Lecture 5.

Theorem of existence and uniqueness of solution

∃! theorem

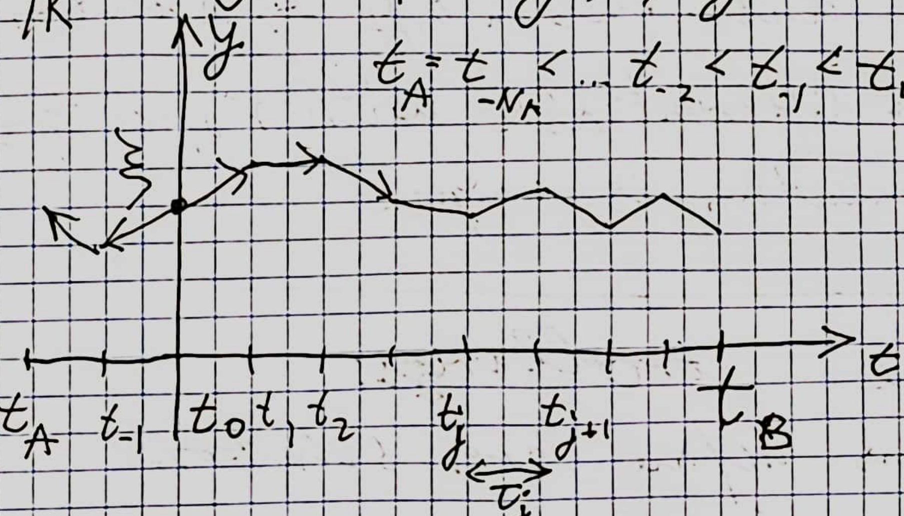
The theorem of ∃! is proved by approximation of solutions, ~~which is proved to~~ converging to them.

The Euler integration method

$$\dot{y} = f(t,y), \quad y(t_0) = \xi, \quad y \in \mathbb{R}^n$$

$$t_A = t_{-N_A} < \dots t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \dots < t_{N_B} = t_B$$

$$\tau_j = t_{j+1} - t_j < \tau$$



$$y(t_0) = \xi, \quad y(t_1) = y(t_0) + f(t_0, y(t_0))\tau_1, \dots$$

$j \geq 0$ $\quad y(t_{j+1}) = y(t_j) + f(t_j, y(t_j))\,\tau_j$ forwards

$j \leq 0$ $\quad y(t_j) = y(t_{j+1}) - f(t_{j+1}, y(t_{j+1}))\,\tau_j$ backwards

$$\boxed{y(t_{**}) = y(t_*) + f(t_*, y(t_*))(t_{**} - t_*)}$$

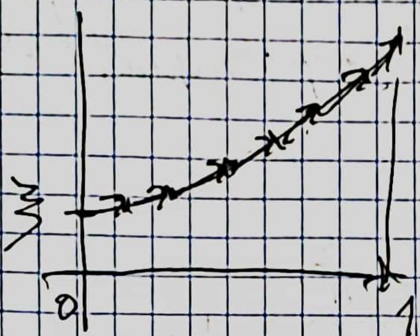
As $\tau \to 0$ the approximations converge

$\underline{\text{Exact formulation}}$: If the solution (f is continuous) is unique, the Euler approximations converge to it uniformly in $[t_A, t_B]$.

Otherwise there is a subsequence which converges uniformly to a solution.

$\underline{\text{Example}}$ $\quad \dot{y} = y \in \mathbb{R}, \quad y_0 = y(0) = \xi$

$t_A = 0, \quad t_B \neq \cancel{1} \quad [0, t_B]$



Let $\tau_j = \tau = \frac{1}{n} t_B$

$y_0 = \xi, \quad t_0 = 0$

$y_1 = y_0 + \frac{t_B}{n} \cdot y_0 = \left(1 + \frac{t_B}{n}\right)\xi$

$y_2 = y_1 + \frac{t_B}{n} y_1 = \left(1 + \frac{t_B}{n}\right)^2 \xi$

$\vdots$

$$y(t_B) = y_n = y_{n-1} + \frac{t}{n} y_{n-1} = \left(1 + \frac{t_B}{n}\right)^n \xi$$

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} \left(1 + \frac{t_B}{n}\right)^n \xi = \xi e^{t_B}$$

$$q = \frac{t}{t_B} \in [0,1] \quad \cancel{\lim_{n\to\infty} y_{\left[\frac{n}{2}\right]} = \lim_{n\to\infty} \left(1 + \frac{t_B}{\left[\frac{n}{2}\right]}\right)^{\left[\frac{n}{2}\right]} \xi = \lim \left(1 + t_B\right.}$$

$$0 < q < 1 \qquad [nq] \le nq < [nq] + 1 \qquad \text{pde pfn}$$

$$\lim_{n\to\infty} y_{[nq]} = \lim_{n\to\infty} \left(1 + \frac{t_B}{n}\right)^{[nq]} \xi =$$

$$= \lim_{n\to\infty} \overbrace{\left(1 + \cancel{f}\frac{t_B q}{nq}\right)^{\frac{nq}{t_B q}}}^{}{}^{\frac{[nq]}{nq} t_B q} \xi =$$

$$= \lim_{n\to\infty} \left(e^1\right)^t \xi = e^t \xi$$

The Euler integration can be used and is used for numeric solution of ODEs

Picard approximation

$\dot{y} = f(t, y)$, $y(t_0) = \xi$, $t_0 \in [t_A, t_B]$

$$y(t) = y(t_0) + \int_{t_0}^{t} \dot{y}(s)\, ds = y(t_0) + \int_{t_0}^{t} f(s, y(s))\, ds$$

The ODE is replaced with the integral one.
A sequence of functional approximations is constructed now.

$$\varphi_0(t) = y(t_0) = \xi$$

$$\varphi_1(t) = \xi + \int_{t_0}^{t} f(s, \varphi_0(s))\, ds$$

$$\varphi_k(t) = \xi + \int_{t_0}^{t} f(s, \varphi_{k-1}(s))\, ds$$

For small enough $t_B - t_A$ get $\varphi_k \rightrightarrows y(t)$, $k \to \infty$ uniformly in t

Example $\dot{y} = y$, $y(0) = \xi$

$$\varphi_0(t) = \xi,$$

$$\varphi_1(t) = \xi + \int_0^t \varphi_0(s)\, ds = \xi + \int_0^t \xi\, ds = (1+t)\xi$$

$$\varphi_2(t) = \xi + \int_0^t (1+s)\xi\, ds = \xi + \left(t + \frac{t^2}{2}\right)\xi = \xi\left(1 + t + \frac{t^2}{2}\right)$$

$$\varphi_{k+1}(t) = \xi + \int_0^t \left(1 + \frac{s}{1} + \frac{s^2}{2} + \dots + \frac{s^k}{k!}\right)\xi\, ds = \xi\left(1 + \frac{t}{1} + \frac{t^2}{2} + \dots + \frac{t^{k+1}}{(k+1)!}\right)$$

$$\varphi_k \rightrightarrows \xi e^t$$

Lipschitz property

~~for~~ $\varphi: \mathbb{R}^m \to \mathbb{R}^n$ is said to posses the Lipschitz property with the Lipschitz constant $L > 0$ if $\forall x_1, x_2 \in \mathbb{R}^m$ (or over the function domain)

$$\|\varphi(x_1) - \varphi(x_2)\| \le L \|x_1 - x_2\|$$

<u>Recalling</u>: $\|x\|$ is called a norm of x

$\|x\| \ge 0$, $\|x\| = 0 \Leftrightarrow x = 0$

$\|x_1 + x_2\| \le \|x_1\| + \|x_2\|$, $\|\lambda x\| = |\lambda| \, \|x\|$

(analogue to the absolute value)

$\Rightarrow \|x_1 + x_2\| \ge \|x_1\| - \|x_2\|$

<u>Example</u> $\varphi: \mathbb{R} \to \mathbb{R}$, φ' bounded, continuous
$|\varphi'(x)| \le L$

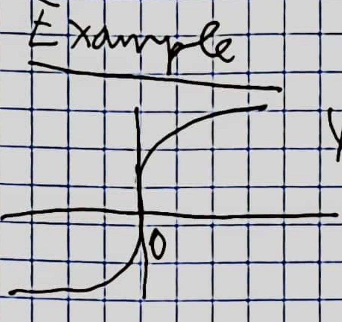
$\Rightarrow \varphi(x_1) - \varphi(x_2) = \varphi'(\theta)(x_1 - x_2)$,

Lagrange: $(x_2 + \theta(x_1 - x_2))$, $\theta \in (0,1)$

$|\varphi(x_1) - \varphi(x_2)| \le L |x_1 - x_2|$ QED

$\Rightarrow$ $\cos x$ is a Lipschitz function, $L = 1$

<u>Example</u> $\varphi(x) = \sqrt[3]{x}$ is not Lipschitz at 0



$\varphi'(x) = \frac{1}{3} x^{-\frac{2}{3}}$, $\lim_{x \to 0} \varphi'(x) = \infty$

Let $x_2 = 0$ $\quad \frac{\varphi(x_1) - \varphi(0)}{x_1 - 0} = \frac{x_1^{\frac{1}{3}}}{x_1} = x_1^{-\frac{2}{3}}$ unbounded

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Proposition

1. Elementary functions

$\ln x$, e^x, $\cos x$, x^2, x^n $(n \in N)$,
their inverse functions are Lipschitzian over each compact interval (closed segment) where they are defined

2. $\varphi_1, \varphi_2 : \mathbb{R} \to \mathbb{R}$ Lipschitzian locally
$\alpha, \beta \in \mathbb{R}$ then

$\alpha\varphi_1 + \beta\varphi_2$, $\varphi_1 \cdot \varphi_2$, φ_1^{-1} (if $\varphi_1' \neq 0$), $\frac{\varphi_1}{\varphi_2}$ (if $\varphi_2 \neq 0$)

$\varphi_1(\varphi_2)$ composition of functions

are Lipschitzian

proofs are straight-forward

Example: $\varphi_1(\varphi_2)$ $\quad x \xmapsto{\varphi_2} y \xmapsto{\varphi_1} z$

$$\frac{d}{dx}\varphi_1(\varphi_2(x)) = \underbrace{\varphi'_{1y}(\varphi_2(x)) \cdot \varphi'_{2x}(x)}_{\text{bounded}}.$$

similarly functions with many arguments are considered

Example: $\cos(x^5 tg(y) + z)$
is Lipschitzian in x, y, z
for $(x, y, z) \in [-100, 100] \times [-1, 1] \times [-1000, 2000]$
(Decartes product)

Lipschitz property with respect to a part of arguments

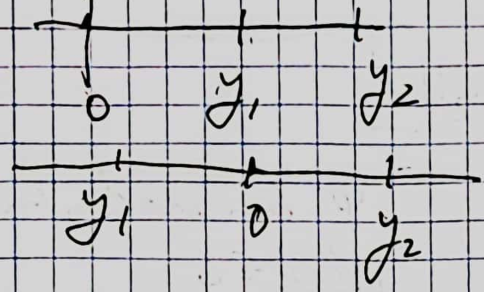
$$\|\varphi(x, y_1) - \varphi(x, y_2)\| \le L\|y_1 - y_2\|$$

$\forall\, x, y_1, y_2$

Examples $\quad |xy| \quad x, y \in [-1, 1]$

$$|xy_1| - |xy_2| = |x|\,(|y_1| - |y_2|)$$
$$\le |x|\,(|y_1| - |y_2|)$$

$$|y_1| - |y_2| \le$$
$$\le |y_1 - y_2|$$



(cases: $y_1 y_2 > 0$, $y_1 y_2 < 0$)

$|y|$ – Lipschitz $L = 1$

$\varphi(x, y) = (x^2+1)\sqrt{|y|}$ non-Lipschitz around $y = 0$

$$\frac{(x^2+1)\sqrt{|y|} - 0}{|y|} = \frac{x^2+1}{\sqrt{|y|}} \text{ unbounded}$$

$x^{\frac{2}{3}} y$ locally Lipshitz in y

$$\left|x^{\frac{2}{3}} y_1 - x^{\frac{2}{3}} y_2\right| \le x^{\frac{2}{3}} |y_1 - y_2|$$

locally bounded

but not Lipschitz in x at $x=0$

$$\left|x_1^{\frac{2}{3}} y - x_2^{\frac{2}{3}} y\right| = \left|x_1^{\frac{2}{3}} - x_2^{\frac{2}{3}}\right| |y| \qquad x_2 = 0 \Rightarrow \frac{x_1^{\frac{2}{3}}}{|x_1|}|y| \text{ unbounded}$$

Examples

1. $e^{x^2 \cdot \ln y} + \operatorname{arctg}\left(\frac{x}{y-1}\right)$

is Lipschitz in x for any $y \in [\varepsilon, 1-\varepsilon]$, $0<\varepsilon<\frac{1}{2}$

is not Lipschitz in y for any x

$y \approx 0$ or $y \approx 1$

2. $(1+\cos x)^{\frac{1}{2}} e^{y}$ is locally Lipschitz in y for any $x \in \mathbb{R}$

is not Lipschitz in x in vicinities of $-\pi + 2k\pi$, $k \in \mathbb{Z}$

Final formulation of the $\exists!$ Theorem

$$\begin{cases} \dot{y} = f(t, y), & y \in \mathbb{R}^n \\ y(t_0) = \xi & \text{initial condition} \end{cases}$$ Cauchy problem

1. Let f be continuous in t, x.

Then there exists a local solution

($\exists\, \varepsilon > 0$: $y: [t_0-\varepsilon, t_0+\varepsilon] \to \mathbb{R}^n$, $y(\cdot)$ differentiable and $y(t_0) = \xi$, $\dot{y}(t) = f(t, y(t))$, $t \in [t_0-\varepsilon, t_0+\varepsilon]$)

$\|f\| \le M$, $|t-t_0| \le \delta_t$, $\|y-\xi\| \le \delta_y$

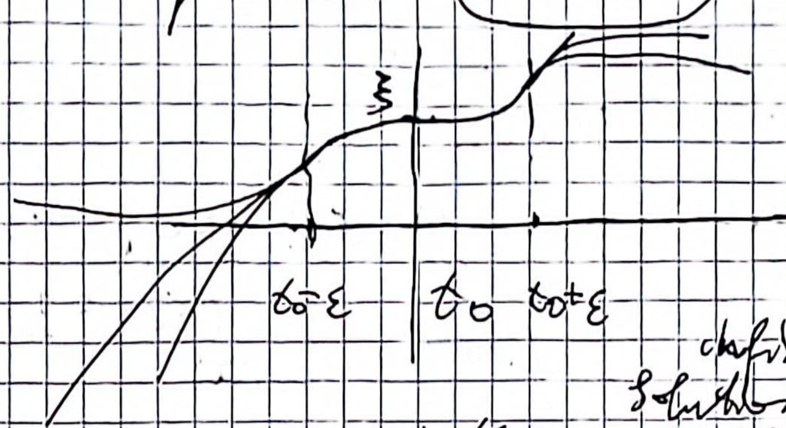
$\varepsilon = \min\left\{\delta_t, \frac{\delta_y}{M}\right\}$

~~2. Under conditions 1. Let~~

2. Let f be not only continuous, but also Lipschitzian in y: $\forall t \ \|f(t, y_1) - f(t, y_2)\| \le L\|y_1 - y_2\|$

Then this local solution is unique for $t \in [t_0-\varepsilon, t_0+\varepsilon]$, $\varepsilon = \min\left\{\frac{\delta_t}{L}, \delta_t, \frac{\delta_y}{M}\right\}$

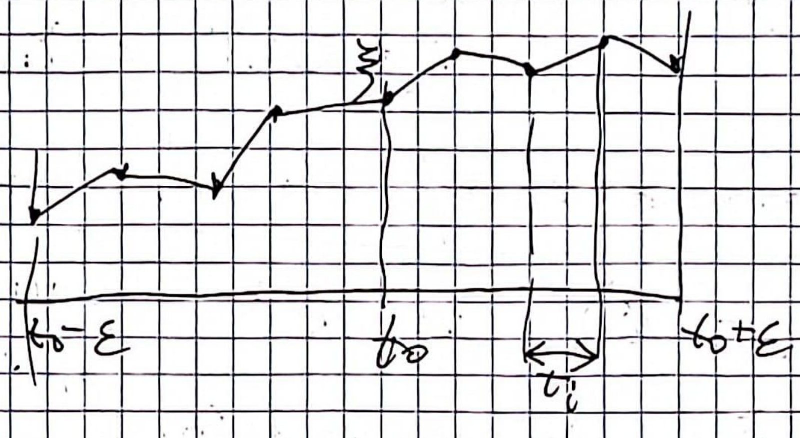
Uniqueness solution meaning:



There can be infinitely many solutions but they coincide over $[t_0-\varepsilon, t_0+\varepsilon]$

Idea of the proof:

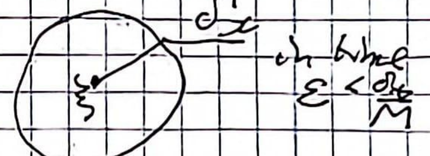
1. An Euler approximation is constructed



$y(t_{i+1}) = y(t_i) + f(t_i, y(t_i))\,\tau_i$

$\max \tau_i \to 0$

No solution can leave the ball



in time $\varepsilon < \frac{\delta_\varepsilon}{M}$

Theorem Arzela-Ascoli

Any infinite sequence $y_k(\cdot): [t_0-\varepsilon, t_0+\varepsilon] \to \mathbb{R}^n$

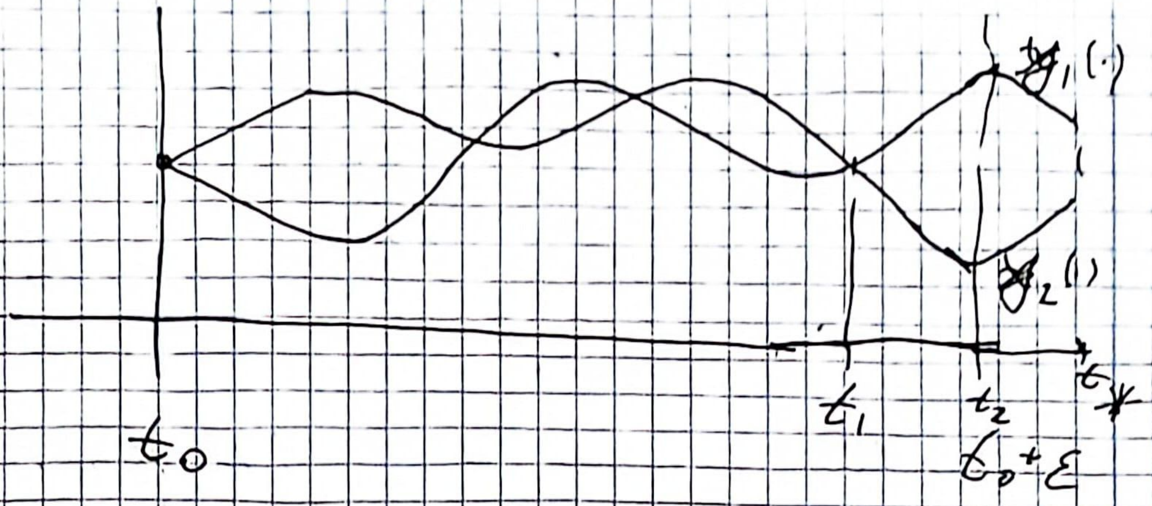
(precompactness property) ~~with~~ bounded $(\|y_k(t)\| \le \|\xi\| + \delta_\varepsilon)$

and Lipschitzian $\|\dot{y}_k(\cdot)\| \le \max\|f\| = M$

contains a subsequence uniformly int converging to a Lipschitzian function

A. it is differentiable B. it satisfies the DE

2. uniqueness:



Assume that there exists t_* ~~t~~ : $y_1(t_*) \neq y_2(t_*)$

Let $t_* > t_0$, $t_0 < t_* \le t_0 + \varepsilon$

Let $t_1 = \sup\limits_{t \in [t_0, t_*]} \{ t \mid y_1(t) = y_2(t) \}$

__Lemma:__ 1. $t_1 < t_*$. Indeed $y_1(t_*) \neq y_2(t_*)$
$\Rightarrow$ also in a vicinity

2. $\forall t \in (t_1, t_*]$: $y_1(t) \neq y_2(t)$
otherwise $t_1 \neq \sup$

Define $t_2 = \arg\max\limits_{t \in [t_1, t_*]} \|y_1(t) - y_2(t)\|$.

$$\|y_1(t_2) - y_2(t_2)\| = \max_{[t_1, t_*]} \|y_1(t) - y_2(t)\| = M_y$$

Then $M_y = \|y_1(t_2) - y_2(t_2)\| = \|$ ~~$\xi + \int$~~

$$= \left\| \cancel{y_1(t_1)} + \int_{t_1}^{t_2} \dot{y}_1(s)\,ds - \cancel{y_2(t_1)} - \int_{t_1}^{t_2} \dot{y}_2(s)\,ds \right\|$$

$$= \left\| \int_{t_1}^{t_2} \left[f(t, y_1(s)) - f(t, y_2(s)) \right] ds \right\|$$

$$\le \left\| \int_{t_1}^{t_2} L \|y_1(s) - y_2(s)\|\,ds \right\| \le \left\| \int_{t_1}^{t_2} L M_y\,ds \right\|$$

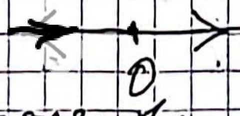
$$\le |t_2 - t_1| L M_y \le |t_* - t_1| L M_y \le \underbrace{|t_* - t_0|}_{\le \varepsilon} L M_y$$

Thus $1 \leq \frac{Bt}{L}(t_* - t_0)$

Now, choose t_* sufficiently close to t_0 so that it will be a contradiction (QED)

Counter example! The requirement is important

$\dot{y} = 3y^{\frac{2}{3}}$



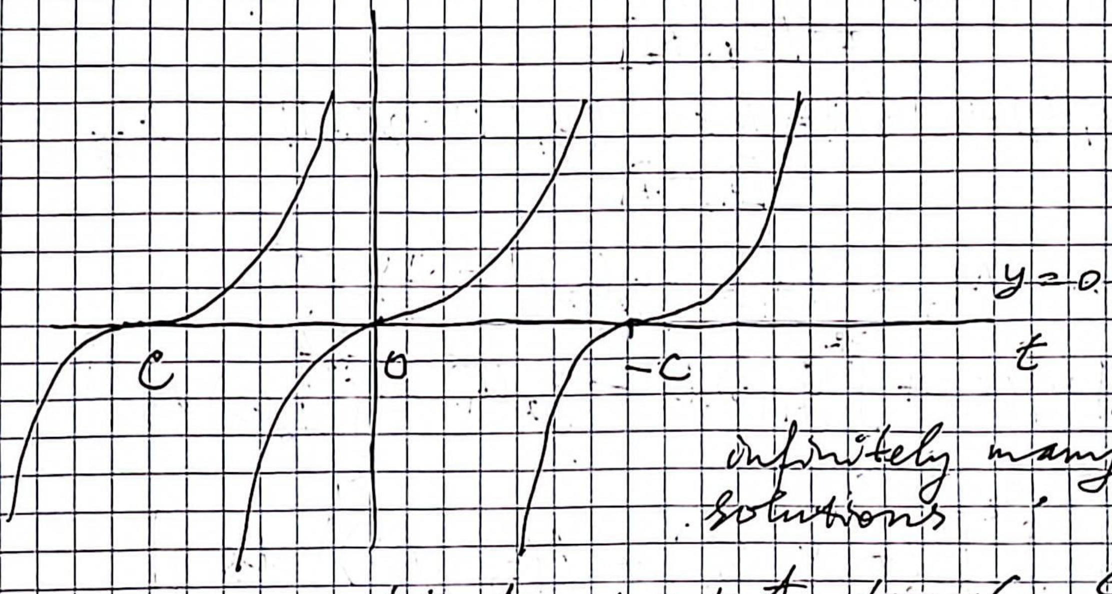
y $\quad y=0$ is a solution

The Lipschitz property holds everywhere except $y=0$

$y \neq 0 \Rightarrow \frac{1}{3} y^{-\frac{2}{3}} \dot{y} = 1$

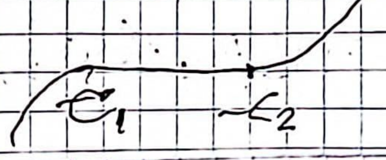
$y^{\frac{1}{3}} = \int \frac{1}{3} y^{-\frac{2}{3}} \underbrace{\dot{y}\,dt}_{dy} = t + C$

$y = (t+C)^3, \quad C \in \mathbb{R}, \quad y = 0$



infinitely many solutions

At any moment $t = -C$, $C \in \mathbb{R}$ the solution can come from below or leave to above, or stay at 0



$$y = \begin{cases} (t+C_2)^3 & t > -C_2 \\ 0 & t \in [-C_1, -C_2] \\ (t+C_1)^3 & t \leq -C_1 \end{cases}$$

The proof of the $\exists!$ theorem in the book by Arnold is based on the Picard approximation and the principle of the unique fixed point of a contraction mapping

We omit it here

Theorem (Arnold) $\begin{cases} \dot{y} = f(t, y) \\ y(t_0) = \xi \end{cases} \Rightarrow y = Y(t, \xi)$

If $f \in C^k$ in t, y (in both!)

then $Y \in C^k$ in t, ξ — Dependence on initial conditions!

Dependence on parameters:

$\begin{cases} \dot{y} = f(t, y, \alpha) \\ y(t_0) = \xi \end{cases}$ $y \in \mathbb{R}^n$, $\alpha \in \mathbb{R}^e$ parameters

$f(\cdot) \in C^k$

$\Leftrightarrow \begin{cases} \dot{y} = f(t, y, \alpha) \\ \dot{\alpha} = 0 \end{cases}$ $\begin{cases} y(t_0) = \xi \\ \alpha(t_0) = \alpha_0 \end{cases}$ init. conditions

$\Rightarrow y = Y(t, \xi, \alpha) \in C^k$

Continuous dependence on the right-hand side

Theorem $\begin{cases} \dot{y}_\varepsilon = f(t, y_\varepsilon) + \varepsilon w(t, y_\varepsilon) \\ y_\varepsilon(t_0) = \xi \end{cases}$ $f, w \in C$

$\lim_{\varepsilon \to 0} y_\varepsilon(t, \xi) = y_0(t, \xi)$ uniformly in $t \in [t_0 - \varepsilon, t_0 + \varepsilon]$