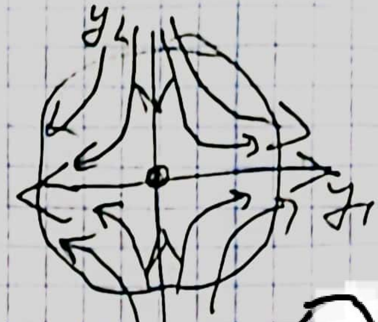
Example: $\begin{cases}\dot y_1 = y_1\\ \dot y_2 = -y_2\end{cases}$



saddle point

1

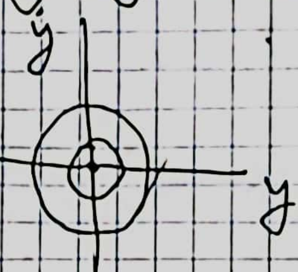
$\begin{cases}\dot y_1 = y_2\\ \dot y_2 = -y_1\end{cases}$

$\ddot y_1 + y_1 = 0$

$\ddot y + y = 0 \Rightarrow \dot y \ddot y + y \dot y = 0$

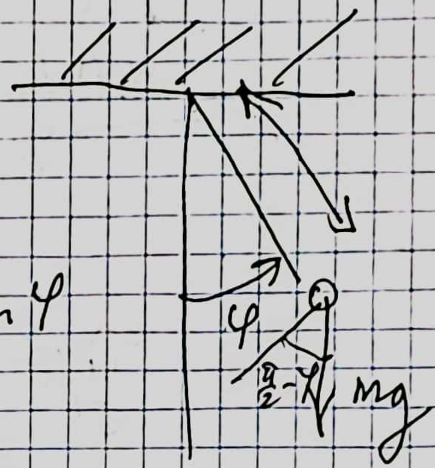
$\left(\frac{\dot y^2}{2} + \frac{y^2}{2}\right)' = 0$

$\dot y^2 + y^2 = \text{const}$



Lecture 7

Pendulum



$$ml^2\ddot\varphi = -mgl\sin\varphi$$

$$\frac{d^2\varphi}{d\left(\sqrt{\frac{g}{l}}\,t\right)^2} = \frac{l}{g}\ddot\varphi = -\sin\varphi$$

New time $t := \sqrt{\frac{g}{l}}\,t$, $\varphi \rightsquigarrow y$

$\ddot y = -\sin y$ mathematical pendulum (Arnold)

introduce a parameter

$$\begin{cases}\ddot y = -\sin y\\ y(0) = a,\ \dot y(0) = a, \quad a \approx 0\end{cases}$$

Solution $y(t) = \Phi(t,a)$, $\Phi(t,0) \equiv 0$ (equilibrium)

$\Phi \in C^\infty$

Taylor for each t:

$$\Phi(t,a) = \Phi(t,0) + \Phi'_a(t,0)a + \underbrace{\frac{1}{2}\Phi''_{aa}(t,\theta a)a^2}_{\text{Remainder},\ \theta\in(0,1)}$$

Let $t \in [-T, T]$, $|a| \le a_0$
for some $T > 0$

Then φ''_{aa} is bounded over this compact

$$\Rightarrow \varphi(t,a) = \varphi'_a(t,0)\cdot a + O(a^2)$$

Let's find $\varphi'_a(t,0) = z$ (notation)

$z = ?$

$$\frac{\partial}{\partial a} \begin{cases} \ddot y = -\sin y \\ y(0) = a, \quad \dot y(0) = a \end{cases}$$

$y = \varphi(t,a)$

$z = y'_a(t)\big|_{a=0} = \varphi'_a(t,0)$

$$\frac{\partial^3}{\partial t^2 \partial a} y = \ddot y'_a = -\cos y \cdot y'_a$$

$$y'_a(0) = 1, \quad \dot y'_a(0) = 1$$

$\Big|_{a=0} \Rightarrow \begin{cases} \ddot z = -\cos 0 \cdot z \\ z(0) = 1, \ \dot z(0) = 1 \end{cases}$

$$\begin{cases} \ddot z = -z \\ z(0) = 1, \ \dot z(0) = 1 \end{cases} \xRightarrow{\text{we will see}} \begin{cases} z = c_1 \cos t + c_2 \sin t \\ z(0) = c_1 = 1 \\ \dot z(0) = c_2 = 1 \end{cases}$$

$$\Rightarrow z = \cos t + \sin t$$

$$y = (\cos t + \sin t)\, a + O(a^2)$$

over $t \in [-T, T]$, $\forall T > 0$

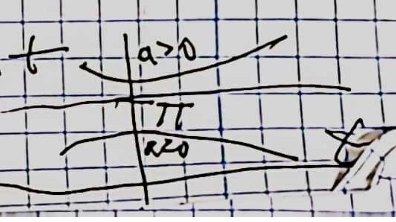
Example
$$\begin{cases} \ddot y = -\sin y \\ y(0) = \pi + a, \quad \dot y(0) = a^2 \end{cases}$$
inverted pendulum

$a \approx 0$, $y = \varphi(t,a) = \pi + z\cdot a + O(a^2)$

$z = \varphi'_a(t,0)$

$$\begin{cases} \ddot z = -\cos\pi \cdot z = z \\ z(0) = 1, \quad \dot z(0) = 0 \end{cases}$$

$z = c_1 \sinh(t) + c_2 \cosh(t) \Rightarrow z = \cosh t$



$$y = \pi + \frac{e^t + e^{-t}}{2} a + O(a^2), \quad t \in [-T, T]$$

Linearization

$\dot{y} = f(y)$, $f(\xi) = 0$ $\quad \xi$ - equilibrium

$\dot{y} = f(\xi) + f'_y(\xi)(y-\xi) + o(y-\xi)$

$$\boxed{\dot{z} = f'_y(\xi)\, z \quad \text{Linearization}}$$

The idea is that $y(t) \approx \xi + z(t)$

$y(t_0) = \xi + z(t_0)$

it is not true!

Example $\quad \dot{y} = y^2$

$y(0) = 0 \Rightarrow y = 0$ is a solution

Linearization: $\dot{z} = 0$

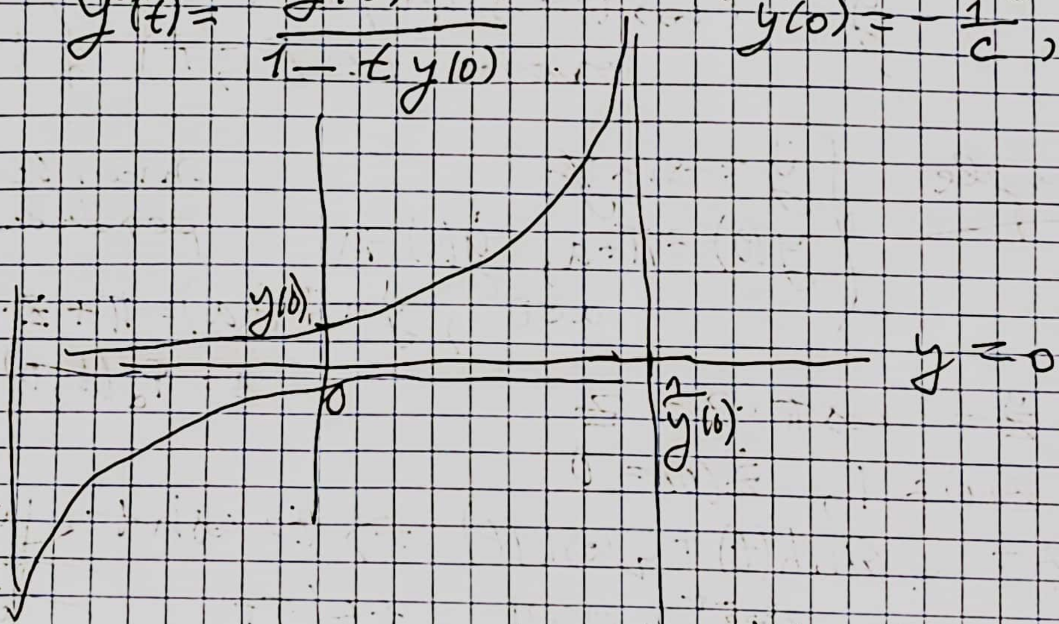
$y(t) \approx y(0) = \text{const}$?

In reality $\dot{y} = y^2$ $\quad \frac{\dot{y}}{y^2} = 1$, $y=0$ solution

$-\frac{1}{y} = c + t$, $y = -\frac{1}{c+t}$

$y(0) = -\frac{1}{c}$, $c = -\frac{1}{y(0)}$

$$y(t) = \frac{y(0)}{1 - t\,y(0)}$$



Linearization mathematically

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Remark

$$\begin{cases} \dot{y} = f(y), \; f(\xi) = 0, \; f \in C^2, \; \varepsilon \in \mathbb{R} \\ y(0) = \xi + \varepsilon v \end{cases} \quad \boxed{y, v \in \mathbb{R}^n} \quad \|v\| = 1$$

$|\varepsilon| < \varepsilon_0, \; t \in [T_1, T]$

$$y = \varphi(t, \varepsilon, v) = \varphi(t, 0, v) + \underbrace{\varphi'_\varepsilon(t, 0, v)}_{\tilde{z}} \varepsilon + O(\varepsilon^2)$$

$$\begin{cases} \dot{y}'_\varepsilon = f'(y) \, y'_\varepsilon \\ y'_\varepsilon(0) = v \end{cases}$$

substitute $\varepsilon = 0$

$$\begin{cases} \dot{\tilde{z}} = f'(\xi) \, \tilde{z} \\ \tilde{z}(0) = v \end{cases} \quad t \in [T_1, T] \qquad y = \xi + \tilde{z} \cdot \varepsilon + O(\varepsilon^2)$$

Exchange: $\tilde{\tilde{z}} = \tilde{z} \cdot \varepsilon, \quad \varepsilon = \text{const}$

$$\begin{cases} \dot{z} = f'(\xi) z \\ z(0) = y(0) - \xi \\ \quad = \varepsilon v \end{cases} \qquad \begin{cases} \dot{\tilde{\tilde{z}}} = f'(\xi) \tilde{\tilde{z}} \\ \tilde{\tilde{z}}(0) = \varepsilon v = \boxed{y(0) - \xi} \end{cases}$$

$$\boxed{\begin{gathered} y - \xi = \tilde{\tilde{z}}(t) + O(\|y(0) - \xi\|^2) \\ t \in [T_1, T] \end{gathered}}$$

Example

$$\dot{y}_1 = \sin(y_1 + y_2 - 1) + y_1 = f_1$$

$$\dot{y}_2 = \cos y_1 - \ln((y_2^2 - 1)^2 + 1) + y_2 - 2 = f_2$$

$\xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ critical point, equilibrium, singular point

$$\frac{\partial f}{\partial x}(\xi) = \begin{pmatrix} f'_{1 x_1}(\xi) & f'_{1 x_2}(\xi) \\ f'_{2 x_1}(\xi), & f'_{2 x_2}(\xi) \end{pmatrix} = \left.\begin{pmatrix} \cos(y_1 + y_2 - 1) + 1, & \cos(\ldots) \\ -\sin y_1, & -\frac{2(y_2^2 - 1) 2 y_2}{(y_2^2 - 1)^2 + 1} + 1 \end{pmatrix}\right|_\xi$$

$$= \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\boxed{\begin{aligned} \dot{z}_1 &= z_1 + z_2 \\ \dot{z}_2 &= z_2 \end{aligned} \quad \text{Linearization}}$$

Linear Systems of DEs

$$\dot{y} = A(t)y + B(t), \quad y \in \mathbb{R}^n$$

non-homogeneous $\quad B: \mathbb{R} \to \mathbb{R}^n$, $B \in C$ continuous

homogeneous $\quad A: \mathbb{R} \to \mathbb{R}^{n \times n}$, $A \in C$

$$\dot{y} = A(t)y$$

Homogeneity: $y \mapsto ky$, $k \neq 0$ preserves the system

$$\cancel{k}\dot{y} = A(t)\cancel{k}y$$

In the detailed way: $\quad a_{ij}, b_i \in C$

$$\begin{cases} \dot{y}_1 = a_{11}(t)y_1 + a_{12}(t)y_2 + \dots + a_{1n}(t)y_n + b_1(t) \\ \dot{y}_2 = a_{21}(t)y_1 + a_{22}(t)y_2 + \dots + a_{2n}(t)y_n + b_2(t) \\ \dots \\ \dot{y}_n = a_{n1}(t)y_1 + a_{n2}(t)y_2 + \dots + a_{nn}(t)y_n + b_n(t) \end{cases}$$

$\boxed{\exists!\ \text{Theorem holds, solutions extendable to } t \in \mathbb{R}}$

Definition Vector functions $f_1, f_2, \dots, f_k : \mathbb{R} \to \mathbb{R}^m$ are called linearily dependent if there exist $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$, $(\lambda_1, \dots, \lambda_k) \neq (0, \dots, 0)$, such that $\lambda_1 f_1 + \lambda_2 f_2 + \dots + \lambda_k f_k = 0 \in \mathbb{R}^m$

meaning: $\forall t \in \mathbb{R} \quad \lambda_1 f_1(t) + \lambda_2 f_2(t) + \dots + \lambda_k f_k(t) = 0$

$\lambda_1 f_1(t) + \lambda_2 f_2(t) + \dots + \lambda_k f_k(t) \equiv 0$ the same

The same definition holds over any set $\subset \mathbb{R}$ (domain of functions)

Homogeneous DEs

Proposition: Solutions of a linear (homogeneous!) DE constitute a ~~vector~~ linear space

Indeed φ_1, φ_2 - solutions

$$\begin{array}{c|l} \lambda_1 & \dot{\varphi}_1 = A(t)\varphi_1 \\ \lambda_2 & \dot{\varphi}_2 = A(t)\varphi_2 \end{array}$$

$$\frac{d}{dt}(\lambda_1\varphi_1 + \lambda_2\varphi_2) = A(t)(\lambda_1\varphi_1 + \lambda_2\varphi_2)$$

Let $\varphi_1(0) = \begin{pmatrix}1\\0\\ \vdots\\0\end{pmatrix}$, $\varphi_2(0) = \begin{pmatrix}0\\1\\ \vdots\\0\end{pmatrix}$, ..., $\varphi_n(0) = \begin{pmatrix}0\\0\\ \vdots\\1\end{pmatrix}$

These solutions exist and are defined for $\forall t \in \mathbb{R}$

Proposition $\varphi_1, \dots, \varphi_n$ constitute a basis of the linear space Y_n of all solutions

Proof

1. $\varphi_1, \dots, \varphi_n$ are linearly independent

Suppose $\lambda_1 \varphi_1(t) + \dots + \lambda_n \varphi_n(t) = 0$ take $t = 0$

get $\begin{pmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_n\end{pmatrix} = 0$

2. $\forall \varphi \in Y_n$ define $\lambda = \varphi(0) = \begin{pmatrix}\lambda_1\\ \lambda_2\\ \vdots\\ \lambda_n\end{pmatrix}$

$\varphi(0) = \lambda_1 \varphi_1(0) + \lambda_2 \varphi_2(0) + \dots + \lambda_n \varphi_n(0)$

$\varphi(t) \equiv \underbrace{\lambda_1 \varphi_1(t) + \lambda_2 \varphi_2(t) + \dots + \lambda_n \varphi_n(t)}_{\text{solution}}$

due to Theorem $\exists!$

QED

Proposition $\varphi_1, \varphi_2, \dots, \varphi_n \in Y_n$, $t_0 \in \mathbb{R}$ (any time instant)

$\varphi_1, \varphi_2, \dots, \varphi_n$ linearly independent $\iff$ $\varphi_1(t_0), \dots, \varphi_n(t_0) \in \mathbb{R}^n$ are linearly independent

Proof "$\Rightarrow$" trivial

"$\Leftarrow$" suppose $\varphi_1(t_0), \dots, \varphi_n(t_0)$ are independent

but $\lambda_1\varphi_1 + \lambda_2\varphi_2 + \dots + \lambda_n\varphi_n = 0$, $\lambda = (\lambda_1 \dots \lambda_n) \neq 0$

take $t = t_0$, let $\varphi(t) = \lambda_1\varphi_1(t) + \dots + \lambda_n\varphi_n(t)$ solution

get $\varphi(t_0) = 0$

but $\varphi_*(t) \equiv 0$ is always a solution

$\Rightarrow \varphi = \varphi_* = 0, \lambda = 0$ QED

Conclusion $\dim Y_h = n$

Definition: Solutions $\varphi_1, \dots, \varphi_n$ which constitute a basis of Y_h are called **fundamental solutions**

The matrix $\Phi(t) = (\varphi_1(t), \dots, \varphi_n(t)) = \begin{pmatrix} \varphi_{11}(t) & \cdots & \varphi_{n1}(t) \\ \varphi_{12}(t) & \cdots & \varphi_{2n}(t) \\ \vdots & & \vdots \\ \varphi_{1n}(t) & \cdots & \varphi_{nn}(t) \end{pmatrix}$ (columns)

is called a fundamental matrix

Obviously $\dot{\Phi} = A(t)\Phi$

$$\dot{\Phi} = (\dot{\varphi}_1 \dots \dot{\varphi}_n) = (A\varphi_1, \dots, A\varphi_n) = A(\varphi_1 \dots \varphi_n) = A\Phi$$

Example $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}^{\bullet} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix}$

Solutions: $\begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} = \varphi_1$, $\varphi_2 = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

$(\varphi_1(0), \varphi_2(0)) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$ unit matrix

$\Phi(t) = \begin{pmatrix} \cos t, & \sin t \\ -\sin t, & \cos t \end{pmatrix}$ is the fundamental matrix

General Solution $Y_h = C_1\varphi_1 + C_2\varphi_2$, $C_1, C_2 \in \mathbb{R}$

Wronskian

Definition: $W[\varphi_1 \dots \varphi_n](t) = \det(\varphi_1(t) \dots \varphi_n(t)) = \det \Phi(t)$ (columns)

Proposition: $\forall t_0 \in \mathbb{R}$, $\varphi_1 \dots \varphi_n$ l. independent $\iff W[\varphi_1 \dots \varphi_n](t_0) \neq 0$

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Conclusion Let $\varphi_1, \varphi_2, \dots, \varphi_n \in Y_h$

Then there are only two options:

$\boxed{\begin{array}{ll} \text{(fundamental solutions)} & 1) \ \forall t \quad W[\varphi_1, \varphi_2, \dots, \varphi_n](t) \neq 0 \\ \text{(lin. dep. solutions)} & 2) \ \forall t \quad W[\varphi_1, \varphi_2, \dots, \varphi_n](t) = 0 \end{array}}$

It only is true for solutions!

Example: $y \in \mathbb{R}^2$, $\dot{y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} y$, $\varphi_1 = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$

$\varphi_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$

$$W[\varphi_1, \varphi_2](t) = \det \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} = 1$$

The case of a scalar linear DE

$$y^{(n)} + a_1(t) y^{(n-1)} + \dots + a_{n-1}(t) \dot{y} + a_n(t) y = b(t) \quad (*)$$

The order is n, $a_1, \dots, a_n, b \in C$

$\boxed{b = 0 \Rightarrow \text{homogeneous}}$

Init. conditions: $y(t_0), \dot{y}(t_0), \dots, y^{(n-1)}(t_0)$

Each solution $\underbrace{\text{is extendable}}_{\text{uniquely}}$ to $t \in \mathbb{R}$

$(*)$ can be equivalently rewritten as

$$\vec{y} = \begin{pmatrix} y \\ \dot{y} \\ \vdots \\ y^{(n-1)} \end{pmatrix}, \quad \dot{\vec{y}} = A(t) \vec{y} + B(t) \quad (**)$$

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \dots & -a_1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ b \end{pmatrix}$$

Once more: Solutions of a homogeneous DE form a linear space of the dimension n

Solutions ~~which~~ $f_1, f_2, \dots, f_n : R \to R$ are fundamental for ~~the~~ $(*)$ 9

iff $\varphi_1 = \begin{pmatrix} f_1 \\ \dot f_1 \\ \vdots \\ f_1^{(n-1)} \end{pmatrix}, \dots, \varphi_n = \begin{pmatrix} f_n \\ \dot f_n \\ \vdots \\ f_n^{(n-1)} \end{pmatrix}$

are fundamental for $(**)$

Definition $W[f_1, f_2, \dots, f_n](t) \overset{def}{=} \cancel{\det[\varphi_1 \dots \varphi_n]}$ $W[\varphi_1 \dots \varphi_n](t)$

$$W[f_1 \dots f_n] = \det \begin{pmatrix} f_1, & f_2, & \dots, & f_n \\ \dot f_1, & \dot f_2, & \dots, & \dot f_n \\ \vdots & & & \\ f_1^{(n-1)}, & f_2^{(n-1)}, & \dots, & f_n^{(n-1)} \end{pmatrix}$$] derivative number $n-1$!

Once more!: $f_1, \dots, f_n$ are fundamental solutions

$\Longleftrightarrow \forall t: W[f_1 \dots f_n](t) \neq 0$

$f_1, \dots, f_n$ are linearly dependent

$\Longleftrightarrow \forall t \; W[f_1 \dots f_n](t) = 0$

The same 2 options!!

Example $\ddot y + y = 0$, $\dot{\vec y} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec y$

General solutions: $y = C_1 \cos t + C_2 \sin t$, $W[\cos t, \sin t] = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1$

Wronskian is often used for arbitrary functions (not solutions of a DE)

$\underbrace{}_{\text{homogeneous}}$

Proposition Functions $f_1(t), \dots, f_n(t)$ are <u>independent</u> over an interval $t \in [a, b]$

if $\cancel{\forall}\ \exists\, t_0 \in [a, b]: W[f_1 .. f_n](t_0) \neq 0$

"for some" **Proof**: let $\lambda_1 f_1 + .. + \lambda_n f_n = 0 \Rightarrow W(t_0) = 0$ contradiction

Example: $f_1 = t^2$, $f_2 = t^3$

$$W[t^2, t^3] = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix}, \quad W[t^2, t^3]\Big|_{t=1} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1$$

$\Rightarrow$ t^2, t^3 are <u>independent</u> over $[-2, 2]$

but $W[t^2, t^3]\Big|_{t=0} = \begin{vmatrix} 0 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$\Rightarrow t^2, t^3$ cannot be a pair of solutions of a DE

$$\ddot{y} + a_1(t)\dot{y} + a_2(t) y = 0, \quad t \in [-2, 2]$$
$$a_1, a_2 \in C$$

Moreover t^2, t^3 cannot be solutions of a DE $\ddot{y} = f(t, y, \dot{y}) \in C^1$ over $[-0.1, 0.1]$ since they have the same init. condition at $t = 0$

$y(0) = 0$, $\dot{y}(0) = 0$!!! ($\exists!$ Theorem)

Example

$$f_1(t) = \begin{cases} t^2, & t \geq 0 \\ 0, & t \leq 0 \end{cases}$$

$f_1, f_2 \in C^1$

$$f_2(t) = \begin{cases} 0, & t \geq 0 \\ t^2, & t \leq 0 \end{cases}$$

$$W[f_1, f_2](t) = \begin{cases} \begin{vmatrix} t^2 & 0 \\ 2t & 0 \end{vmatrix} = 0, & t \geq 0 \\ \begin{vmatrix} 0 & t^2 \\ 0 & 2t \end{vmatrix} = 0, & t \leq 0 \end{cases}$$

$W \equiv 0$!

Nevertheless f_1, f_2 linearly independent

Suppose $\lambda_1 f_1(t) + \lambda_2 f_2(t) \equiv 0 \quad (\lambda_1, \lambda_2) \neq (0,0)$

$\lambda_1 \neq 0, \ t > 0 \Rightarrow \lambda_1 t^2 \equiv 0$

$\lambda_2 \neq 0, \ t < 0 \Rightarrow \lambda_2 t^2 \equiv 0$

contradiction!

It also means that f_1 and f_2 cannot simultaneously satisfy an equation of the form $y'' + a_1(t) y' + a_2(t) y = 0 \quad a_1, a_2 \in C$

$t \in [-1, 1]$

We use as synonimical systems of DEs and a (vector) DE.

Non-homogeneous linear DEs

$$\underbrace{\dot y = A(t)y}_{\text{homogeneous part}} + B(t) \qquad y\in\mathbb{R}^n,\ A, B \in C$$

Proposition Let $y_h \in Y_h$ then $Y_h + y_h = Y_h$

Proof Indeed $g\in Y_h \Rightarrow g - y_h \in Y_h \Rightarrow g \in Y_h + y_h$

$\cancel{\Leftarrow g + y_h \in Y_h \Rightarrow g\in Y_h - y_h}$ not needed

$g = g - y_h + y_h \in Y_h \Leftarrow g - y_h \in Y_h \Leftrightarrow g\in Y_h + y_h$

Proposition
Let $y_p \in Y$ the set of all solutions of $\dot y = A(t)y + B(t)$

Then $Y = Y_h + y_p$

Proof 1) $y\in Y \Rightarrow y - y_p \in Y_h$

$$\begin{aligned}\dot y &= Ay + B\\ \dot y_p &= Ay_p + B\\ \hline (y-y_p)' &= A(y-y_p)\end{aligned}$$

2) $y_h\in Y_h \Rightarrow y_h + y_p \in Y$

$$\begin{aligned}\dot y_h &= Ay_h\\ + \quad \dot y_p &= Ay_p + B\\ \hline (y_h+y_p)' &= A(y_h+y_p)+B\end{aligned}$$

QED

In particular: $y_{p1}, y_{p2}\in Y \Rightarrow y_{p1}-y_{p2}\in Y_h$

Example $\dot y = \begin{pmatrix}0&1\\-1&0\end{pmatrix}y + \begin{pmatrix}1-e^t\\ e^t+t+1\end{pmatrix}$ $\quad y_p = \begin{pmatrix}t+1\\e^t\end{pmatrix}$

$$Y = c_1\begin{pmatrix}\cos t\\-\sin t\end{pmatrix} + c_2 \begin{pmatrix}\sin t\\ \cos t\end{pmatrix} + \begin{pmatrix}t+1\\e^t\end{pmatrix}$$

but also $Y =$ ———— 〃 ———— $+\begin{pmatrix}t+1\\e^t\end{pmatrix} + \begin{pmatrix}-\cos t\\ \sin t\end{pmatrix}$

$\ddot{y} + y = e^t$ search $y_p = a e^t$ obviously $y_p = \frac{1}{2} e^t$

$y = C_1 \cos t + C_2 \sin t + \frac{1}{2} e^t$

$$\begin{cases} \dot{y}_1 = y_2 + 1 \\ \dot{y}_2 = -y_1 - 2 \end{cases}$$ search for $y_p = \begin{pmatrix} a \\ b \end{pmatrix}$

$$\left.\begin{matrix} 0 = b + 1 \\ 0 = -a - 2 \end{matrix}\right] \Rightarrow y_p = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$$

$$y = C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} + \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \quad C_1, C_2 \in \mathbb{R}$$

General Solution

Solution of homogeneous DE with constant coefficients

$\dot{y} = Ay$, $A \in \mathbb{R}^{n \times n}$, $y \in \mathbb{R}^n$
constant matrix

Particular case $y \in \mathbb{R}$, $A \in \mathbb{R}$, $y(t_0) = \xi$

$$\Rightarrow y = e^{A(t - t_0)} y(t_0)$$

The same formula holds for $y \in \mathbb{R}^n$ but it contains exponent of matrix.

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots \quad x \in \mathbb{R}$

$e^A = I + \frac{A}{1!} + \frac{A^2}{2!} + \dots \quad A \in \mathbb{R}^{n \times n}, \ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$e^{At} = I + \frac{1}{1!} At + \frac{1}{2!} A^2 t^2 + \dots, \quad \frac{d}{dt} e^{At} = A e^{At}$

The proofs are very similar to the scalar case

Example $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ calculate e^{Jt}

$$\begin{pmatrix} \dot y_1 \\ \dot y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ -y_1 \end{pmatrix} \qquad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad e^{At}$$

Then $y = e^{At} y(0)$ is the solution for $\dot y = Ay$, $y(0) = \xi$

$$J^0 = I, \quad J^1 = J, \quad J^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$$

$$J^3 = -J, \quad J^4 = I$$

$$\begin{tikzcd} I \arrow[r, "\times J"] & J \arrow[d, "\times J"] \\ -J \arrow[u, "\times J"] & -I \arrow[l, "\times J"] \end{tikzcd}$$

$$e^{Jt} = I + \frac{1}{1!} Jt - \frac{1}{2!} t^2 I - \frac{1}{3!} J t^3 + \frac{1}{4!} t^4 I + \frac{1}{5!} J t^5 \ldots$$

$$= I\left(1 - \frac{1}{2!} t^2 + \frac{1}{4!} t^4 - \ldots\right) + J\left(\frac{1}{1!} t - \frac{1}{3!} t^3 + \frac{1}{5!} t^5 + \ldots\right)$$

$$= I \cos t + J \sin t = \begin{pmatrix} \cos t, & \sin t \\ -\sin t, & \cos t \end{pmatrix}$$

$$y(0) = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \Rightarrow y(t) = \begin{pmatrix} \cos t, & \sin t \\ -\sin t, & \cos t \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} =$$

$$= C_1 \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + C_2 \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$$