Lecture 8 Homogeneous linear time-invariant (LTI) DEs

Refresher

~~Recall~~

$$\dot{y} = Ay, \quad y \in \mathbb{R}^n, \quad y(t_0) = \xi \implies y(t) = e^{A(t-t_0)}\xi$$

$$e^A = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \ldots = \lim_{N\to\infty} \sum_{i=0}^{N} \frac{1}{i!}A^i$$

$$e^{At} = I + \frac{1}{1!}At + \frac{1}{2!}A^2t^2 + \ldots$$

The theory is built exactly as in the scalar case.

$$\|A\| \stackrel{\text{def}}{=} \sup_{y \neq 0} \frac{\|Ay\|}{\|y\|} = \max_{\|y\|=1} \|Ay\|$$

Then $\|Ay\| \le \|A\|\,\|y\|$, $\|AB\| \le \|A\|\,\|B\|$

That's the norm induced by $\|\cdot\|$ for vectors

In particular, Euclidian norm $\|y\| = \left(\sum_{i=1}^{n} y_i^2\right)^{\frac{1}{2}}$

$$\|Ay\|^2 = \langle Ay, Ay\rangle = (Ay)^T Ay = y^T A^T A y$$

$$\|A\| = \cancel{\max_{\lambda \in \ldots} \operatorname{Spec}(A^TA)} \max_{\lambda \in \operatorname{Spec} A^TA} |\lambda|^{\frac{1}{2}}$$

In the complex case $\langle x, y\rangle = x^* y$, $x^* = \overline{x}^T$, $A^* = \overline{A}^T$

conjugate transpose

Hermitian transpose

$$\|A\| = \max_{\lambda \in \operatorname{Spec} A^*A} |\lambda|^{\frac{1}{2}}$$

$f, f_i : \mathbb{R}^m \to \mathbb{R}^k$ $\quad \sum_{i=1}^{\infty} f_i(u) = f(u)$, $u \in \Omega$

Definition The series converges uniformly in Ω if $\forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}: \ \forall N \ge N_0\ \forall u \in \Omega$

$$\left\| \sum_{i=1}^{N} f_i(u) - f(u) \right\| \le \varepsilon$$

Definition Absolute convergence

$\exists \sum_{i=1}^{\infty} \|f_i(u)\|$

number

Theorem Let $\sum_{i=1}^{\infty} \dot{f}_i(t) = \dot{f}(t), t \in [a,b]$

converges uniformly and $\sum_{i=1}^{\infty} f_i(t_0) < \infty$

converges at one point to

$t_0 \in [a,b]$

Then $\sum_{i=1}^{\infty} f_i(t) = f(t)$ converges uniformly as well.

Theorem $\sum_{i=1}^{\infty} c_i < \infty$, $\|f_i(u)\| \le c_i$

$u \in \Omega$

$\Rightarrow \sum_{i=1}^{\infty} f_i(u)$ converges absolutely and uniformly

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Theorem The series $e^A = I + \frac{1}{1!}A + \frac{1}{2!}A^2 + \dots$

converges absolutely and uniformly in ~~each~~ every ball $\|A\| \le R$, $R \ge 0$

Proof $\left\| \frac{A^k}{k!} \right\| \le \frac{1}{k!} R^k$, $\sum_{k=0}^{\infty} \frac{1}{k!} R^k = e^R$

Example $A = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$, $A^k = \begin{pmatrix} a^k & 0 & 0 \\ 0 & b^k & 0 \\ 0 & 0 & c^k \end{pmatrix}$

$\Rightarrow e^A = \begin{pmatrix} e^a & 0 & 0 \\ 0 & e^b & 0 \\ 0 & 0 & e^c \end{pmatrix}$, diagonal matrix

Theorem $\frac{d}{dt} e^{At} = A e^{At}$, $t \in [-T, T]$

uniformly converges

Proof $\frac{d}{dt}\frac{A^k t^k}{k!} = \frac{A^k k}{k!} t^{k-1} =$

$$= A \frac{A^{k-1} t^{k-1}}{(k-1)!}$$

$$\frac{d}{dt}\sum_{i=0}^{\infty}\frac{A^i t^i}{i!} = A\sum_{i=\phi}^{\infty}\frac{A^{i-1}t^{i-1}}{(i-1)!} = Ae^{At}$$

Conclusion $\left.\begin{matrix} y' = Ay \\ y(t_0) = \xi \end{matrix}\right| \Rightarrow y(t) = e^{A(t-t_0)}\xi$ substitution ($\exists!$ Theorem)

Theorem $B = KAK^{-1}$

$$e^B = Ke^AK^{-1}$$

Proof $e^B = I + \frac{1}{1!}(KAK^{-1}) + \ldots$

$$(KAK^{-1})^i = \underbrace{KA\cancel{K^{-1}}\cancel{K}A\cancel{K^{-1}}\ldots KAK^{-1}}_{i} = KA^iK^{-1}$$

QED

Refresher Eigenvalue, eigenvector

$Ay = \lambda y$, $y \neq 0$ is called eigenvector

λ is eigenvalue

$$(A-\lambda I)y = 0 \Rightarrow \underbrace{\det(A-\lambda I)}_{\text{characteristic polynomial}} = 0$$

Eigenvectors corresponding to different eigenvalues are linearly independent

~~Suppose~~

$\dot{y} = Ay, \; y(t_0) = \xi$

Suppose $e_1, e_2, \dots, e_n$ is a basis of eigen vectors with the eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ 4

Then $A\underbrace{[e_1, e_2, \dots, e_n]}_{\text{columns}} = AE = [\lambda_1 e_1, \dots, \lambda_n e_n]$

$= E\Lambda, \quad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_n \end{pmatrix}$

$AE = E\Lambda, \quad A = E\Lambda E^{-1}$

$\exp[A(t-t_0)] = E e^{\Lambda(t-t_0)} E^{-1} =$

$= E \begin{pmatrix} e^{\lambda_1(t-t_0)} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n(t-t_0)} \end{pmatrix} E^{-1}$

$y(t) = E \begin{pmatrix} e^{\lambda_1(t-t_0)} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n(t-t_0)} \end{pmatrix} \underbrace{E^{-1}\xi}_{C}$

$C = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \in \mathbb{R}^n$

$= \underbrace{(e_1 \; e_2 \cdots e_n)}_{\text{columns}} \underbrace{\begin{pmatrix} c_1 e^{\lambda_1(t-t_0)} \\ c_2 e^{\lambda_2(t-t_0)} \\ \vdots \\ c_n e^{\lambda_n(t-t_0)} \end{pmatrix}}_{\text{one column}} =$

$= c_1 e_1 e^{\lambda_1(t-t_0)} + c_2 e_2 e^{\lambda_2(t-t_0)} + \dots + c_n e_n e^{\lambda_n(t-t_0)}$

$\boxed{\text{General solution}}$ $\quad c_1, c_2, \dots, c_n \in \mathbb{R}$

Example

$\dot{y}_1 = 3y_1 - y_2$

$\dot{y}_2 = 4y_1 - 2y_2$, $\quad A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$

$p(\lambda) = \begin{vmatrix} 3-\lambda, & -1 \\ 4, & -2-\lambda \end{vmatrix} = \lambda^2 - \lambda - 6 + 4 = \lambda^2 - \lambda - 2$

$= (\lambda - 2)(\lambda + 1)$

$$\lambda = 2 \quad \begin{pmatrix} 3-2 & -1 \\ 4 & -2-2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

$$y_1 - y_2 = 0 \qquad y_1 = y_2, \; e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$4y_1 - 4y_2 = 0$$

$$\lambda = -1 \quad \begin{pmatrix} 3+1 & -1 \\ 4 & -2+1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} 4 & -1 \\ 4 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \qquad y_2 = 4y_1$$

$$e_2 = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix}, \quad E^{-1} = \frac{1}{3}\begin{pmatrix} 4 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} \frac{4}{3}, & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix}$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

useful formula

$$y = e^{At} y(0) = e^{EAE^{-1}t} y(0) = E e^{At} E^{-1} y(0)$$

$$y = \begin{pmatrix} 1 & 1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix} \underbrace{\begin{pmatrix} \frac{4}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}}_{C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$$

Cauchy problem solution

also arbitrary

$$y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 1 \\ 4 \end{pmatrix} e^{-t}, \quad c_1, c_2 \in \mathbb{R}$$

General solution

one of the forms ...

Another (equivalent) way:

$\dot{y} = Ay$ search for the solution in the form

$y = e^{\lambda t} w, \; w \neq 0$

$\lambda e^{\lambda t} w = A e^{\lambda t} w$

$(A - \lambda I) w = 0 \quad \det(A - \lambda I) = 0$

$\lambda = \lambda_1, \lambda_2, \dots, \lambda_n$ different

$w = e_1, e_2, \dots, e_n$ → lin. indep.

⇒ n linearly independent solutions $e^{\lambda_i t} e_i$

⇒ fundamental solutions

$y = c_1 e^{\lambda_1 t} e_1 + \dots + c_n e^{\lambda_n t} e_n, \; c_1, c_2, \dots, c_n \in \mathbb{R}$

Remark: nothing changes in the complex case (over $\mathbb{C}$)

<u>Example</u>

$$\begin{cases} \dot{y}_1 = 3y_1 - 6y_2 \\ \dot{y}_2 = 5y_1 - 8y_2 \end{cases} \qquad \begin{vmatrix} 3-\lambda & -6 \\ 5 & -8-\lambda \end{vmatrix} = \lambda^2 + 5\lambda + 6 = (\lambda+3)(\lambda+2)$$

$\lambda_1 = -3$ $\quad \begin{cases} 6y_1 - 6y_2 = 0 \\ 8y_1 - 8y_2 = 0 \end{cases} \quad y_1 = y_2, \; e_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda_2 = -2$ $\quad \begin{matrix} 5y_1 - 6y_2 = 0 \\ 5y_1 - 6y_2 = 0 \end{matrix} \quad 5y_1 = 6y_2, \; e_2 = \begin{pmatrix} 6 \\ 5 \end{pmatrix}$

$$y = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 6 \\ 5 \end{pmatrix} e^{-2t}, \; c_1, c_2 \in \mathbb{R}$$

over $\mathbb{C}$: ~~get~~ $c_1, c_2 \in \mathbb{C}$

The case of the roots multiplicity for characteristic polynomials

In the general case we get

$$\dot{y} = Ay, \quad A = E\Lambda E^{-1}, \quad y = Ee^{\Lambda t}E^{-1}y(0)$$

In general Λ is in the Jordan form

$$\Lambda = \begin{pmatrix} \Lambda_1 & & & 0 \\ & \Lambda_2 & & \\ & & \ddots & \\ 0 & & & \Lambda_k \end{pmatrix}, \qquad \Lambda_i = \begin{pmatrix} J_{i1}^{m_{i1}} & & 0 \\ & \ddots & \\ 0 & & J_{ik_i}^{m_{ik_i}} \end{pmatrix}$$

$\det(A - \lambda I)$

$$P(\lambda) = (-1)^n (\lambda - \lambda_1)^{m_1} \cdots (\lambda - \lambda_k)^{m_k} \Rightarrow P_i = (\lambda - \lambda_i)^{m_i}$$

$$m_i = m_{i1} + m_{i2} + \dots + m_{ik_i}$$

Jordan block size

$$J_m = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} + \lambda I_m = \underbrace{\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}}_{m} \Big\} m$$

$$P(\lambda) = (\lambda - s)^m = (-1)^m (s - \lambda)^m$$

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_k t} \end{pmatrix} \qquad e^{J_m t} = ?$$

Proof!

$$\begin{array}{c|c} & \begin{matrix} l & k \end{matrix} \\ \hline \begin{matrix} m \\ n \end{matrix} & \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \end{array} \cdot \begin{array}{c|c} \begin{matrix} u & v \end{matrix} & \\ \hline \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} & \begin{matrix} l \\ k \end{matrix} \end{array} =$$

Recursively applied

$$= \begin{array}{c|c} \begin{pmatrix} A_1A_2 + B_1C_2 & A_1B_2 + B_1D_2 \\ C_1A_2 + D_1C_2 & C_1B_2 + D_1D_2 \end{pmatrix} & \begin{matrix} m \\ n \end{matrix} \\ \hline \begin{matrix} u & \qquad v \end{matrix} & \end{array}$$

$$e^{\Lambda t} = \begin{pmatrix} e^{\Lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\Lambda_k t} \end{pmatrix}, \quad e^{\Lambda_j t} = \begin{pmatrix} e^{J_{im_1}} & & 0 \\ & \ddots & \\ 0 & & e^{J_{im_{k_j}}} \end{pmatrix}$$

QED

<u>Theorem</u> $AB = BA \Rightarrow e^A e^B = e^{A+B}$

The proof is algebraically the same as for the scalar case $e^\alpha e^\beta = \left(1+\frac{\alpha}{1!}+\dots\right)\left(1+\frac{\beta}{1!}+\frac{\beta^2}{2!}+\dots\right)$
$= \left(1+\frac{\alpha+\beta}{1!}+\frac{(\alpha+\beta)^2}{2!}+\dots\right)$

$$J_m = \lambda I_m + \Delta_m, \quad I_m = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad \Delta_m = \begin{pmatrix} 0 & I_{m-1} \\ 0 & 0 \end{pmatrix}$$

$$I_m \Delta_m = \Delta_m I_m, \quad e^{(\lambda I_m + \Delta_m)t} = e^{\lambda I_m t} \cdot e^{\Delta_m t}$$
$$= e^{\lambda t} \underbrace{\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}}_{I_m} \cdot e^{\Delta_m t}$$

$$e^{\Delta_m t} = ?$$

$$e^{\Delta_m t} = I_m + \frac{\Delta_m}{1!} t + \frac{\Delta_m^2}{2!} t^2 + \dots + \frac{\Delta_m^k}{k!} t^k + \dots$$

$$\Delta_m = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}, \quad \Delta_m^2 = \begin{pmatrix} 0 & 0 & 1 & & 0 \\ & \ddots & & \ddots & \\ & & \ddots & & 1 \\ & & & \ddots & 0 \\ 0 & & & & 0 \end{pmatrix}, \quad \Delta_m^k = \begin{pmatrix} \overbrace{0 \; \dots}^{k} & 0 & 1 & 0 \\ & \ddots & & \ddots \\ & & \ddots & 1 \\ & & & 0 \\ 0 & & & \vdots \\ & & & 0 \end{pmatrix}$$

$$\Delta_m^m = 0 \qquad \text{Proof: } e_1 = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, e_m = \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$\Delta_m: \; e_m \mapsto e_{m-1} \mapsto e_{m-2} \mapsto \dots \mapsto e_1 \mapsto 0$$

$$e^{\Delta_m t} = I_m + \frac{\Delta_m}{1!} t + \frac{\Delta_m^2}{2!} t^2 + \dots + \frac{\Delta_m^{m-1}}{(m-1)!} t^{m-1}$$

$$= \begin{pmatrix} 1 & t & \dots & \frac{t^{m-1}}{(m-1)!} \\ & \ddots & \ddots & \vdots \\ & & \ddots & t \\ 0 & & & 1 \end{pmatrix} \qquad e^{J_m t} = e^{\lambda t} \cdot e^{\Delta_m t} = \begin{pmatrix} e^{\lambda t} & \frac{t}{1!} e^{\lambda t} & \dots & \frac{t^{m-1}}{(m-1)!} e^{\lambda t} \\ & \ddots & \ddots & \vdots \\ & & \ddots & \frac{t}{1!} e^{\lambda t} \\ 0 & & & e^{\lambda t} \end{pmatrix}$$

Example

$$e^{\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}} \overset{(t=1)}{=} \begin{pmatrix}e^3 & e^3\\ 0 & e^3\end{pmatrix}$$

$$e^{\begin{pmatrix}3 & 1\\ 0 & 3\end{pmatrix}t} = \begin{pmatrix}e^{3t} & te^{3t}\\ 0 & e^{3t}\end{pmatrix}$$

One Jordan block J_m corresponds to the chain of eigenvectors!

$(A-\lambda I)e_1 = 0$ $\quad \Delta_m : e_1 \mapsto 0$

$(A-\lambda I)e_2 = e_1$ generalized eigenvector $\quad \Delta_m : e_2 \mapsto e_1$

$(A-\lambda I)e_3 = e_2$ $\quad \Delta_m : e_3 \mapsto e_2$

...

$(A-\lambda I)e_m = e_{m-1}$ $\quad \Delta_m : e_m \mapsto e_{m-1}$

Example

$$\begin{cases}\dot y_1 = y_1 - y_2\\ \dot y_2 = y_1 + 3y_2\end{cases} \qquad \begin{vmatrix}1-\lambda & -1\\ 1 & 3-\lambda\end{vmatrix} = \lambda^2 - 4\lambda + 4 = (\lambda-2)^2$$

$$\lambda = 2 \qquad \overset{A-2I}{\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = 0, \quad y_1 + y_2 = 0, \quad e_1 = \begin{pmatrix}1\\ -1\end{pmatrix}$$

There is no second eigenvector (not proportional to e_1)

$$\begin{pmatrix}-1 & -1\\ 1 & 1\end{pmatrix}\begin{pmatrix}y_1\\ y_2\end{pmatrix} = \begin{pmatrix}1\\ -1\end{pmatrix}$$

$$\begin{aligned}-y_1 - y_2 &= 1\\ y_1 + y_2 &= -1\end{aligned} \overset{y_2=0}{\Longrightarrow} y_1 = -1, \quad e_2 = \begin{pmatrix}-1\\ 0\end{pmatrix}$$

not unique!

any vector $e_2 + \alpha e_1$ fits

$$E = (e_1\ e_2) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix}, \quad E^{-1} = -\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A = E \Lambda E^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix}$$

$$y = e^{At} y(0) = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}}_{C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}}$$

$$y = \begin{pmatrix} e^{2t}, & te^{2t} - e^{2t} \\ -e^{2t}, & -te^{2t} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} =$$

General solution

$$= C_1 e^{2t} \begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} t-1 \\ -t \end{pmatrix}$$

<u>Example</u>

$$A = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}, \quad y' = Ay$$

$$y = e^{At} y(0)$$

$$e^{At} = \left(\begin{array}{ccc|c} e^{2t} & te^{2t} & \frac{t^2}{2} e^{2t} & 0 \\ 0 & e^{2t} & te^{2t} & 0 \\ 0 & 0 & e^{2t} & 0 \\ \hline 0 & 0 & 0 & e^{3t} \end{array}\right)$$

DE in the $\mathbb{C}$-case

$$\dot{y} = Ay, \quad y \in \mathbb{C}^n, \quad A \in \mathbb{C}^{n\times n}$$

Obviously, if $\bar{A} = A$ real matrix

$$\dot{y} = Ay \Leftrightarrow \dot{\bar{y}} = A\bar{y}$$ conjugate

Correspondingly $Ae = \lambda e \Leftrightarrow A\bar{e} = \bar{\lambda}\bar{e}$

<u>Example</u>

$$\begin{cases} \dot{y}_1 = y_1 - 5y_2 \\ \dot{y}_2 = 2y_1 - y_2 \end{cases} \qquad \det\begin{pmatrix} 1-\lambda & -5 \\ 2 & -1-\lambda \end{pmatrix} = \lambda^2 + 9$$

$\lambda = \pm 3i$

$$\lambda = -3i \qquad \begin{pmatrix} 1+3i & -5 \\ 2 & -1+3i \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0$$

multiply by $-1-3i \Rightarrow$ $\quad -2(1+3i) \quad 10$

$10 = -2\cdot(-5)$

proportional

$\cancel{e_1} =$

$$(1+3i)y_1 = 5y_2$$

$$e_1 = \begin{pmatrix} 5 \\ 1+3i \end{pmatrix}$$

$$\lambda = 3i \Rightarrow e_2 = \begin{pmatrix} 5 \\ 1-3i \end{pmatrix}$$

$$y = c_1\begin{pmatrix} 5 \\ 1+3i \end{pmatrix}\underbrace{e^{(-3i)t}}_{\cos(3t) - i\sin(3t)} + c_2\begin{pmatrix} 5 \\ 1-3i \end{pmatrix}\underbrace{e^{3it}}_{\cos(3t) + i\sin(3t)}$$

$$\cancel{y = \underbrace{\tilde{c}_1}_{c_1+c_2} \underbrace{\mathrm{Re}}_{\text{real part}}\left[\begin{pmatrix} 5 \\ 1+3i \end{pmatrix}e^{-3it}\right] + \underbrace{\tilde{c}_2}_{c_1-c_2} \underbrace{\mathrm{Im}}_{\text{imaginary part}}\left[\begin{pmatrix} 5 \\ 1+3i \end{pmatrix}e^{-3it}\right].}$$

Moreover, if $\bar{e}_1 = e_2$, $\bar{\lambda}_1 = \lambda_2$

$$\varphi_1(t) = e_1 e^{\lambda_1 t}, \quad \varphi_2(t) = \overline{\varphi_1(t)} = e_2 e^{\lambda_2 t}$$

$$y = C_1 \varphi_1(t) + C_2 \varphi_2(t) = C_1 \varphi_1(t) + C_2 \bar{\varphi}_1(t)$$

$C_1, C_2 \in \mathbb{C}$

φ_1, φ_2 – fundamental solutions

$$\mathrm{Re}\, \varphi_1 = \frac{1}{2}(\varphi_1 + \varphi_2), \quad \mathrm{Im}\, \varphi_1 = \frac{1}{2i}(\varphi_1 - \varphi_2)$$

also fundamental solutions

$$\varphi_1 = \mathrm{Re}\, \varphi_1 + i\, \mathrm{Im}\, \varphi_1, \quad \varphi_2 = \mathrm{Re}\, \varphi_1 - i\, \mathrm{Im}\, \varphi_2$$

real functions!!

$$y = \tilde{C}_1 \mathrm{Re}\, \varphi_1(t) + \tilde{C}_2 \mathrm{Im}\, \varphi_1(t)$$

$\tilde{C}_1, \tilde{C}_2 \in \mathbb{C}$, $\tilde{C}_1, \tilde{C}_2 \in \mathbb{R}$ over $\mathbb{R}$!!

<u>Example</u>

$$\begin{cases} \dot{y}_1 = 3y_1 - y_2 \\ \dot{y}_2 = 2y_1 + y_2 \end{cases} \qquad \begin{vmatrix} 3-\lambda & -1 \\ 2 & 1-\lambda \end{vmatrix} = \lambda^2 - 4\lambda + 5$$

$\lambda = 2 \pm i$

$$\lambda_1 = 2 - i \qquad \begin{pmatrix} 1+i & -1 \\ 2 & -1+i \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = 0 \qquad (1+i) y_1 = y_2$$

$$e_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix}$$

$$\lambda_2 = 2 + i \Rightarrow e_2 = \begin{pmatrix} 1 \\ 1-i \end{pmatrix}$$

$$\varphi_1 = \begin{pmatrix} 1 \\ 1+i \end{pmatrix} e^{(2-i)t} = e^{2t} \begin{pmatrix} 1 \\ 1+i \end{pmatrix} (\cos t - i \sin t)$$

$$= e^{2t} \begin{pmatrix} \cos t & - i \sin t \\ \cos t + \sin t & + i(\cos t - \sin t) \end{pmatrix}$$

$$y = C_1 e^{2t} \begin{pmatrix} \cos t \\ \cos t + \sin t \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} \sin t \\ \cos t - \sin t \end{pmatrix}$$

$C_1, C_2 \in \mathbb{R}$ over $\mathbb{R}$, $C_1, C_2 \in \mathbb{C}$ over $\mathbb{C}$!