Lecture 3

<u>Example</u> $y^2 + x^2 y' = xyy'$

$$y' = \frac{y^2}{xy - x^2}$$

$$y' = \frac{\left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right) - 1}, \quad \frac{y}{x} = z, \quad y = xz$$

$$y' = xz' + z = \frac{z^2}{z-1}$$

<u>checking</u>

$xy - x^2 = 0$

$x = 0$ $y = x$

not a solution

not a solution $x^2 + x^2 = x^2$

$$z' = \frac{1}{x}\left(\frac{z^2}{z-1} - z\right) = \frac{1}{x}\,\frac{z^2 - z^2 + z}{z-1} = \frac{1}{x}\,\frac{z}{z-1}$$

$$\frac{z-1}{z}\, z' = \frac{1}{x}$$

$z-1=0$ not a solution
(We don't acquire it)
$z=0$ is a pitaron
$\boxed{y=0}$

$$\int \left(1 - \frac{1}{z}\right) z'\,dx = \int \frac{dx}{x} = \ln|x| + C_1$$

$$\int \left(1 - \frac{1}{z}\right) dz = z - \ln|z| + C_2$$

$$\frac{1}{|z|} e^{z} = e^{C_1 - C_2} |x|$$

$$\frac{1}{z} e^{z} = \tilde{C}_1 x \qquad \tilde{C}_1 \neq 0 \quad , \; z=0 \; (y=0)$$

$$\frac{x}{y} e^{\frac{y}{x}} = \tilde{C}_1 x$$

Solution
$$\begin{cases} e^{\frac{y}{x}} - \tilde{C}_1 y = 0, \; \tilde{C}_1 \neq 0 \\ \text{or} \\ y = 0 \end{cases}$$

<u>Remark</u> ~~On the way we have got~~ we would get

$$y^2 dx + (x^2 - xy)\,dy = 0 \quad \text{for } y = y(x)$$

That equation has a solution $x = x(y) = 0$
which appears if y is considered independent

It is not a solution of the original DE

Exact ODE of the 1st order

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

is called <u>exact</u> if there is a function $F(x,y)$ such that

$$dF = M(x,y)\,dx + N(x,y)\,dy$$

In other words: $\frac{\partial F}{\partial x} = M(x,y)$, $\frac{\partial F}{\partial y} = N(x,y)$

In that case $F(x,y) = C$ is the solution
$C \in \mathbb{R}$

Rigorously:

<u>Proposition</u> Let M, N be continuous $M^2 + N^2 \neq 0$, then $F(x,y) = F(x, y(x))$ is a constant on any solution $y(x)$.

The same for $x(y)$

Proof: Let $N \neq 0 \Rightarrow \frac{M}{N} + y' = 0$
$y = y(x)$
$$y' = -\frac{M}{N}$$ in vicinity

$$\frac{d}{dx} F(x, y(x)) = M + N \cdot \left(-\frac{M}{N}\right) = M - M = 0$$

Let $M \neq 0$, $x = x(y)$ $\quad x' + \frac{N}{M} = 0$

in vicinity

$$\frac{d}{dy} F(x(y), y) = Mx' + N = -M\frac{N}{M} + N = 0$$

QED

<u>Example</u> $(3x^2 + y)dx + x\,dy = 0$

$$\frac{\partial}{\partial x}(x^3 + xy)dx + \frac{\partial}{\partial y}(x^3 + xy)dy = 0$$

Solution: $\boxed{x^3 + xy = C}$

How to check the exactness of the equation?

$$dF(x,y) = F'_x(x,y)\,dx + F'_y(x,y)\,dy$$

If $F''_{xx}, F''_{xy}, F''_{yx}, F''_{yy}$ are continuous $F \in C^2$

then $F''_{xy}(x,y) \equiv F''_{yx}(x,y)$ (known)

In that case $F'_x = M$, $F'_y = N$

implies $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Theorem Let M, N be C^1 (continuously differentiable) in $\mathbb{R}^2$. Then the DE is exact

No holes!

iff $M'_y(x,y) \equiv N'_x(x,y)$ $(*)$

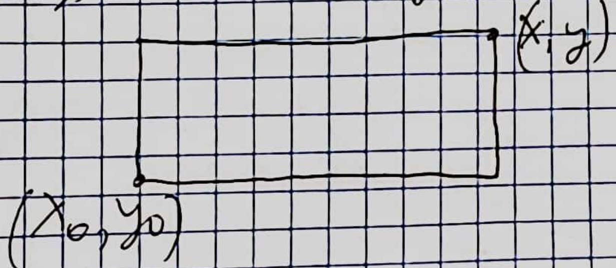
Proof 1. Exact $\Rightarrow$ $(*)$ already proved

2. Prove $(*)$ $\Rightarrow$ Exactness

Search for $F(x,y)$ or any primitive / function

$$F'_x = M(x,y) \Rightarrow F(x,y) = \int_{x_0}^{x} M(x,y)\,dx + \varphi(y)$$

unknown



Indeed $\left(F - \int_{x_0}^{x} M\,dx\right)'_x = 0$

$= M - M$

$\Rightarrow \varphi'_x = 0$

$$F'_y = \int_{x_0}^{x} M'_y(x,y)\,dx + \varphi'(y) = N(x,y)$$

$$\varphi'(y) = N(x,y) - \int_{x_0}^{x} M'_y(x,y)\,dx$$

<u>Claim</u> $\xi(x,y) = N(x,y) - \int_{x_0}^{x} M'_y(x,y)\,dx$

does not depend on x: $\xi(x,y) = \tilde{\xi}(y)$

<u>Indeed</u> $\xi'_x = N'_x - M'_y \equiv 0$

Then $\varphi(y) = \int_{y_0}^{y} \left[N(x,y) - \int_{x_0}^{x} M'_y(x,y)\,dx \right] dy + C$

$$\boxed{F(x,y) = \int_{x_0}^{x} M'_y(x,y)\,dx + \int_{y_0}^{y} \left[N - \int_{x_0}^{x} M'_y\,dx \right] dy + C}$$

QED (quod erat demonstrandum)
latin
"which was to be demonstrated"

<u>Example</u>

$$2xy\,dx + (x^2 - y^2)\,dy = 0$$

$$\frac{\partial}{\partial y}(\underset{\overset{\shortparallel}{M}}{2xy}) = 2x = \frac{\partial}{\partial x}(x^2 - y^2)$$

$F'_x = 2xy$, $F'_y = x^2 - y^2 = N$, $F = ?$

$F(x,y) = x^2 y + \varphi(y)$, $\cancel{\varphi(y)}$

$F'_y = x^2 + \varphi'(y) = N = x^2 - y^2$

$\varphi'(y) = -y^2$, $\varphi(y) = -\frac{y^3}{3} + C_1$

$$F(x,y) = x^2 y - \frac{y^3}{3} + C_1$$

Solution: $\boxed{x^2 y - \frac{y^3}{3} = C}$

(was already applied to the linear DEs)

Integration factor

(was already applied to linear DEs)

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

The function $\mu(x,y)$ is called the integration factor if

$\mu M\,dx + \mu N\,dy = 0$ is exact

i.e. $$\boxed{\frac{\partial}{\partial y}(\mu M) = \frac{\partial}{\partial x}(\mu N)}$$

$$\mu'_y M + \mu M'_y = \mu'_x N + \mu N_x$$

It is a PDE We leave it here.

Exchange of variables

Example

$$x\,dx = (x\,dy + y\,dx)\sqrt{1+x^2}$$

$$d\sqrt{1+x^2} = \frac{x\,dx}{\sqrt{1+x^2}} = x\,dy + y\,dx = d(xy)$$

i.e. $\mu = \dfrac{1}{\sqrt{1+x^2}}$ - integration ~~varial~~ factor

Solution

$$\boxed{xy - \sqrt{1+x^2} = C}$$

Example $(x^2+y^2+1)yy' + (x^2+y^2-1)x = 0$

$$\frac{1}{2}\left[(x^2+y^2)(y^2)' + (y^2)' + (x^2+y^2)(x^2)' - (x^2)'\right] = 0$$

$$(x^2+y^2)(x^2+y^2)' + (y^2-x^2)' = 0$$

Solution $\frac{1}{2}(x^2+y^2)^2 + y^2 - x^2 = C$ First Integral!

Look at the system of ODEs

$$\begin{cases} \dot{x} = f(x,y)\,\xi(t,x,y) \\ \dot{y} = g(x,y)\,\xi(t,x,y) \end{cases} \qquad \begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \end{cases}$$

initial conditions

$f \neq 0$, $\xi \neq 0$
do not vanish

Then the solution $(x(t), y(t))$ has the same path as

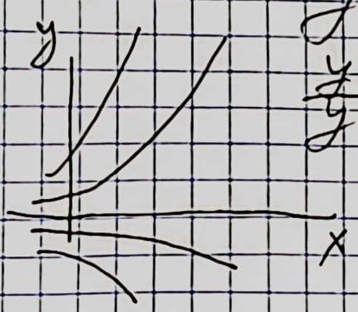
$$y' = \frac{g(x,y)}{f(x,y)}, \quad y(x_0) = y_0$$

Proof: $t'_x = \frac{1}{f\xi}$ Theorem on inverted function

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = g\xi \frac{1}{f\xi} \qquad \text{QED}$$

Example $\begin{cases} \dot{x} = \frac{1}{x}(x^2+y^2t) \\ \dot{y} = xy(x^2+y^2t) \end{cases}$ to find the curves $t \geq 1$

$y' = x^2 y$ $\quad y \neq 0$ $\qquad y = 0$ solution



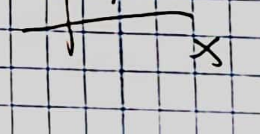
$\frac{y'}{y} = x^2$ $\qquad \ln|y| = \frac{x^3}{3} + C_1$

$$\boxed{y = C e^{\frac{x^3}{3}} \quad C \in \mathbb{R}}, \quad C = \pm e^{C_1}, 0$$

Inverse application

$$\frac{dy}{dx} = \frac{ax+by+c}{dx+ey+f}, \quad dx+ey+f \neq 0 \quad \Rightarrow \quad \begin{cases} \dot{x} = dx+ey+f \\ \dot{y} = ax+by+c \end{cases}$$

the same trajectories!



$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = y'_t\, t'_x$$

System of DEs

The unknown function is $t \mapsto y(t) \in \mathbb{R}^n$ vector

or $x \mapsto y(x)$

$$y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{pmatrix} \text{ is to be found.}$$

Example

$$\begin{cases} x^2 y_1'' + y_2' - y_1 y_2' + \cos(x y_1) = 0 \\ y_2' + y_1 = \operatorname{arctg}(y_1 - y_2) \end{cases} \qquad y = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix} = ?$$

One can always rewrite a system of DEs as a system of 1-st order DEs

Equivalency of a 1-st order system ODEs and one high-order ODE

$$f(y^{(n)}, y^{(n-1)}, \ldots, \dot{y}, y, t) = 0 \qquad y(t) = ?$$

Denote $y_0 = y$, $y_1 = \dot{y}$, ..., $y_{n-1} = y^{(n-1)}$, get:

$$\begin{cases} \dot{y}_0 = y_1 \\ \dot{y}_1 = y_2 \\ \ldots \\ \dot{y}_{n-2} = y_{n-1} \\ f(\dot{y}_{n-1}, y_{n-1}, y_{n-2}, \ldots, y_1, y_0, t) = 0 \end{cases} \qquad \vec{y}(t) = \begin{pmatrix} y_0(t) \\ y_1(t) \\ \vdots \\ y_{n-1}(t) \end{pmatrix} = ?$$

"Theorem"

Obviously each solution $y(t)$ of the DE generates the solution $y_0(t) = y(t)$, $y_1(t) = \dot{y}(t)$, ...

and vice versa $\vec{y}(t)$ generates solution $y = y_0(t)$

example $\cos(xy'' - (y')^3) = 0$ (order 2)

$\Leftrightarrow$ $y_0 = y,\ y_1 = y'$

$$\begin{cases} y_0' = y_1 \\ \cos(x y_1' - (y_1)^3) = 0 \end{cases}$$

standard form:

$$\dot{y} = f(t, y),\ y \in \mathbb{R}^n \quad (\text{or } y' = f(x, y))$$

$$\begin{cases} \dot{y}_1 = f_1(t, y_1, y_2, \dots, y_n) \\ \dot{y}_2 = f_2(t, y_1, y_2, \dots, y_n) \\ \dots \\ \underbrace{\dot{y}_n}_{y} = \underbrace{f_n(t, y_1, y_2, \dots, y_n)}_{f} \end{cases}$$

Initial condition

$$y(t_0) = \xi$$

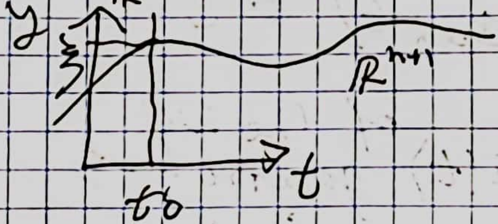
or

$$\begin{cases} y_1(t_0) = \xi_1 \\ y_2(t_0) = \xi_2 \\ \dots \\ y_n(t_0) = \xi_n \end{cases}$$

Solution $y(t)$ corresponds to its trajectory $y(t) \in \mathbb{R}^n$ in the phase space $y \in \mathbb{R}^n$

or $(t, y(t)) \in \mathbb{R}^n$ in the extended phase space $\mathbb{R}^{n+1}$

Synonims: phase = state



Natural understanding of the standard form: "GPS"

Some oracle defines the velocity $\dot{y}$ at each point and time

The "Ways" theorem: a solution exists and is unique!

The Cauchy Problem:

$$\begin{cases} \dot{y} = f(t, y), & y \in \mathbb{R}^n \\ y(t_0) = \xi \in \mathbb{R}^n \end{cases}$$

Initial condition

$y(t) = ?$

Correspondingly the Cauchy problem for a scalar DE

$$\Phi(y^{(n)}, y^{(n-1)}, \dots, \dot{y}, y, t) = 0, \quad y \in \mathbb{R}$$

has the initial conditions

$$\underbrace{y(t_0) = \xi_0, \ \dot{y}(t_0) = \xi_1, \dots, y^{(n-1)}(t_0) = \xi_{n-1}}_{n \text{ scalar conditions}}$$

Theorem of existence and uniqueness

$\exists!$ Theorem (not the final formulation)

1. $f(t, y)$ is continuous in t, y $\Rightarrow \exists$ local solution
2. $f(t, y)$ is continuously differentiable with respect to y $\Rightarrow$ the solution is unique

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$\Rightarrow$ Simple corollaries:

1. Initial conditions define each solution

Thus the general solution of the system $\dot{y} = f(t, y)$, $y \in \mathbb{R}^n$ or of the nth order DE $\Phi(y^{(n)}, \dots, \dot{y}, y, t) = 0$ has to have exactly n scalar parameters

2. In all previous 1st order examples and methods we actually have found <u>all</u> solutions, since they had exactly 1 parameter and each initial condition could be satisfied. There are still additional singular solutions at the points where the $\exists!$ theorem does not hold.

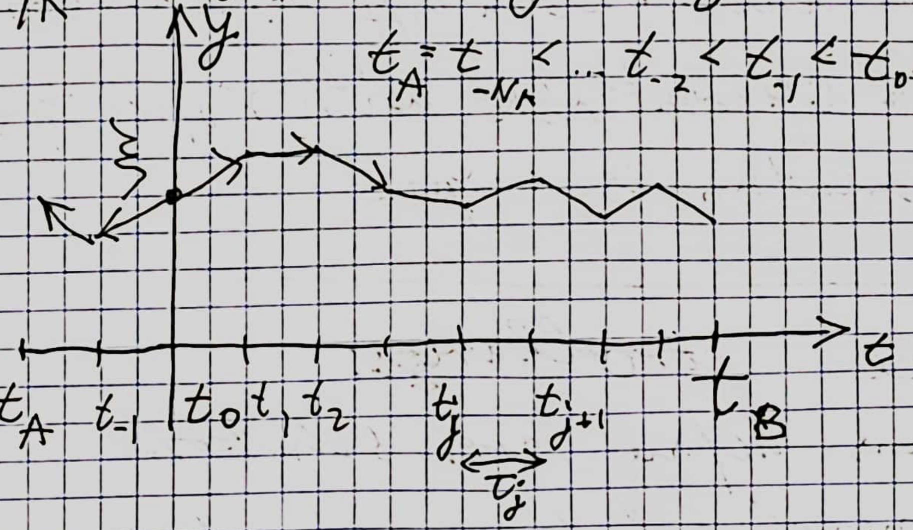
The theorem of $\exists!$ is proved by approximation of solutions, ~~which is proved to~~ converging to them.

The Euler integration method

$$\dot{y} = f(t,y),\quad y(t_0) = \xi,\quad y \in \mathbb{R}^n$$

$$t_A = t_{-N_A} < \dots t_{-2} < t_{-1} < t_0 < t_1 < t_2 < \dots < t_{N_B} = t_B$$

$$\tau_j = t_{j+1} - t_j < \tau$$



$$y(t_0) = \xi,\quad y(t_1) = y(t_0) + f(t_0, y(t_0))\,\tau_1, \dots$$

$j \geq 0$ $\quad y(t_{j+1}) = y(t_j) + f(t_j, y(t_j))\,\tau_j$ forwards

$j \leq 0$ $\quad y(t_j) = y(t_{j+1}) - f(t_{j+1}, y(t_{j+1}))\,\tau_j$ backwards

$$\boxed{y(t_{**}) = y(t_*) + f(t_*, y(t_*))(t_{**} - t_*)}$$