Lecture 4 Special 25/01-2024

Not for exam

Mathematical Biology

System by Lotka-Volterra (1920-1926)

Predator-Prey dynamics
(see Wikipedia and Arnold)

Assumption: there is a lot of the food for the prey. It is a small ecologic system with only 2 species interacting

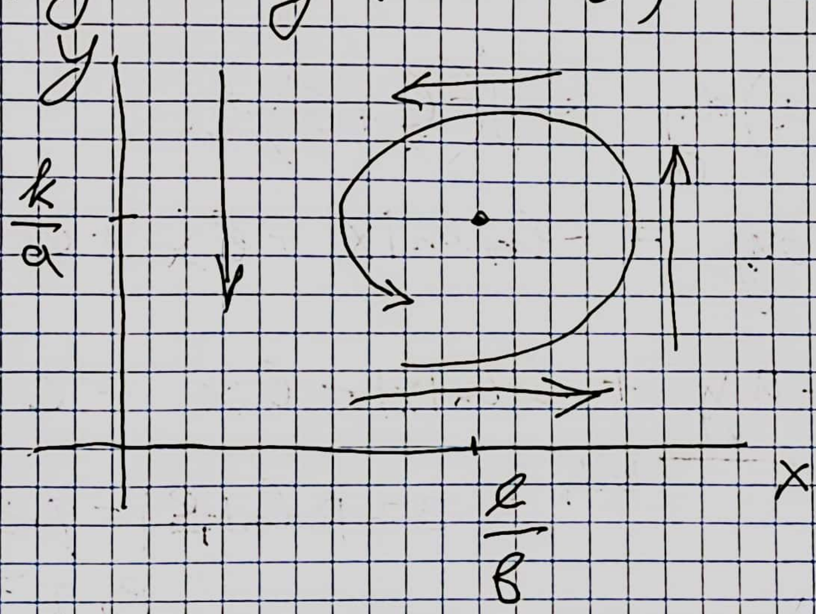
y predator density

x – prey density

$$\begin{cases} \dot{x} = kx - axy & k, a > 0 \\ \dot{y} = -\ell y + bxy & \ell, b > 0 \end{cases}$$

$$\begin{cases} \dot{x} = x(k - ay) \\ \dot{y} = y(bx - \ell) \end{cases}$$

extinction is possible: $x = y = 0$



the phase space

$(x_*, y_*) = \left(\frac{\ell}{b}, \frac{k}{a}\right)$ equilibrium

constant solution!

Let $x \neq 0$, $y \neq \frac{k}{a}$ then $\dot{x} \neq 0$

$t = t(x)$

$$\frac{dt}{dx} = \frac{1}{x(k-ay)}$$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{y(-l+bx)}{x(k-ay)}$$

Separate variables:

$$\frac{k-ay}{y}\,dy = dx\,\frac{(-l+bx)}{x}$$

here $dy = y'dx$

The same result is obtained for $\dot{y} \neq 0$, $t = t(y)$, $x \neq \frac{l}{b}$

Thus we get that whenever

$x \neq \frac{l}{b}$, $y \neq \frac{k}{a}$

$$\frac{k-ay}{y}\,dy + \frac{l-bx}{x}\,dx = 0$$

taking integral with respect to x, $dy = y'dx$

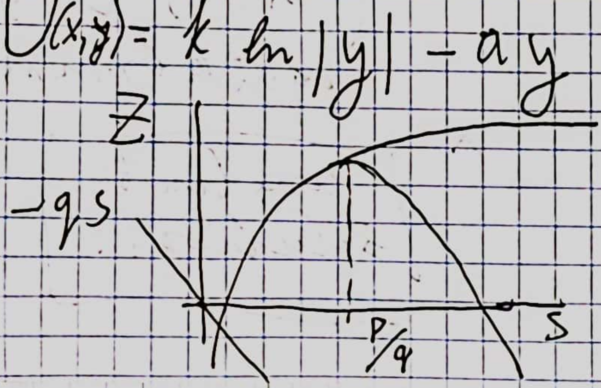
or to y, $dx = x'dy$

get

$$\int\left(\frac{k}{y}-a\right)dy = \int\left(-\frac{l}{x}+b\right)dx$$

$$k\ln|y| - ay = -l\ln|x| + bx + C$$

$$U(x,y) = k\ln|y| - ay + l\ln|x| - bx = C$$



$z = p\ln|s| - qs$

$p, q > 0$

$z\left(\frac{p}{q}\right) = p\ln p - q$

$$z' = \frac{p}{z} - q = 0 \Rightarrow z = \frac{p}{q}$$

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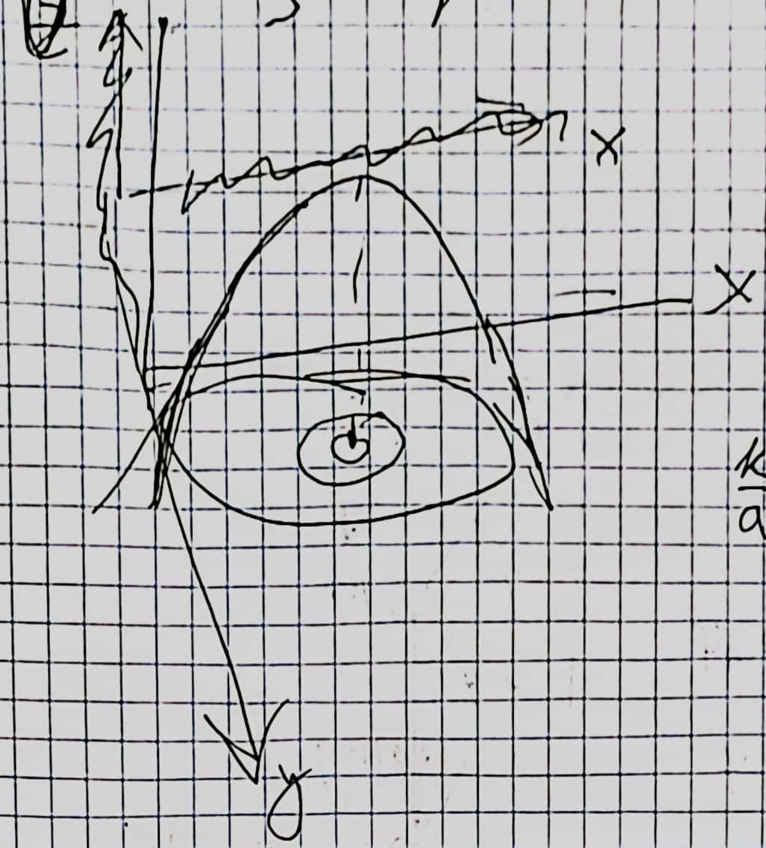
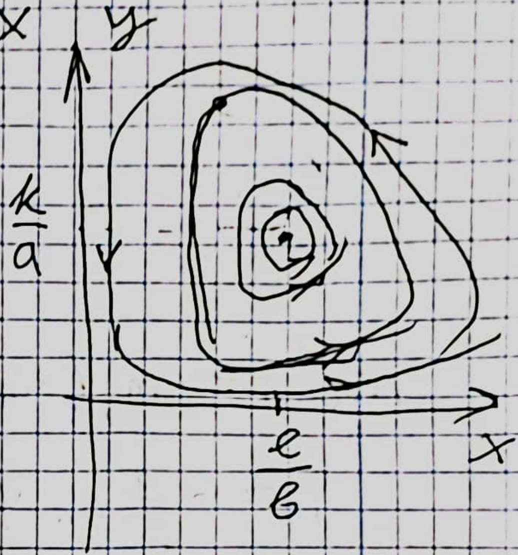
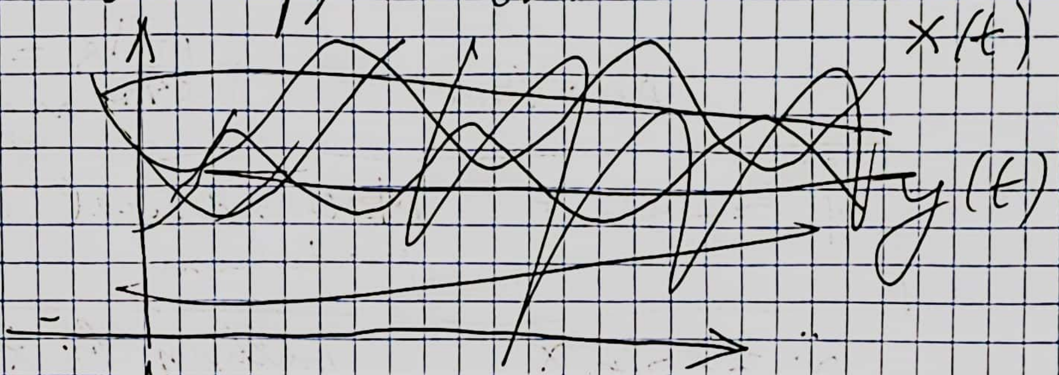
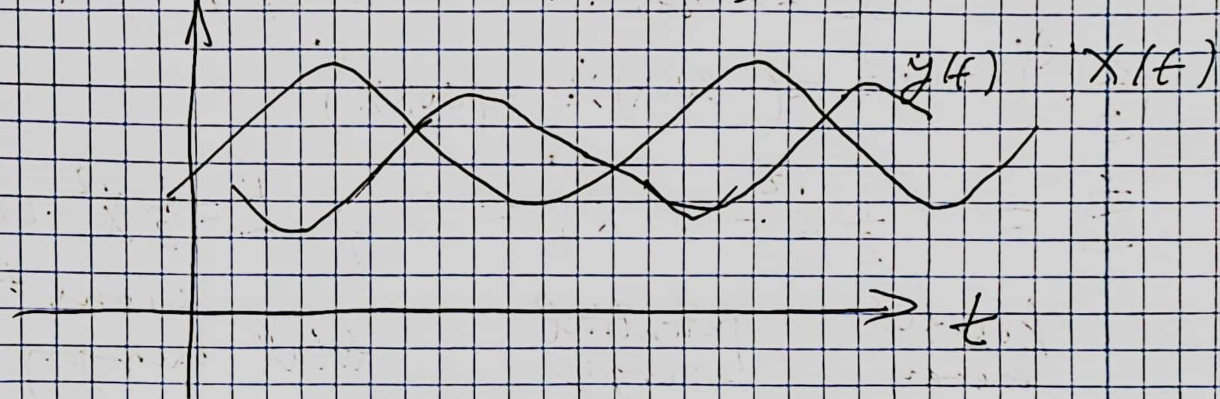




~~Now suppose that~~





Another interpretation:

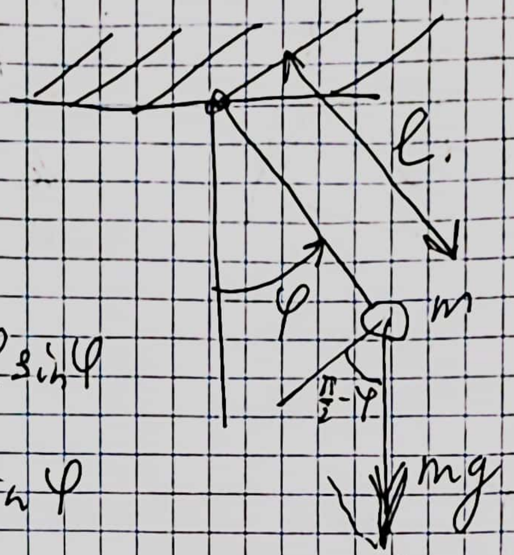
x – population, y – infected with a potentially fatal disease

Let parameters $a(t)$, $b(t)$, $k(t)$, $l(t)$ be periodic functions of t, $T = 1$ year

So we get characteristic waves! influenza, covid-19, etc. (flu)

Pendulum DE

Arnold



$$m\ell^2\ddot{\varphi} = -mg\ell\sin\varphi$$

$$\ddot{\varphi} = -\frac{g}{\ell}\sin\varphi$$

$$\left(\frac{d}{\sqrt{\frac{g}{\ell}}\,dt}\right)^2\varphi = -\sin\varphi \qquad \tilde{t} = \sqrt{\frac{g}{\ell}}\,t$$

new time

$$\sqrt{\frac{m}{s^2 m}}\, s = 1$$ no meas. units

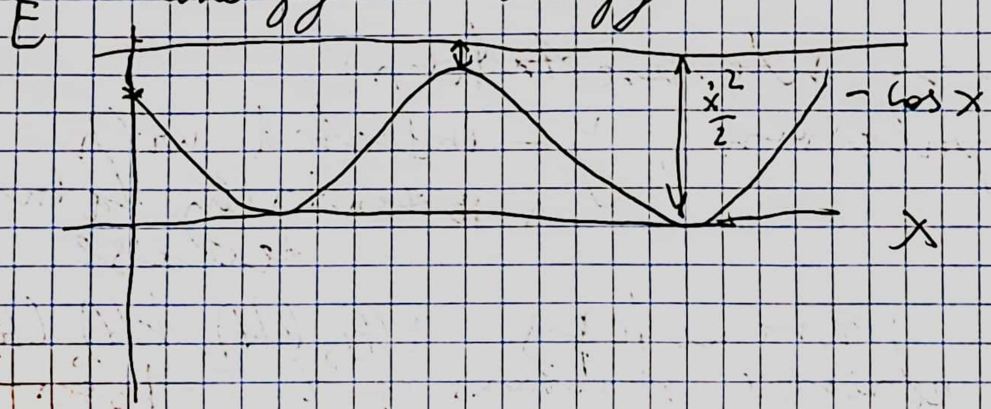
$$\boxed{\ddot{\varphi} = -\sin\varphi \quad \text{Mathematical pendulum}}$$

$$\ddot{x} = -\sin x \qquad \text{multiply by } \dot{x}$$

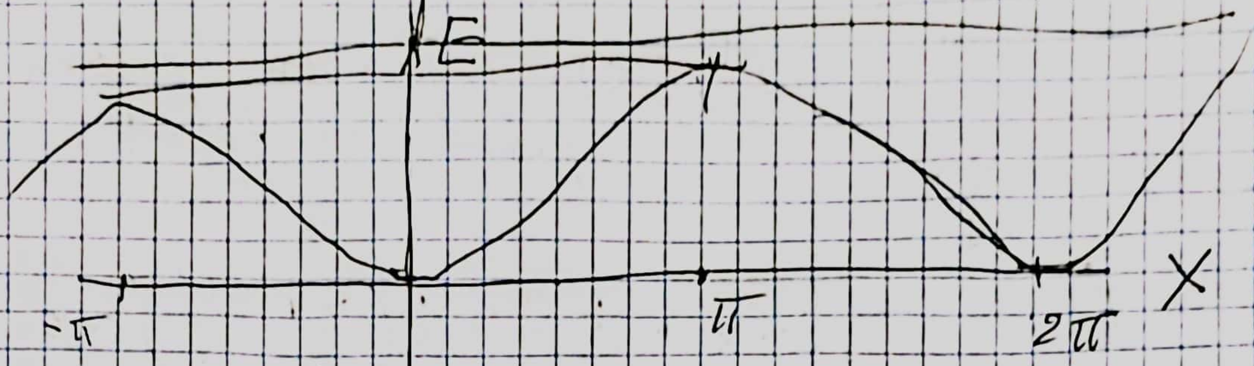
$$\left(\frac{1}{2}\dot{x}^2\right)^{\cdot} = \dot{x}\ddot{x} = (-\sin x)\dot{x} = (\cos x)^{\cdot}$$

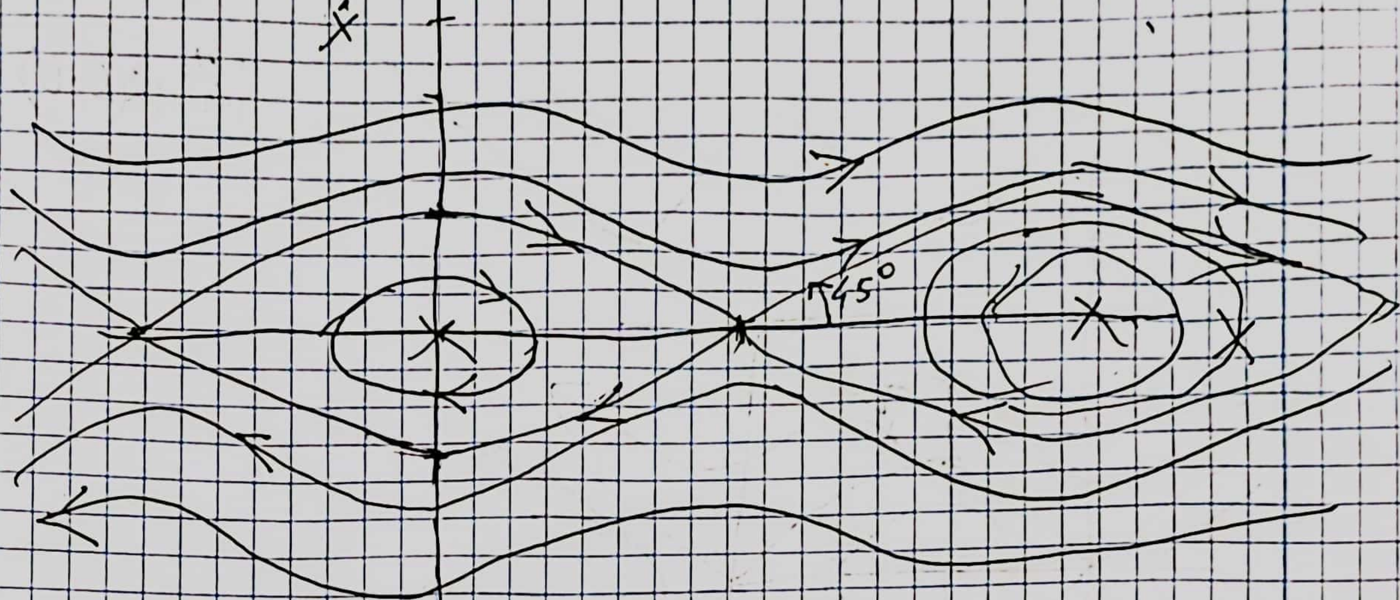
$$\dot{E} = \left(\frac{1}{2}\dot{x}^2 - \cos x\right)^{\cdot} = 0, \quad E = \frac{1}{2}\dot{x}^2 - \cos x = \text{const}$$

kinetic energy — potential energy — total energy

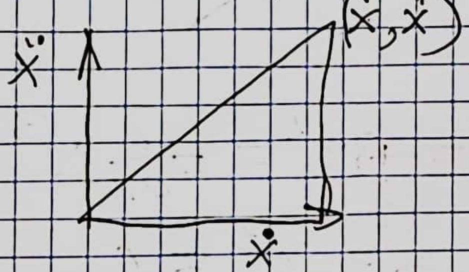


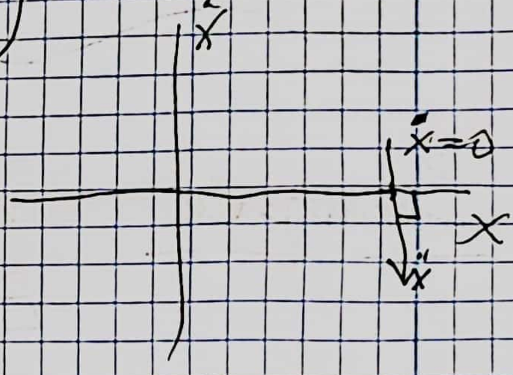
First Integral





$$\dot{x} = \pm\sqrt{2(E+\cos x)}$$





$x = \pi$, $\cos\pi = -1$ $\quad E = \frac{\dot{x}^2}{2} - \cos x = 1$

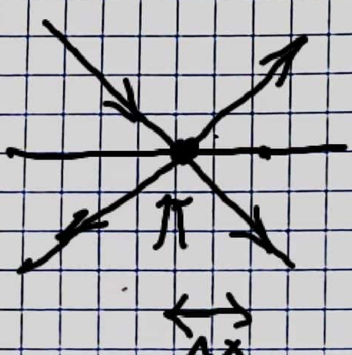
$x = \pi + \Delta x$, $\cos x = -1 + \frac{\Delta x^2}{2} + O(\Delta x^4)$

There are 5 solutions here

$$1 = E = \frac{\dot{x}^2}{2} + 1 - \frac{\Delta x^2}{2} + O(\Delta x^4)$$

$$\frac{\dot{x}^2}{2} = \frac{\Delta x^2}{2} + O(\Delta x^4)$$

$$\dot{x}^2 = \Delta x^2 + O(\Delta x^4)$$



$$|\dot{x}| = |\Delta x|\left(1 + O(\Delta x^2)\right)^{\frac{1}{2}} = |\Delta x|\left(1 + \tfrac{1}{2}O(\Delta x^2)\right)$$

$$\dot{x} = \pm\Delta x + O(\Delta x^3)$$

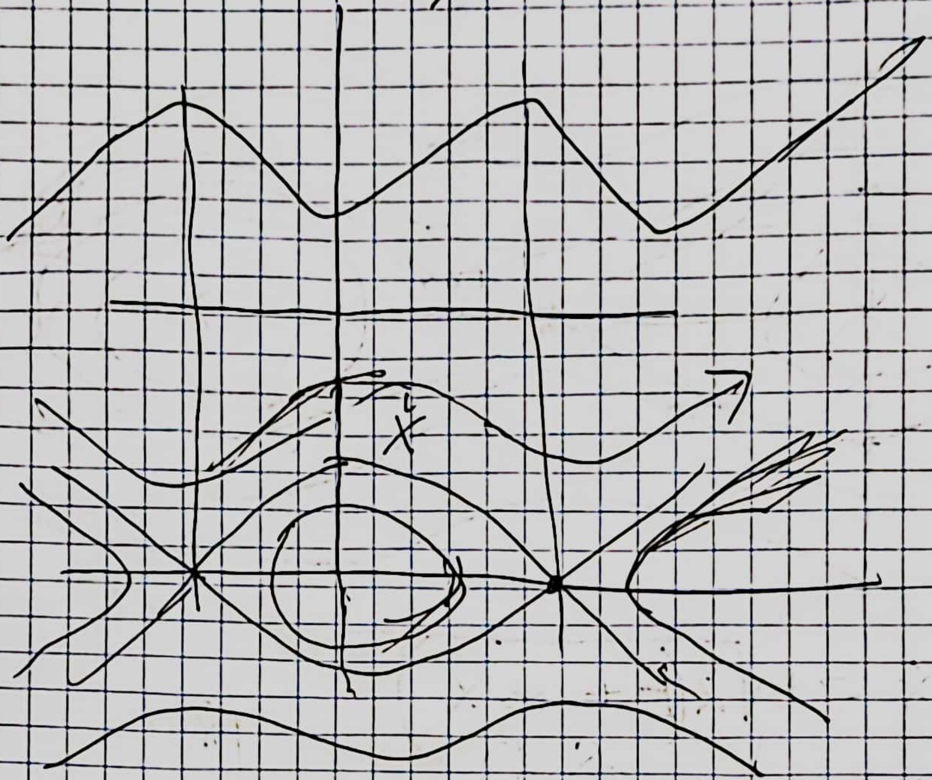
$$\ddot{x} = -\frac{\partial U}{\partial x}(x)$$ Newton Equation

potential field

$$E = U + \frac{\dot{x}^2}{2}$$

$$\dot{E} = -\frac{\partial U}{\partial x}\cdot\dot{x} + \dot{x}\ddot{x} = \dot{x}\left(\ddot{x} + \frac{\partial U}{\partial x}\right) = 0$$

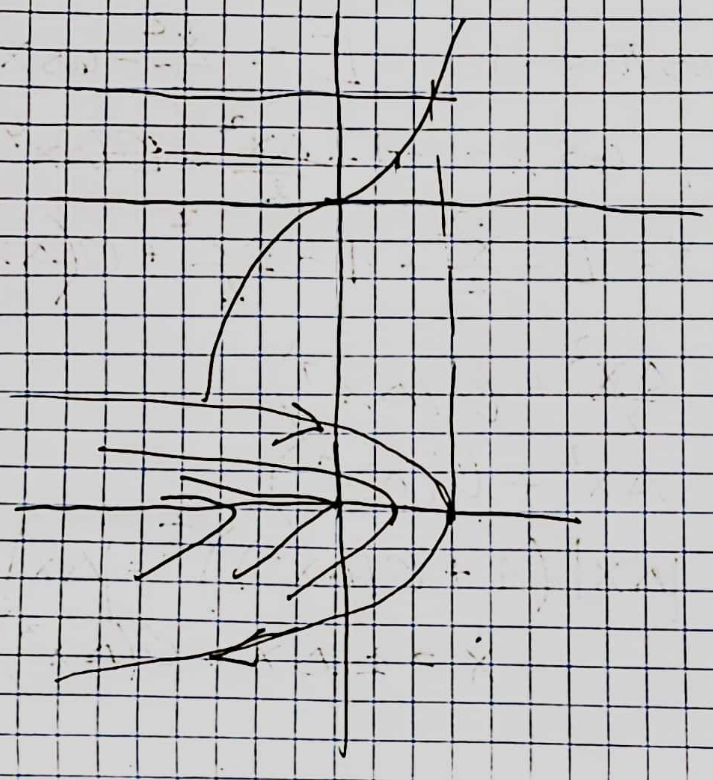
$$|\dot{x}| = \sqrt{2(E-U)}$$



Example

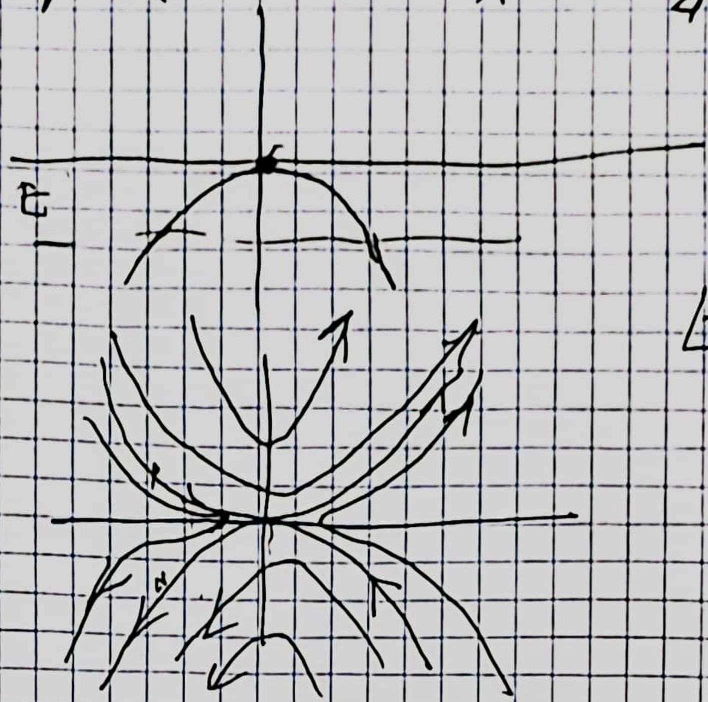
$$\ddot{x} = -3x^2 = -\frac{\partial}{\partial x}x^3, \quad U = +x^3$$

$$E = \frac{\dot{x}^2}{2} + x^3 = C$$



Example

$$\ddot{x} = x^3 = -\frac{1}{4}\frac{\partial}{\partial x} x^4$$



$$\frac{\dot{x}^2}{2} - \frac{x^4}{4} = E : \text{const}$$

$$E = 0 \Rightarrow \quad \dot{x} = \pm \frac{1}{\sqrt{2}} x^2$$

Example