Lecture 2

1 We'll see that ~~O~~ODE in the normal form $\dot{y} = f(t, y)$, $y \in \mathbb{R}^n$

almost always has a ~~unique~~ solution

y is called the phase variable, state

$y \in \mathbb{R}^n$ – the phase space, state space

(t, y) is called the extended phase

$\in \mathbb{R}^{n+1}$ the extended state

The Cauchy Problem

$$\begin{cases} \dot{y} = f(t, y), & y \in \mathbb{R}^n, \ t \in \mathbb{R} \\ y(t_0) = \xi \end{cases}$$

Standard form

initial condition

If f is smooth then a solution locally exists and is unique (will be proved)

Sometimes ODE of the first order is written in the form

$$M(x, y)\,dx + N(x, y)\,dy = 0$$

It means that x and y have exchangeable meaning.

One may search for $y(x)$

or for $x(y)$

If x is independent the $dy = y'dx$

otherwise $dx = x'(y)\,dy$

$M(x,y)dx + N(x,y)y'dx = 0$ | $M(x,y)x'dy + N(x,y)dy = 0$

$M(x,y) + N(x,y)y' = 0$ | $M(x,y)x' + N(x,y) = 0$

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Methods of solving Elementary ODE of 1st order

1. ODE with separable variables

Separable ODEs

$$y' = f(x)g(y)$$

check: $g(y) = 0 \Rightarrow y = \xi_1, \xi_2, \dots$ values

$\Rightarrow \boxed{y(t) = \xi_i \text{ - constant solutions}}$

Suppose $y \neq \xi_i$, $i = 1, 2, \dots \Rightarrow g(y) \neq 0$

$$\frac{y'}{g(y)} = f(x) \qquad \text{Recall: } y = y(x)$$

$$\int \frac{y'(x)\,dx}{g(y(x))} = \int f(x)\,dx = F(x) + C_2$$

$$\int \frac{dy}{g(y)} = G(y) + C_1$$

$\Rightarrow$ $G(y) = F(x) + C$ $\quad (C = C_2 - C_1)$

General Solution:

Thus $I(x,y) = G(y) - F(x) = C = \text{const}$

A function which is kept constant over solutions of the ODE is called the first Integral

It is suggested to express $y = y(x)$ but it is not required

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<u>Example</u> $y' = \alpha y$ $\quad \alpha \in \mathbb{R}$ $\quad y = 0$ solution

$$\frac{y'}{y} = \alpha$$

$$\int \frac{dy}{y} = \int \frac{y'dx}{y} = \int \alpha\, dx = \alpha x + c_2$$

$$\ln|y| + c_1 \qquad \ln|y| = \alpha x + \underbrace{c_2 - c_1}_{c}$$

$$y = 0, \quad |y| = \hat{c}e^{\alpha x}, \quad \hat{c} = e^{c_2 - c_1}$$

$$\Rightarrow \boxed{y = \tilde{c}e^{\alpha x}}$$ the general solution

<u>Example</u>

$$\dot{N} = (\lambda - \mu)N - \varepsilon N^2 \quad \text{population}$$

particular solutions: $N = 0$, $N = \frac{\lambda - \mu}{\varepsilon}$

Let $N \neq 0$

$$\int \frac{dN}{N(\lambda - \mu - \varepsilon N)} = \int \frac{\dot{N}\, dt}{(\lambda - \mu)N - \varepsilon N^2} = \int dt = t + c_1$$

$$= \frac{1}{\tilde{\lambda}} \int \left[\frac{1}{N} + \frac{1}{\frac{\tilde{\lambda}}{\varepsilon} - N}\right] dN$$

$$\tilde{\lambda} = \lambda - \mu > 0$$

$$c_2 + \ln|N| - \ln\left|N - \frac{\tilde{\lambda}}{\varepsilon}\right| = \tilde{\lambda} t + \tilde{\lambda} c_1$$

$$\ln\left|\frac{N}{N - \frac{\tilde{\lambda}}{\varepsilon}}\right| = \tilde{\lambda} t + c, \quad c = \tilde{\lambda} c_1 - c_2$$

$$\frac{N}{\left|N - \frac{\tilde{\lambda}}{\varepsilon}\right|} = \tilde{c}e^{\tilde{\lambda} t} \qquad N = 0,\ N = \frac{\tilde{\lambda}}{\varepsilon} \text{ solutions}$$

$$N > \frac{\tilde{\lambda}}{\varepsilon} \Rightarrow N = \frac{\tilde{\lambda}}{\varepsilon} \frac{1}{\underbrace{1 - \frac{1}{c} e^{-\tilde{\lambda} t}}_{>1}}$$

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$$N < \frac{\tilde{\lambda}}{\varepsilon} \Rightarrow N = \frac{\tilde{\lambda}}{\varepsilon} \frac{1}{1 + \frac{1}{c} e^{-\tilde{\lambda} t}}$$

Solutions: (General Solution)

$$\begin{cases} N = \frac{\lambda - \mu}{\varepsilon} \frac{1}{1 + C_3 e^{-(\lambda-\mu)t}}, \quad C_3 \in \mathbb{R} \\ N = 0 \end{cases}$$

Linear ODE of the 1st order

$y' + a(x) y = b(x)$ non homogeneous DE

Method of the parameter variation

Step I: $y' + a(x) y = 0$ (homogeneous DE)

$y \neq 0 \Rightarrow \frac{y'}{y} = -a(x)$ $\qquad y = 0$ – solution

$$\ln|y| = -\int a(x)dx = -\int_{x_0}^{x} a(x)dx + C$$

$$y = \pm e^{C} e^{-\int_{x_0}^{x} a(x)dx}, \quad y = 0$$

$$y = \tilde{C} \underbrace{e^{-\int_{x_0}^{x} a(x)dx}}_{y_h(x)} \quad \text{general solution}$$

Step 2: Variation of the coefficient

Let $y(x) = \tilde{C}(x) e^{-\int_{x_0}^{x} a(x)dx} = \tilde{C}(x) y_h(x)$

$y_h(x) = e^{-\int \dots dx} \neq 0$ $\qquad \tilde{C}(x)$ – differentiable

does not vanish!

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$$y' = \tilde{c}' y_{hp} + \tilde{c} y'_{hp}$$ $\qquad \underline{y'_{hp} + a(x) y_{hp} = 0}$

substitute into ODE

$$\tilde{c}' y_{hp} + \underbrace{\tilde{c} y'_{hp} + a \tilde{c} y_{hp}}_{= 0} = b$$

y_{hp} - a particular homogeneous solution

$$\tilde{c}'(x) y_{hp}(x) = b(x)$$

$$\tilde{c}'(x) = \frac{b(x)}{y_{hp}(x)}$$

$\ne 0$

$$\tilde{c}(x) = \int_{x_0}^{x} \frac{b(x)}{y_{hp}(x)} dx + C_1$$

$$y = y_{hp}(x) \int_{x_0}^{x} \frac{b(x)}{y_{hp}(x)} dx + C_1 y_{hp}(x),$$

$C_1 \in \mathbb{R}$

General Solution! is the set of ALL solutions

Example $\quad y' - 2y = 4 - x, \quad y(0) = 0$

Init. condition

1. $y' - 2y = 0 \Rightarrow y = ce^{2x}, \quad c \in \mathbb{R}$

2. $y = c(x) e^{2x}, \quad c(x) = ?$

$$c' e^{2x} + \underbrace{c \cdot 2e^{2x} - 2ce^{2x}}_{= 0} = 4 - x$$

$$c'(x) = (4 - x) e^{-2x}$$

$$C(x) = \int (4e^{-2x} - x e^{-2x})\,dx$$

$$= -2e^{-2x} - \int \underbrace{x}_{u} \underbrace{e^{-2x}\,dx}_{dv} \quad \underset{\text{by parts}}{=}$$

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$$= -2e^{-2x} + \frac{1}{2} x e^{-2x} - \frac{1}{2}\int e^{-2x}\,dx$$

$$= -2e^{-2x} + \frac{1}{2} x e^{-2x} + \frac{1}{4} e^{-2x} + C_1$$

$$\boxed{y = C(x)\, e^{2x} = \frac{1}{2}x - 1\frac{3}{4} + C_1 e^{2x}}$$ General Solution

$$y(0) = 0 \Rightarrow \boxed{C_1 = 1\frac{3}{4},\ y = \frac{1}{2}x - 1\frac{3}{4} + 1\frac{3}{4} e^{2x}}$$

The Cauchy Problem Solution

The Integration factor

Second Method

$$y' + a(x)\, y = b(x)$$

1. Multiply by $\mu(x)$ (unknown!)

$$\underbrace{\mu(x) y' + a(x)\mu(x) y}_{\text{let it be } [\mu(x) y]'} = \mu(x) b(x)$$

let it be $[\mu(x) y]' = \mu y' + \mu' y$

$$\Rightarrow \mu' = a(x)\mu \qquad \frac{\mu'}{\mu} = a(x),\ \mu = e^{\int_{x_0}^{x} a(x)\,dx} > 0$$

one solution is enough

$$\mu = e^{A(x)}$$

2. $[\mu(x) y]' = b(x)$, $\mu y = \int_{x_0}^{x} b(x)\,dx + C_1$

$$y(x) = \frac{1}{\mu(x)} \int_{x_0}^{x} b(x)\,dx + C_1 \frac{1}{\mu(x)}$$

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$$y = e^{-A(x)} \int_{x_0}^{x} e^{A(x)} b(x)\,dx + C_1 e^{-A(x)}$$

It is Easy to check that $\mu = \frac{1}{y_{LP}}$

Example (the same)

$$y' - 2y = 4 - x$$

$$\mu y' - 2\mu y = (4-x)\mu \qquad \mu \neq 0$$

$$\mu' = -2\mu, \quad \mu = e^{-2x}, \text{ substitute}$$

$$(e^{-2x} y)' = (\mu y)' = (4-x)\mu = (4-x)e^{-2x}$$

$$e^{-2x} y = \int (4-x) e^{-2x} dx$$

$$y = e^{2x} \underbrace{\int (4-x)\, e^{-2x} dx}_{C(x)} = -\tfrac{1}{2}x - 1\tfrac{3}{4} + C_1 e^{2x}$$

Bernoulli Equation

$$y' + a(x)y = b(x)y^n \qquad n \neq 1, 0$$

(already ~~seen~~ studied)

$$\frac{y'}{y^n} + a(x)\frac{y}{y^n} = b(x) \qquad y \equiv 0 \text{ solution}$$

$$Z = y^{-(n-1)}, \quad Z' = -(n-1)\frac{1}{y^n} y'$$

$$-\frac{1}{n-1} Z' + a(x) Z = b(x) \quad \text{already known}$$

Example $\dot{N} = -(\lambda-\mu)N - \varepsilon N^2, \quad Z = \frac{1}{N}$

$$-\frac{\dot{N}}{N^2} + (\lambda-\mu) N^{-1} = \varepsilon \qquad \boxed{\dot{Z} + (\lambda-\mu)Z = \varepsilon},$$

$$Z = \frac{\varepsilon}{\lambda-\mu} + Ce^{-(\lambda-\mu)t}, \quad N = \frac{1}{Z}$$

Homogeneous Equation

$$M(x,y)\,dx + N(x,y)\,dy = 0$$

The homogeneity degree s:

$\exists s \in \mathbb{R}\ \forall k > 0\ \forall x, y \in \mathbb{R}$

$$M(kx, ky) = k^s M(x,y)$$
$$N(kx, ky) = k^s N(x,y)$$

Examples:

$s=1$: $(x+y)\,dx + y\,dy = 0$

$s=2$: $(x^2+xy)\,dx + \frac{y^3}{x}\,dy = 0$, $x \neq 0$

$s=3$: $y^3\,dx + xy^2\,dy = 0$

$s=0$: $tg\,\frac{x}{y}\,dx + \ln\frac{x}{y}\,dy = 0$, $y, x \neq 0$, $yx > 0$

$s=-1$: $\frac{x+y}{x^2+y^2}\,dx + \frac{y^7}{x^8}\,dy = 0$

$s=\frac{1}{3}$: $(x+y)^{\frac{1}{3}}\,dx + \frac{(x^2+y^2)^{\frac{1}{3}}}{(x-y)^{\frac{1}{3}}}\,dy = 0$

$$y' = f\left(\frac{y}{x}\right), \quad x \neq 0$$

Example:

$$y' = \sin\frac{y}{x}$$

$$f\left(\frac{y}{x}\right)dx - dy = 0$$

$s = 0$, $\quad dy = y' \cdot dx$

x is the independent variable

Claim (almost a proposition/theorem)

The both forms are equivalent (almost)

Let x be the independent variable, $y = y(x)$

check that $x = \xi_1, \xi_2, \ldots, \xi_k, \ldots$ constants which satisfy $N(\xi, y) \equiv 0 \Rightarrow x = x(y) = \xi_k$ - a solution $dx = 0$

Now $x \neq \xi_1, \xi_2, \ldots$, $\quad dy = y'\,dx$ (actually $dx = \Delta x$)

The ~~identity~~ ODE is true for any $dx \Rightarrow$ cancel it

$$M(x,y) + N(x,y)\,y' = 0, \quad y' = -\frac{M(x,y)}{N(x,y)}$$

$$y' = \frac{M\left(1, \frac{y}{x}\right) \cdot x^3}{N\left(1, \frac{y}{x}\right) \cdot x^3}$$

$$\boxed{y' = \frac{M\left(1, \frac{y}{x}\right)}{N\left(1, \frac{y}{x}\right)}}$$

actually $x > 0$ should be required ☹ ignore

Backwards $\boxed{(QED)}$ $y' = f\left(\frac{y}{x}\right) \Rightarrow f\left(\frac{y}{x}\right)dx - dy = 0$

~~So~~ Solution: $z = \frac{y}{x}$, $y = xz$, $y' = z + xz'$

$$z + xz' = f(z)$$

$$xz' = f(z) - z$$

$$\frac{z'}{f(z) - z} = \frac{1}{x}$$

$f(z) - z = 0$

$z = \xi_1, \xi_2, \dots$

$y = \xi_i \cdot x$ solutions

$$F(z) + C_2 = \int \frac{dz}{f(z) - z} = \int \frac{z' dx}{f(z) - z} = \int \frac{dx}{x} = \ln|x| + C_1$$

Solutions
$$\begin{cases} \ln|x| = F(z) + \tilde{C} & \tilde{C} = C_2 - C_1 \\ \ln|x| - F\left(\frac{y}{x}\right) = \tilde{C} & \\ \frac{y}{x} = \xi_1, \xi_2, \dots & y = \xi_1 x, \xi_2 x, \dots \end{cases}$$

Example $y^2 + x^2 y' = xyy'$

checking

$$y' = xz' + z = \frac{z^2}{z-1}$$