Lecture 6 $\exists!$ Theorem

Cauchy problem $\begin{cases} \dot{y} = f(t,y) \\ y(t_0) = \xi \end{cases}$, $f: \mathbb{R}^{n+1} \to \mathbb{R}^n$ locally defined

$\xi \in \mathbb{R}$ initial condition

Theorem 1. $f \in C$ continuous $\Rightarrow \exists$ local solution

2. Lipschitz condition, $L > 0$

$\forall t, y_1, y_2$ locally around (t_0, ξ) $\quad \|f(t,y_1) - f(t,y_2)\| \leq L \|y_1 - y_2\|$

$\Rightarrow$ unique local solution

In fact we get $\exists!$ $\Rightarrow$ exists a solution, as a function of time and init. conditions

$y = \Phi(t, t_0, \xi)$ it is called the phase flow

Theorem (Arnold) $\underset{(\text{in } t, \xi)}{f \in C^k} \Rightarrow \underset{(\text{in } t, t_0, \xi)}{\Phi \in C^k}$

The proof is not simple

Remark Since all norms in finite-dimensional vector spaces are equivalent

$$\left(\forall \|\cdot\|_1, \|\cdot\|_2 : \exists \alpha, \beta_{>0} : \forall z : \begin{array}{l} \|z\|_1 \leq \alpha \|z\|_2 \\ \|z\|_2 \leq \beta \|z\|_1 \end{array} \right)$$

the Lipschitz condition does not depend on the choice of norms, but the Lipschitz constant L surely changes

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Example: $\begin{cases} \dot{y}_1 = \arctan(e^{t}+e^{-t})\,y_1 - (\sin t)\,y_2 \\ \dot{y}_2 = \dfrac{t^2-t}{t^2+|t|}\,y_1 - (t^2+1)\,y_2 \end{cases}$

can $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{*} = \begin{pmatrix} \sin t \\ \cos^2 t - 1 \end{pmatrix}$, $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}_{**} = \begin{pmatrix} e^{t}-1 \\ t^2+t \end{pmatrix}$

be simultaneously solutions of the system for $t \in [-1, 1]$?

Answer Impossible. the system satisfies the $\exists!$ conditions, but both solutions $*$ and $**$ vanish at $t=0$

Example Let $\varphi(s_1, s_2, s_3) \in C^1$ $\Rightarrow$ lipschitzian
$\varphi : \mathbb{R}^3 \to \mathbb{R}$

Consider $\ddot{y} = \varphi(t, y, \dot{y})$

can this equation have two solutions

$y_1(t) = \sin t$, $y_2(t) = \sin t - \cos t + 1$

for $t \in [-0.1, 0.1]$?

No Indeed $y_1(0) = 0$, $\dot{y}_1(0) = 1$

$y_2(0) = 0$, $\dot{y}_2(0) = 1$

contradicts the $\exists!$ Theorem

Dependence on parameters

$$\begin{cases} \dot y = f(t,y,p) \\ y(t_0) = \xi \end{cases} \qquad f \in C^k,\ p \in \mathbb{R}^\ell \text{ parameter}$$

$$\Rightarrow y = \varphi(t, t_0, \xi, p)$$

Rewrite
$$\begin{cases} \dot y = f(t,y,p) \\ \dot p = 0 \\ y(t_0) = \xi,\ p(t_0) = p_0 \end{cases} \Rightarrow \varphi \in C^k \text{ in } t, t_0, \xi, p_0 = p$$

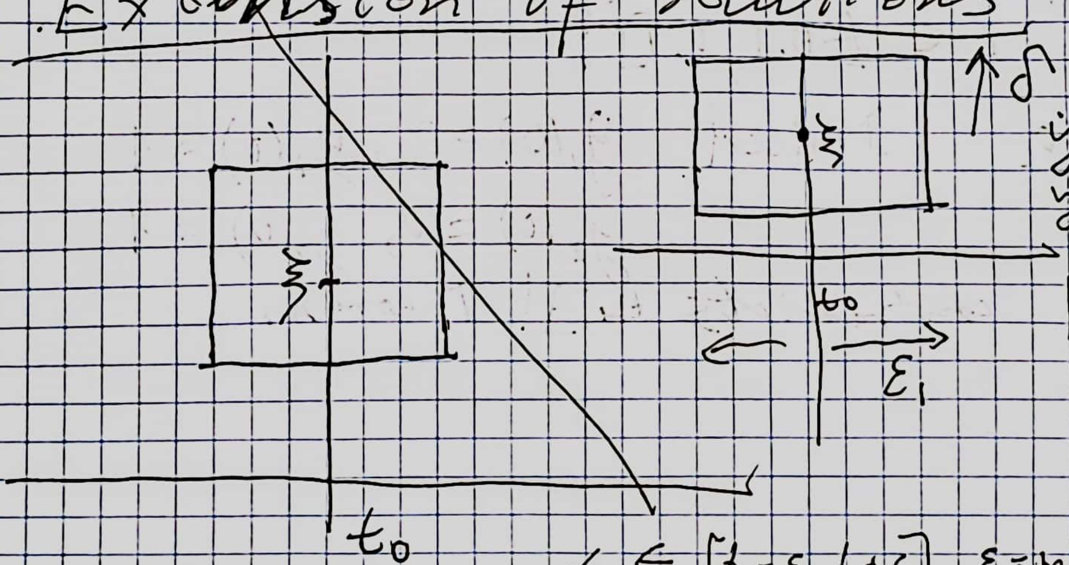
Continuous dependence on the Right-Hand Side

$$\begin{cases} \dot y = f(t,y) + w(t,y) \\ y(t_0) = \xi + w_0 \end{cases} \qquad f, w \in C,\ \|f\| \le M,\ \|w\| \le \varepsilon_1,\ \|w_0\| \le \varepsilon_1$$

$\Rightarrow$ the local solutions uniformly in t converge to the solutions for $\varepsilon_1 = 0$ as $\varepsilon_1 \to 0$

without proof.

Extension of solutions

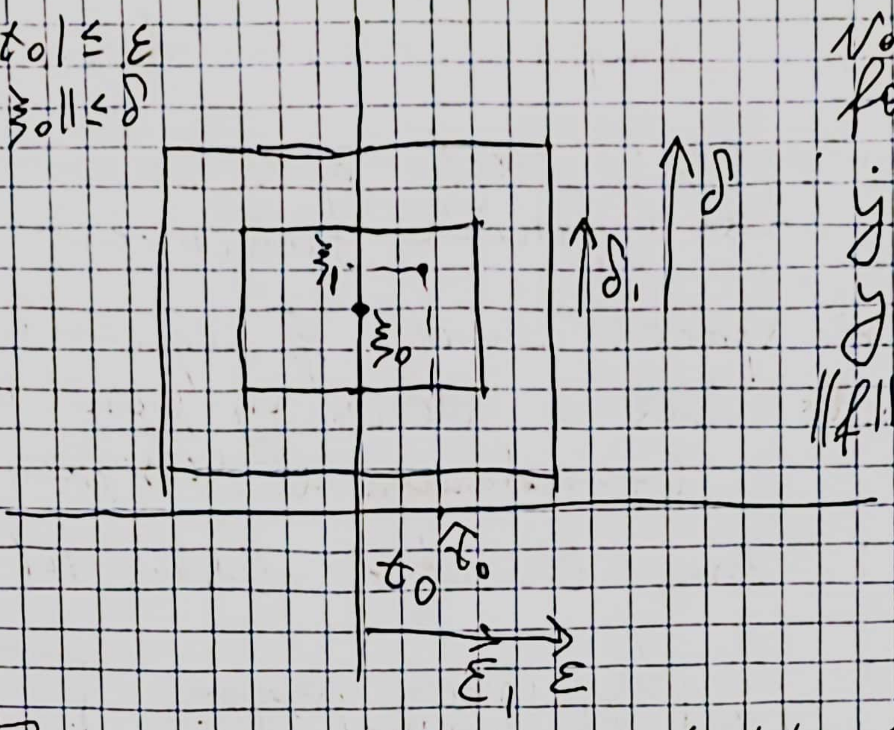


$\dot y = f(t,y)$
$y(t_0) = \xi$
$\|f\| \le M$
L

$t \in [t_0 - \varepsilon, t_0 + \varepsilon]$, $\varepsilon = \min\left[\varepsilon_1, \frac{1}{2L}, \frac{\delta}{M}\right]$

$|t - t_0| \le \varepsilon$

$\|y - \xi_0\| \le \delta$



Nonstandard formulation

$\dot{y} = f(t, y)$

$y(\hat{t}_0) = \xi$

$\|f\| \le M, \quad L$

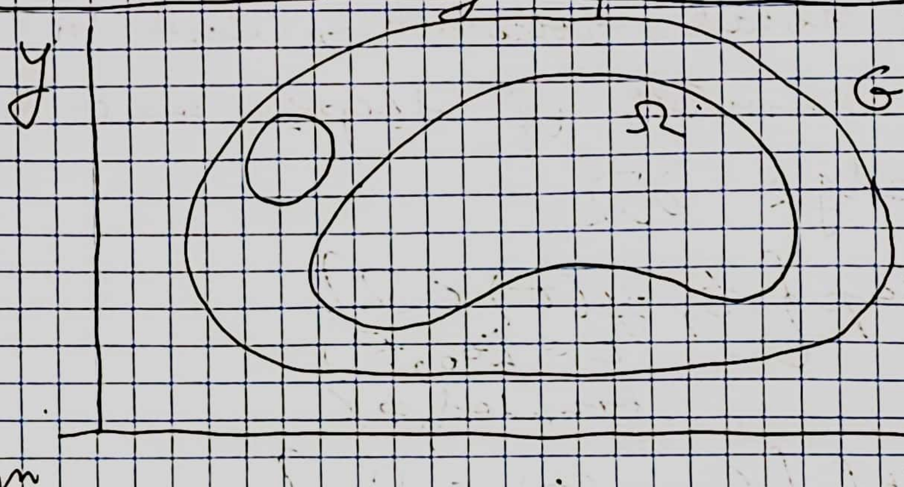
$\|f(t, y_1) - f(t, y_2)\| \le L \|y_1 - y_2\|$

Theorem For any initial condition $y(\hat{t}_0) = \xi_*$, $\|\xi - \xi_0\| \le \delta_1$ there exists a unique solution defined over $[\hat{t}_0 - \varepsilon_*, \hat{t}_0 + \varepsilon_*]$

$$\varepsilon_* = \min\left[\varepsilon - \varepsilon_1, \frac{\delta - \delta_1}{M}\right]$$

One can take $\delta_1 = \frac{\delta}{2}$

Solution extension up to the boundary of a compact



$\Omega \subset G \subset \mathbb{R}^{n+1}$ (Ω compact, G open)

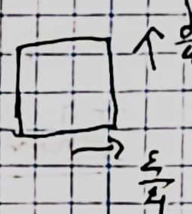
$\dot{y} = f(t, y)$

Theorem Let $\dot{y} = f(t, y)$ satisfy the local $\exists!$ theorem at each point of the compact Ω. Then each local solution can be in a unique way extended till the boundary $\partial\Omega$ in the both time directions

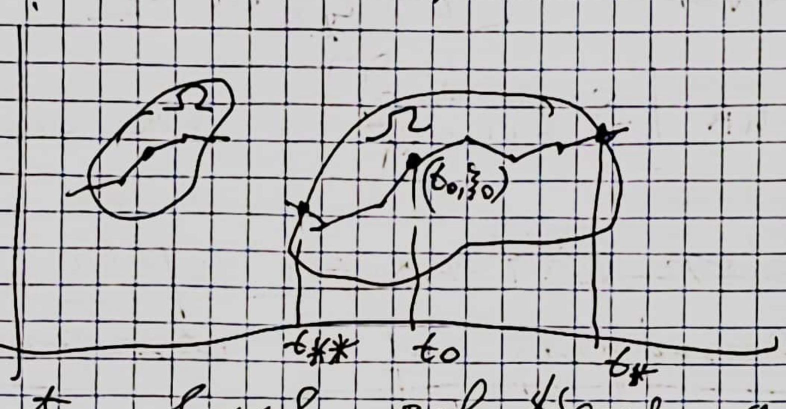
<u>Recall</u>: Compactness definitions

1. Topologic: each open cover has a finite subcover
2. Full Metric space: each ~~of~~ sequence of points has a density point in the set (fundamental subsequence)
3. $\mathbb{R}^n$: Any bounded closed set

Choose a finite open cover and the minimal δ, ε



Then at each point there is a unique solution defined over some ε_p in time forward and back.



Connect local solutions and in a finite number of steps come to the edge $\partial\Omega$

$$t_* = \sup\left\{t,\ \underset{\tau \in [t_0, t]}{y(\tau) \in \Omega}\right\}$$
bounded, $\neq \emptyset$

$$t_{**} = \inf\left\{t,\ \underset{\tau \in [t, t_0]}{y(\tau) \in \Omega}\right\}$$

Obviously $y(t_*) \in \partial\Omega$, $y(t_{**}) \in \partial\Omega$

Uniqueness the same as previously. QED

~~Auton~~
Example Prove that all solutions of the DE

$$\dot{y} = a(t,y), \quad y \in \mathbb{R}^n, \quad a \in C, \quad \|a\| \le M, \quad \text{locally Lipshitz in } y$$

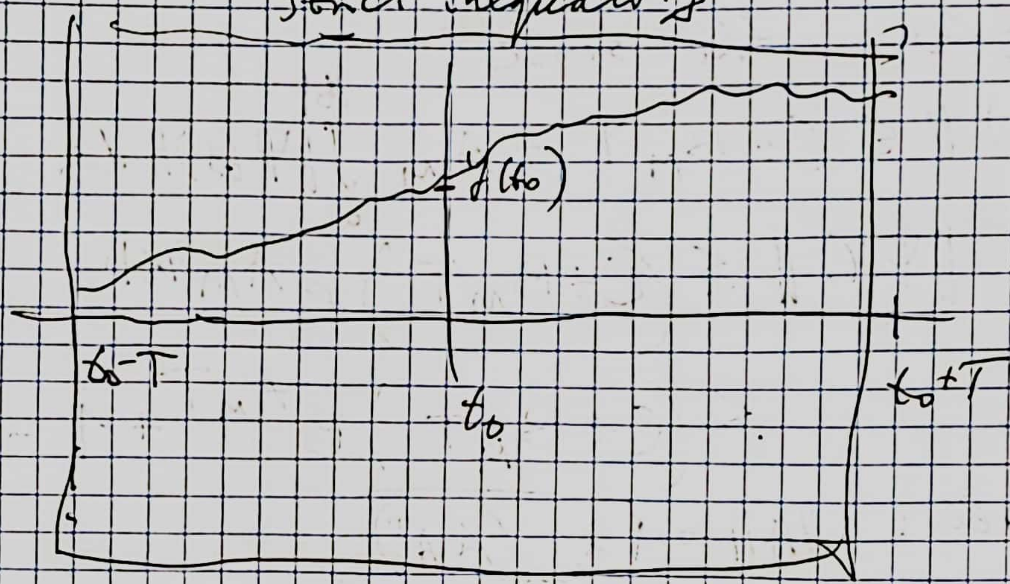
can be extended to $t \in \mathbb{R}$

$$y(t) = y(t_0) + \int_{t_0}^{t} a(t,y)\, dt$$

$$\|y(t)\| < \|y(t_0)\| + |t - t_0| \, M + 1$$

strict inequality

$$|t - t_0| \le T$$



the solution is extended till the boundary, i.e, till $t = t_0 + T,\ t_0 - T$

True for any $T > 0$ QED

Similarly all solutions are extended till $\pm\infty$ in time

$$\dot{y} = A(t)\, y + B(t) \qquad A, B \in C$$

Lemma Gronwall - Bellman ~~$\mu \in C$~~

$\varphi(t) \in C$, $\quad \varphi(t) \le \lambda + \int_0^t \mu \, \varphi(s)\, ds \quad \forall t > 0, \ \lambda \ge 0, \ \mu \ge 0$

$\Rightarrow \quad \varphi(t) \le \lambda e^{\mu t} \quad t \ge 0$

Let $y(0) = \xi$, $t_0 = \xi$

$$\varphi(t) = \xi + \int_0^t A(s)\varphi(s)\,ds + \int_0^t B(s)\,ds$$

Let $t \in [-T, T]$, $\|\xi\| \le C$, $\max_{[-T,T]} \|B\| = B_M$, $\max_{[-T,T]} \|A\| = A_M$

$\Rightarrow$ 1. $t > 0$ $\cancel{\|\varphi\| \le C}$

$$\|\varphi(t)\| \le C + B_M T + \int_0^t \|A(s)\|\|\varphi(s)\|\,ds$$

$$\le C + B_M T + A_M \int_0^t \|\varphi(s)\|\,ds$$

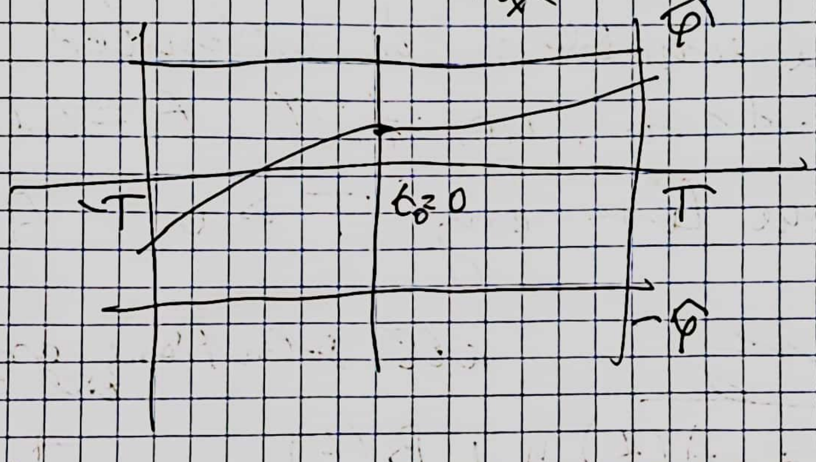
2. $t < 0$ the same

$$\|\varphi(t)\| \le C + B_M T + A_M \int_0^{|t|} \|\varphi(s)\|\,ds$$

$$\Rightarrow \|\varphi(t)\| \le (C + B_M T)\, e^{A_M |t|}$$

$$< (C + B_M T)\, e^{A_M T} + 1 = \hat{\varphi}$$

Recall: $\|A\| = \sqrt{\lambda_{max}(AA^T)}$



QED

Gronwall-Bellman Lemma (simplified)

$$\varphi(t) \le \lambda + \int_0^t \mu\, \varphi(s)\, ds \qquad \forall t > 0$$

$$\varphi \in C \qquad \lambda, \mu \ge 0$$

If there was identity we would get $\begin{cases} \dot\varphi = \mu \varphi \\ \varphi(0) = \lambda \end{cases} \Rightarrow \varphi(t) = \lambda e^{\mu t}$

<u>Claim</u> $\varphi(t) \le \lambda e^{\mu t}$

Proof: $z \stackrel{def}{=} \lambda + \int_0^t \mu\, \varphi(s)\, ds$, $\varphi(t) \le z(t)$

$v(t) \stackrel{def}{=} z(t) - \varphi(t) \ge 0$

$$\begin{cases} \dot z = \mu \varphi(t) = \mu(z(t) - v(t)) \\ z(0) = \lambda \end{cases}$$ linear!

$z_h = c e^{\mu t}$, $c \in \mathbb{R}$

$z = c(t) e^{\mu t}$, $\dot z = \cancel{c \mu e^{\mu t}} + \dot c e^{\mu t} = \mu(\cancel{c e^{\mu t}} - v)$

$$\dot c = -\mu v e^{-\mu t}$$

$$c(t) = \underset{\lambda}{c_0} - \mu \int_0^t v(s) e^{-\mu s} ds$$

$z = c_0 e^{\mu t} - \ldots e^{\mu t}$, $[\ldots] \ge 0$

$$\varphi(t) \le \varphi(t) + v(t) = z(t) = \lambda e^{\mu t} - \underset{\ge 0}{\ldots}$$

QED

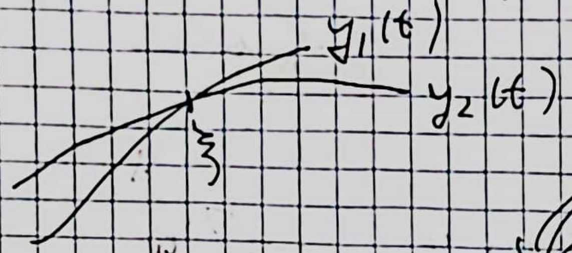
Extension solutions in the autonomous case

$$\begin{cases}\dot{y}=f(y)\\ y(t_0)=\xi\end{cases} \quad \text{or} \quad \begin{cases}\dot{y}_*=f(y_*)\\ y_*(0)=\xi\end{cases}$$

$$y(t)\equiv y_*(t-t_0)$$ shift in time

Indeed $y(t_0)=y_*(t_0-t_0)=y_*(0)=\xi$

$$\dot{y}(t)=\dot{y}_*(t-t_0)\cdot 1=f(y_*(t-t_0)) \quad \text{QED}$$



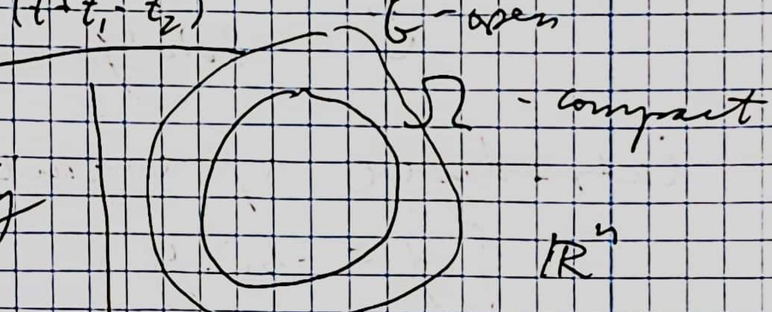
intersection $y_1(t_1)=y_2(t_2)$ is impossible

$\hat{y}_1(t)=y_1(t-t_1)$ solution, $\hat{y}_2(t)=y_2(t-t_2)$ solution

The same curve: $y_1(t-t_1)\equiv y_2(t-t_2) \Leftarrow \hat{y}_1(0)=\hat{y}_2(0)$

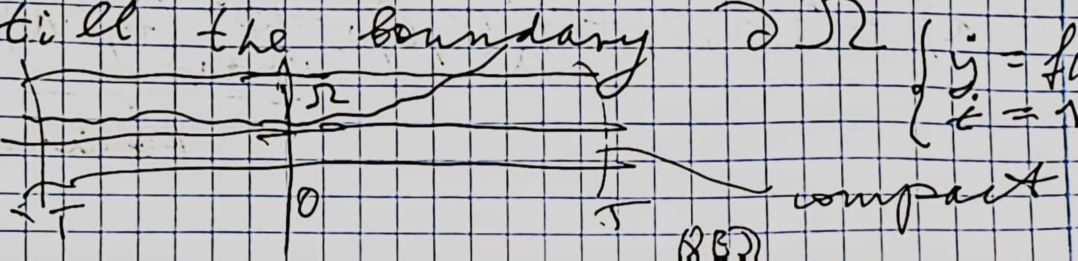
shift $y_1(t)=y_2(t+t_1-t_2)$

$$\begin{cases}\dot{y}=f(x) & \text{Lipsch locally}\\ y(0)=\xi\in\mathbb{R}^n\end{cases}$$



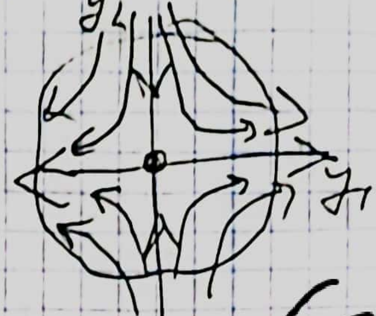
Each solution is extendable in each direction till infinity of time or till the boundary $\partial\Omega$

$$\begin{cases}\dot{y}=f(y)\\ \dot{t}=1\end{cases}$$



Example!

$$\begin{cases} \dot{y}_1 = y_1 \\ \dot{y}_2 = -y_2 \end{cases}$$

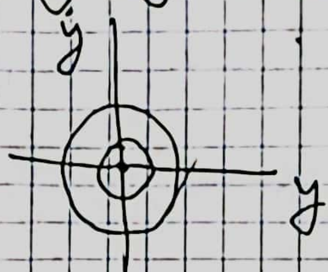


saddle point

$$\begin{cases} \dot{y}_1 = y_2 \\ \dot{y}_2 = -y_1 \end{cases}$$

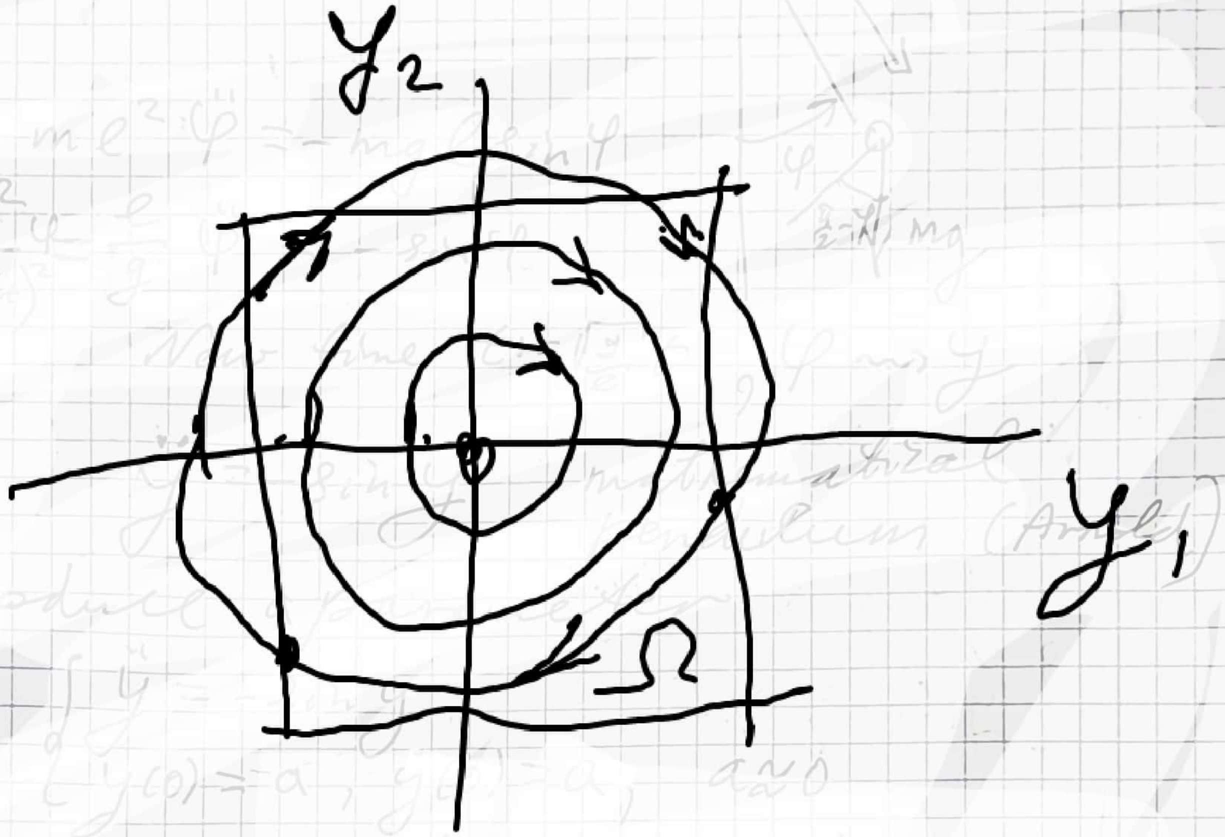
$\ddot{y}_1 + y_1 = 0$

$\ddot{y} + y = 0 \Rightarrow \dot{y}\ddot{y} + y\dot{y} = 0$



$\left(\frac{\dot{y}^2}{2} + \frac{y^2}{2}\right)^{\cdot} = 0$

$\dot{y}^2 + y^2 = const$



Trajectory can enter and leave the region.

Trajectory can remain forever in the region.

Equilibrium is possible.