begin{pmatrix} \dot{\tilde{C}}_1 \\ \vdots \\ \dot{\tilde{C}}_n \end{pmatrix} = \Phi^{-1} B$$

$$\parallel$$

$$\dot{\tilde{C}}$$

$\dot{\tilde{c}} = \Phi^{-1}(t)B(t)$, $\tilde{c}(t) = \int_{t_0}^{t} \Phi^{-1}(s)B(s)\,ds$

Example

$\Rightarrow \boxed{\vec{y} = \Phi(t)c + \Phi(t)\tilde{c}(t)} \to$ $\tilde{c}(t_0) = 0$

$\vec{y}_p = \Phi(t)\tilde{c}$, $\vec{y}_p(t_0) = 0$

$t > 0$, $\ddot{y} - \dot{y} - 2y = te^{t}$ $\quad y(1) = 1$, $\dot{y}(1) = 0$ $\quad t_0 = 1$

Search for solution of the homogeneous DE

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$y = e^{\lambda t}$, $\ddot{y} - \dot{y} - 2y = 0$

(Soon we'll see a better way) $\lambda^2 e^{\lambda t} - \lambda e^{\lambda t} - 2e^{\lambda t} = 0$

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

$\lambda = 2$, $\lambda = -1$ $\quad f_1 = e^{2t}$, $f_2 = e^{-t}$

$$\vec{y}_h = \begin{pmatrix} y_h \\ \dot{y}_h \end{pmatrix} = \begin{pmatrix} c_1 e^{2t} + c_2 e^{-t} \\ c_1 2e^{2t} + c_2(-e^{-t}) \end{pmatrix} = c_1 \begin{pmatrix} e^{2t} \\ 2e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-t} \\ -e^{-t} \end{pmatrix}$$

$$\dot{\vec{y}} = A\vec{y} + B, \quad A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}, \quad B(t) = \begin{pmatrix} 0 \\ te^{t} \end{pmatrix}$$

$$\vec{y}_p = \Phi(t)\tilde{c}(t), \quad \Phi(t) = \begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix}, \quad \dot{\Phi} = A\Phi$$

$$\cancel{\dot{\Phi}\tilde{c}} + \Phi\dot{\tilde{c}} = \cancel{A\Phi\tilde{c}} + B, \quad \Phi\dot{\tilde{c}} = B$$

$$\begin{cases} \dot{\tilde{c}}_1 e^{2t} + \dot{\tilde{c}}_2 e^{-t} = 0 \\ \dot{\tilde{c}}_1 2e^{2t} + \dot{\tilde{c}}_2(-e^{-t}) = te^{t} \end{cases}$$

$$\dot{\tilde{c}}_1 = \frac{\begin{vmatrix} 0 & e^{-t} \\ te^{t} & -e^{-t} \end{vmatrix}}{-e^{2t}e^{-t} - 2e^{2t}e^{-t}} = \frac{-t}{-3e^{t}} = \frac{1}{3}te^{-t}$$

$$\dot{\tilde{c}}_2 = \frac{\begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & te^{t} \end{vmatrix}}{-3e^{t}} = -\frac{1}{3}te^{2t}$$

$$\tilde{C}_1(t) = \frac{1}{3}\int_1^t te^{-t} = -\frac{1}{3}te^{-t}\Big|_1^t + \frac{1}{3}\int_1^t e^{-t}\,dt =$$

$$= -\frac{1}{3}te^{-t} + \frac{1}{3}e^{-1} \cancel{+} - \frac{1}{3}e^{-s}\Big|_1^t =$$

$$= -\frac{1}{3}te^{-t} + \frac{1}{3}e^{-1} - \frac{1}{3}e^{-t} + \frac{1}{3}e^{-1} =$$

$$= -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + \frac{2}{3}e^{-1}$$

$$\tilde{C}_2(t) = -\frac{1}{3}\int_1^t se^{2s}\,ds = -\frac{1}{6}se^{2s}\Big|_1^t + \frac{1}{6}\int_1^t e^{2s}\,ds =$$

$$= -\frac{1}{6}te^{2t} + \frac{1}{6}e^{2} + \frac{1}{12}e^{2t} - \frac{1}{12}e^{2} =$$

$$= -\frac{1}{6}te^{2t} + \frac{1}{12}e^{2t} + \frac{1}{12}e^{2}$$

General Solution

$$y = C_1 e^{2t} + C_2 e^{-t} + \tilde{C}_1(t)e^{2t} + \tilde{C}_2(t)e^{-t}$$

Cauchy problem: $\tilde{C}_1(1) = \tilde{C}_2(1) = 0$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} y(1) \\ \dot{y}(1) \end{pmatrix} = \underbrace{\begin{pmatrix} e^{2t} & e^{-t} \\ 2e^{2t} & -e^{-t} \end{pmatrix}}_{\Phi}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + 0$$

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \frac{1}{-3e^{t}}\begin{pmatrix} -e^{-t} & -e^{-t} \\ -2e^{2t} & e^{2t} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} e^{-2t} \\ 2e^{t} \end{pmatrix}$$

Next we will learn much simpler method

Remark Instead of $\int_1^t$ one could use indefinite integrals, $\int_0^t$ calculations are more complicated

Solutions of scalar LTI DEs
Quasipolynomials

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = 0 \qquad y(t) \in \mathbb{R}$$

$a_1, a_2, \dots, a_n \in \mathbb{R}$ (or $\mathbb{C}$)

$$\vec{y} = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad \dot{\vec{y}} = A\vec{y} \quad A = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & & & 1 \\ -a_n & -a_{n-1} & \dots & -a_1 \end{pmatrix}$$

Proposition $\det(A - \lambda I) = (-1)^n(\lambda^n + a_1\lambda^{n-1} + \dots + a_n)$

$n = 1 \Rightarrow$ trivial. Induction

$$\Delta_n = \det \begin{pmatrix} -\lambda & 1 & & \\ 0 & -\lambda & 1 & \\ \vdots & & \ddots & 1 \\ -a_n & -a_{n-1} & \dots & -\lambda - a_1 \end{pmatrix} = -\lambda \underbrace{\Delta_{n-1}}_{(-1)^{n-1}(\lambda^{n-1} + a_1\lambda^{n-2} + \dots + a_{n-1})} + (-1)^n(-a_n) =$$

$$= (-1)^n(\lambda^n + a_1\lambda^{n-1} + \dots + a_n)$$

QED. Very convenient!

Proposition There is only one eigenvector to any eigenvalue $\Rightarrow$ one (1) Jordan block

Indeed Let $\lambda \in \operatorname{Spec} A$, $y_0 = 1 \Rightarrow y$

$\lambda\vec{y} = A\vec{y}$

$$\boxed{\begin{aligned} \lambda y_0 &= y_1 \\ \lambda y_1 &= y_2 \\ &\dots \\ \lambda y_{n-2} &= y_{n-1} \end{aligned}}$$

$\boxed{\lambda \neq 0}$: $\lambda y_1 = y_0 = 1 \Rightarrow y_1 = 1/\lambda$, $\lambda y_2 = y_1 = \frac{1}{\lambda} \Rightarrow y_2 = \frac{1}{\lambda^2}$, ...

$\boxed{\lambda = 0}$: $y_0 = 1$, $y_1 = 0$, ..., $y_{n-1} = 0$

$$\begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \\ \dots \end{pmatrix} = A\vec{y} = \lambda\vec{y} \Rightarrow \vec{y} = \begin{pmatrix} 1 \\ \frac{1}{\lambda} \\ \vdots \\ \frac{1}{\lambda^{n-1}} \end{pmatrix}$$

QED

ODE $P\left(\frac{d}{dt}\right)y = 0$ $\quad p(\lambda)$ - characteristic polynomial

$$P\left(\frac{d}{dt}\right) = \left(\frac{d}{dt}\right)^n + a_1\left(\frac{d}{dt}\right)^{n-1} + \dots + a_{n-1}\frac{d}{dt} + a_n I_0, \quad I_0 : f \mapsto f \text{ unit operator}$$

Quasi polynomial:

$$P_{\alpha,k}(t) = e^{\alpha t}\left[d_0 + d_1 t + \dots + d_{k-1} t^{k-1}\right] \in L_{\alpha,k}$$

$\dim L_{\alpha,k} = k$ linear space

Basis: $e^{\alpha t}, \frac{t}{1!}e^{\alpha t}, \dots, \frac{t^{k-1}}{(k-1)!}e^{\alpha t}$

$$\frac{d}{dt} - \alpha I_0 : \frac{t^{k-1}}{(k-1)!}e^{\alpha t} \mapsto \frac{t^{k-2}}{(k-2)!}e^{\alpha t} \mapsto \dots \mapsto e^{\alpha t} \mapsto 0$$

Proof: $\frac{d}{dt}\left(\frac{t^s}{s!}e^{\alpha t}\right) - \alpha\frac{t^s}{s!}e^{\alpha t} =$

$$= \frac{s t^{s-1}}{s!}e^{\alpha t} + \cancel{\frac{t^s}{s!}\alpha e^{\alpha t}} - \cancel{\alpha\frac{t^s}{s!}e^{\alpha t}}$$

QED

$L_{\alpha,k}$ are solutions of $\left(\frac{d}{dt} - \alpha I_0\right)^k y = 0$ and only they are

Proof Solutions $\Rightarrow$ trivial

all of them $\Leftarrow$ dimension

Why independent? $c_1 e^{\alpha t} + \dots + c_{k} \frac{t^{k-1}}{(k-1)!}e^{\alpha t} = 0$

Apply $\left(\frac{d}{dt} - \alpha I_0\right)^{k-1} \Rightarrow c_{k} e^{\alpha t} = 0 \Rightarrow c_{k} = 0$

and so on

Theorem: Let $\alpha_1 \dots \alpha_m$ be different

$$P(\lambda) = (\lambda - \alpha_1)^{m_1} \dots (\lambda - \alpha_k)^{m_k}$$

$$Y = L_{\alpha_1,m_1} \oplus L_{\alpha_2,m_2} \oplus \dots \oplus L_{\alpha_k,m_k}$$

Direct sum $\Leftrightarrow$ only intersect at 0

$$m_1 + m_2 + \dots + m_k = n$$

Proof commutable $\left(\frac{d}{dt}-\alpha I_0\right)\left(\frac{d}{dt}-\beta I_0\right)$

$$p(\lambda) = q(\lambda)(\lambda-\alpha_1)^{m_1} \qquad y = y_1 + y_2 + \dots + y_k$$

$$p\left(\frac{d}{dt}\right)y_j = q\left(\frac{d}{dt}\right)\left(\frac{d}{dt}-\alpha_j I_0\right)^{m_j} y_j = 0$$

They only intersect at 0

Proof of commutation: $e_1, \dots, e_m = e^{\alpha t}, \frac{t}{1!}e^{\alpha t}, \dots, \frac{t^{m-1}}{(m-1)!}e^{\alpha t}$

$$\frac{d}{dt}e_s = \alpha e_s + e_{s+1} \qquad \frac{d}{dt}:$$

$$\frac{d}{dt}e_1 = \alpha \alpha_1$$

in that basis $\begin{pmatrix} \alpha & 1 & & 0 \\ & \alpha & \ddots & \\ & & \ddots & 1 \\ 0 & & & \alpha \end{pmatrix}$

$$\left(\frac{d}{dt}-\beta I_0\right)e_s = (\alpha-\beta)e_s + e_{s-1}$$

bijection: a one-to-one mapping

$\det\begin{pmatrix} \alpha-\beta & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \alpha-\beta \end{pmatrix} \neq 0$

Example 1. $(\lambda^2-1)^2 = \lambda^4 - 2\lambda^2 + 1$
$= (\lambda-1)^2(\lambda+1)^2$

$$y^{IV} - 2\ddot{y} + y = 0$$

Solution: $y = e^{t}(c_1+c_2 t) + e^{-t}(c_3+c_4 t)$

$c_1, c_2, \dots, c_4 \in \mathbb{R}$ (or $\mathbb{C}$)

The same logic is true over $\mathbb{C}$

2. $p(\lambda) = (\lambda+1)^3$

$$\dddot{y} + 3\ddot{y} + 3\dot{y} + y = 0$$

Solution $y = e^{-t}(c_1 + c_2 t + c_3 t^2)$ General Solution