Ordinary Differential Equations-1

ODE-1, 2023-24

Thursday, Shenkar Physics-104
$11^{10} - 14^{00}$

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Lecture 1. ODE 1 04/01-2024

Ordinary Differential Equations

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Reception hours: Sunday $15^{10}-16^{00}$

site at my homepage

http://www.tau.ac.il/~levant/ode1

Literature:

Arnold V.I. ODE

Boyce, DiPrima Elementary DEs and Boundary Value Problems

Newton: Anagram ~ "It's useful to solve DEs"

DEs describe gradual changes of some object parameters. The numeral parameters are assumed determining the object.

The idea is that the changing law allows to completely determine the object.

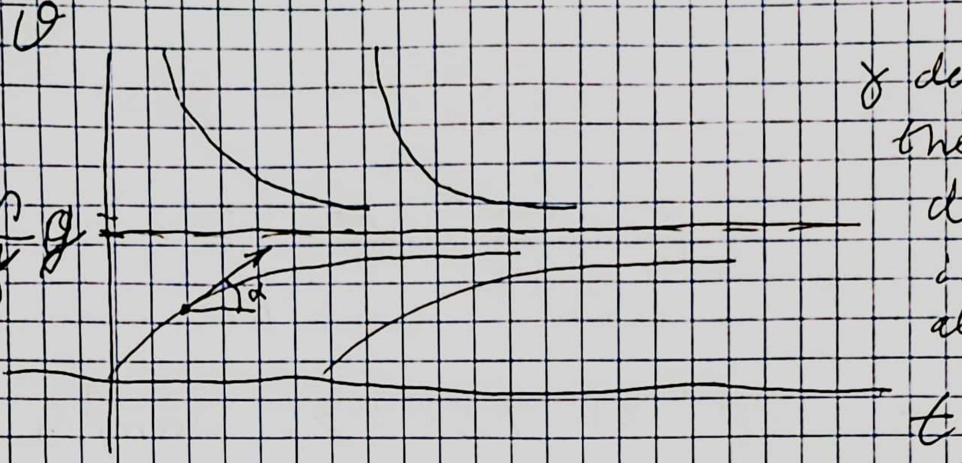
Example Free fall

$$\frac{dv}{dt} = \underbrace{mg}_{\text{gravitational force}} - \underbrace{\gamma v}_{\text{friction}}$$

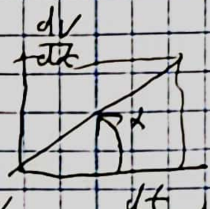
v - velocity
t - time
m - mass [kg]
g - acceleration $9.8\ \frac{m}{s^2}$
γ - friction coefficient

②

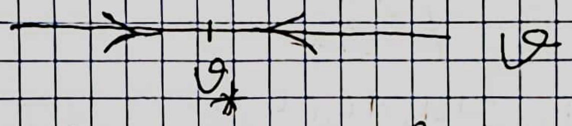
$\vartheta_* = \frac{m}{\gamma} g$



γ depends on the atmosphere density, i.e. the altitude



$tg\,\alpha = \frac{dV}{dt}$ $\quad \frac{dt}{dt} = 1$



$\vartheta(t) = \vartheta_*$ solves the equation

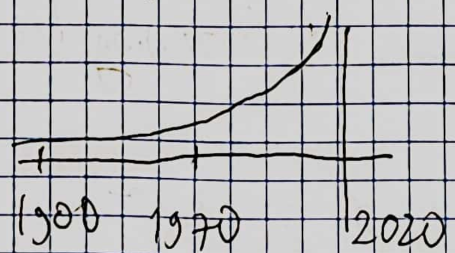
$$\frac{d\vartheta}{dt} = 0 = mg - \gamma \underbrace{\frac{mg}{\gamma}}_{\vartheta_*}$$

Example Human population of the Earth

Let N be the number of people.

$N \in \mathbb{N} = \{1, 2, 3, \dots\}$, but if measured in millions becomes float $N \in \mathbb{R}$, $N > 0$

1960–1970: It was claimed that $N \to \infty$ at $t = 2020$



$$N = \frac{k}{2020 - t}$$

Since $N(2019) = 7000 \cdot 10^6$
$k = 7000$ mln

$$\frac{dN}{dt} = \frac{d}{dt}\frac{k}{2020-t} = k\frac{1}{(2020-t)^2} = \frac{1}{k}N^2$$

$N^2 \sim$ the number of pairs. Each time a meeting of a man and a woman takes place $\Rightarrow$ they reproduce, It is not realistic. It means that in average each woman has unbounded number of children (proportional to N) 3

In reality the number of children varies in $[0, 20]$ (exaggerated)

$$\frac{dN}{dt} = (\lambda - \mu) N = 0.01 N$$

grows indefinitely

birth rate 0.02 mortality 0.01

Suppose there is some illness, infection like a common cold, Rhinitis (usual corona) with small mortality rate

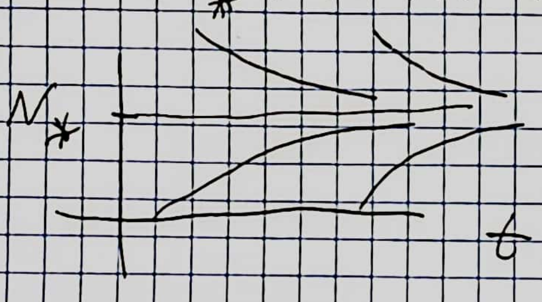
$$\frac{dN}{dt} = (\lambda - \mu) N - \varepsilon N^2$$

, food is unbounded

$N(t) = N_*$ solution, $N_* = \frac{\lambda - \mu}{\varepsilon}$

$N_* \approx 10\,000$ mln $\Rightarrow 10^{10} = 10^{-2}/\varepsilon$

$$\boxed{\varepsilon = 10^{-12}}$$



We have already seen the picture

We will see that $v \to v_*$, $N \to N_*$ asymptotically, but never gets the value.

We see that the future can be predicted due to the DE math. model

Wrong model leads to wrong prediction

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Some notions and notation

1. $\mathcal{O}$ (Big o), o (little o) — Landau notation

$f, g : \mathbb{R} \to \mathbb{R}$, $x \in \mathbb{R}$, $a \in \mathbb{R} \cup \{\infty\} \cup \{-\infty\}$

In the context one has always to know

$$x \to a$$

$f(x) = o(g(x))$, $f = o(g)$

means that f is infinitesimally small compared to g in the vicinity of a.

Rigorously:
$$V_\varepsilon(a) = \begin{cases} (a-\varepsilon, a+\varepsilon) & a \in \mathbb{R} \\ (\varepsilon, \infty) & a = \infty \\ (-\infty, -\varepsilon) & a = -\infty \end{cases} \qquad \varepsilon > 0$$

$$f = o(g) \Leftrightarrow \forall \delta > 0\ \exists \varepsilon > 0:$$
$$x \neq a,\ x \in V_\varepsilon(a) \to |f(x)| \leq \delta |g(x)|$$

if $g(x) \neq 0 \Rightarrow \lim\limits_{x \to a} \frac{f(x)}{g(x)} = 0$

$f(x) = \mathcal{O}(g(x))$ means that f and g are of the same order of magnitude

$$\exists c > 0\ \exists \varepsilon > 0:$$
$$x \neq a,\ x \in V_\varepsilon(a) \longrightarrow |f(x)| \leq c\,|g(x)|$$

"=" is actually "belongs to" ~~to~~ the set of functions

Examples: $\boxed{\text{Let } x \to 0}$ then

$\ln|x| = o\left(\frac{1}{x}\right) \in O\left(\frac{1}{x}\right)$, $\ln|x| \in O\left(\frac{1}{x}\right)$

set! set!

$1 = o(\ln|x|)$, $1 = O(\ln|x|)$

$1 = O(1000)$, $1 \neq o(1000)$

$\cos x \neq O(\sin x)$: $\nexists C$ $\left|\frac{\cos x}{\sin x}\right| \leq C$, $x \to 0$

$\sin x = o(\cos x)$, $\sin x = O(\cos x)$

$x^7 = o(x^3)$, $x^3 = o(1)$, $x^7 = O(1)$

Denote $f = o(g)$ as $f < g$

$\frac{1}{|x|^k} < \frac{1}{x^3} < \frac{1}{x^2} < \frac{1}{x} < \ln|x| < x^2 < x < 1$

$k > 3$

$e^{-\frac{1}{|x|}} < x < |x|^{\frac{1}{2}} < 1$

$\boxed{x \to \infty}$

$\cos x = O(1)$, $\sin x = O(1)$

$\cos x \neq O(\sin x)$, $\sin x \neq O(\cos x)$

$1000 < \ln x < x^{\frac{1}{2}} < x < x^k < e^x$

$k > 1$

$\cos x = o(e^x)$

2. Differential, derivative

$f'(x_0) = f'_x(x_0) = \frac{df}{dx}(x_0)$, $f'(t_0) = \frac{df}{dt}(t_0) = \dot{f}(t_0)$

(from physics)

$f'_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0)$

I single out "t" (time)

(6) $$\frac{df(x_0)}{dx} = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x}$$

$$\forall \delta > 0 \ \exists \varepsilon > 0 : \ \underset{\Delta x \neq 0}{\Delta x \in V_\varepsilon(0)} \Rightarrow \left| \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} - f'(x_0) \right| < \delta$$

$$\Leftrightarrow \left| \frac{f(x_0+\Delta x)-f(x_0) - f'(x_0)\Delta x}{\Delta x} \right| < \delta$$

$$\Leftrightarrow f(x_0+\Delta x)-f(x_0)-f'(x_0)\Delta x = o(\Delta x)$$

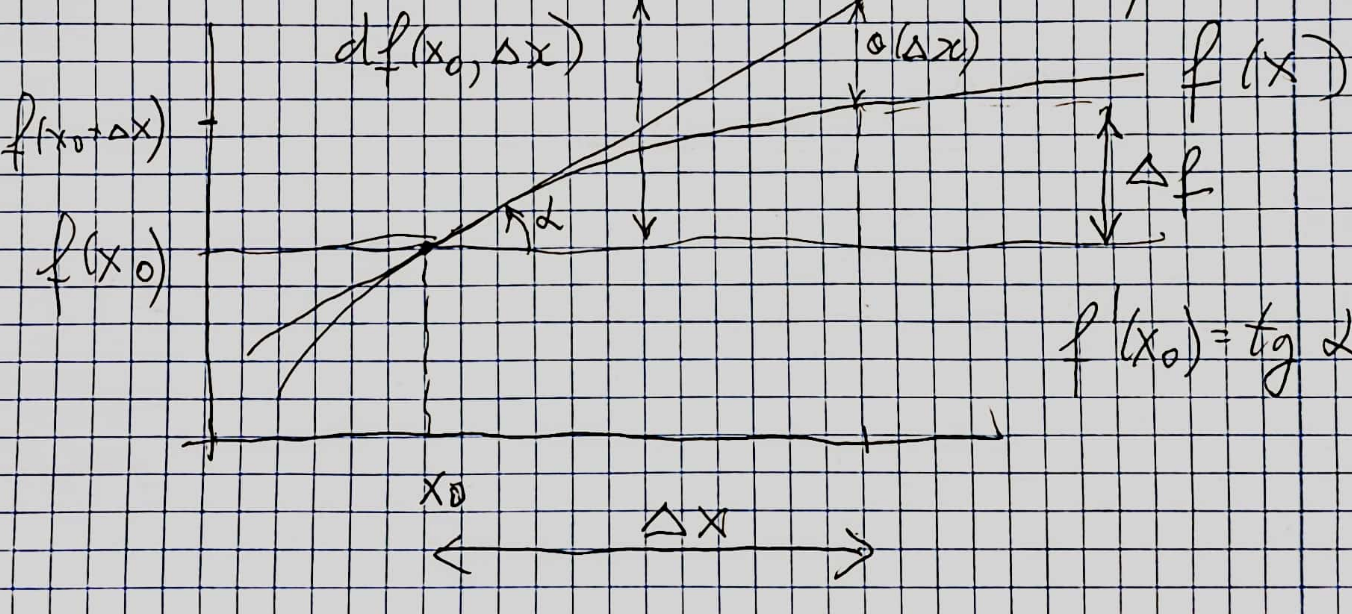
$$\boxed{\underbrace{f(x_0+\Delta x)-f(x_0)}_{\Delta f} = f'(x_0)\Delta x + o(\Delta x)}$$

Equivalent definition

differential
$$\boxed{df = df(x_0, \Delta x) \stackrel{def}{=} f'(x_0)\Delta x}$$

$$\boxed{\Delta f = df(x_0, \Delta x) + o(\Delta x)}$$

$\Delta x \in \mathbb{R}$ is not small! $\Delta x = 10000$ is acceptable



$f'(x_0) = tg\,\alpha$

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Leibnitz: df is written instead of $df(x, \Delta x)$
$df(x)$

Then if x is an independent variable

x is also the function $f(x) \equiv x$

$$\Rightarrow \quad dx = \frac{dx}{dx} \cdot \Delta x = 1 \cdot \Delta x = \Delta x$$

$$\Rightarrow \quad df = f'(x)\Delta x = f'(x)dx$$

$$\boxed{df(x) = f'(x)dx}$$

Excellent Grand!

Suppose $x = x(t)$, t_0 - a value of t

$$df = \frac{d}{dt} f(x(t_0)) \cdot \underset{dt}{\underset{\parallel}{\Delta t}} = \frac{df}{dx}(x(t_0)) \underbrace{\frac{dx}{dt}(t_0) \cdot \Delta t}_{dx}$$

$$\Delta x = \dot{x}(t_0)\Delta t + o(\Delta t)$$

in different sense!

$$df = \frac{df}{dx}(x(t_0))\dot{x}(t_0)dt$$

$$df(\overset{x(t_0)}{\overset{\parallel}{x_0}}, \Delta x) - df(x(t_0)) = f'_x(x_0)\, o(\Delta t) = o(\Delta t)$$

Brilliant mockery! (Leibniz)

A few variables:

$$\Delta f(x, y) = \underbrace{f'_x(x, y)\overset{dx}{\overset{\parallel}{\Delta x}} + f'_y(x, y)\overset{dy}{\overset{\parallel}{\Delta y}}}_{df(x, y, \Delta x, \Delta y)} + o(\Delta x) + o(\Delta y)$$

<u>Lagrange Theorem</u>: $f(x + \Delta x) = f'(x + \theta\Delta x)\Delta x$
$= df(x + \theta\Delta x, \Delta x)$

f' is ~~continuous~~ exists between x, $x+\Delta x$, $\theta \in (0, 1)$

Taylor

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$f: \mathbb{R} \to \mathbb{R}$, differentiable k times in $[x, x+\Delta x]$ or $[x+\Delta x, x]$

$$f(x_0+\Delta x) - f(x_0) = \frac{df(x_0, \Delta x)}{1!} + \dots + \frac{d^k f(x_0, \Delta x)}{k!} + o(\Delta x^k)$$

Peano Remainder

if $f^{(k+1)}$ exists between x, $x+\Delta x$ then

$$\Delta f = \frac{df}{1!} + \dots + \frac{d^k f}{k!} + \frac{d^{k+1} f}{(k+1)!}(x_0+\theta\Delta x, \Delta x)$$

$\theta \in (0,1)$

the remainder in the Lagrange form

Here $d^2 f = d(f' \Delta x) = df' \cdot \Delta x = f'' \Delta x^2$

$d^\ell f(x_0, \Delta x) = d^\ell f = f^{(\ell)}(x_0) \Delta x^\ell$

many variables

$$d^2 f(x, y, \Delta x, \Delta y) = d(f'_x \Delta x + f'_y \Delta y) =$$

$$= df'_x \Delta x + df'_y \Delta y = f''_{xx} \Delta x^2 + 2 f''_{xy} \Delta x \Delta y + f''_{yy} \Delta y^2$$

$$f''_{xy} = f''_{yx}$$

$f: \mathbb{R}^m \to \mathbb{R}$ both Taylor expansions hold

$f: \mathbb{R}^m \to \mathbb{R}^n$ only the Peano form

Types of DEs

DE is an equation for a function $y: \mathbb{R}^n \to \mathbb{R}^n$ to be found. It can be a system of equations containing the ~~F~~ unknown function and its derivatives

ODE contains only derivatives with respect to one variable $y: \mathbb{R} \to \mathbb{R}^n$ which can be considered the time.

Partial DE (PDE) also contains partial derivatives

$$y_1' + \cos(y_2' \cdot y_1) = 0 \qquad \text{ODE} \quad y = y(x)$$

$$y'_{1x_1} + y'_{2x_2} \cdot \sin(y_1 + x_2) = 0 \qquad \text{PDE}$$

$$y_1(x_1, x_2) = ? \quad y_2(x_1, x_2) = ?$$

Normal form of ODE

$$\begin{cases} \dot{y}_1 = f_1(t, y_1, y_2, \dots, y_n, \cancel{t}) \\ \dot{y}_2 = f_2(t, y_1, y_2, \dots, y_n, \cancel{t}) \\ \dots \\ \dot{y}_n = f_n(t, y_1, y_2, \dots, y_n, \cancel{t}) \end{cases} \qquad y(t) = ? \quad y = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix}$$

$$\dot{y} = f(t, y) \qquad \text{normal form}$$

the highest order of derivative is called the order of DE

It is the system of the 1st-order ODEs in the standard form

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Example:
$$\begin{cases} \dot{y}_1 = y_1 - y_2 + \cos(t) \\ \dot{y}_2 = y_2 \cdot t \end{cases}$$

Example
$$\begin{cases} u''_{xx} + u''_{yy} = 0 \quad \text{PDE} \\ \text{Laplace DE} \quad u(x,y) = ? \end{cases}$$

Example
$$\operatorname{arctg}(\dddot{y}) + y^7 \dot{y} = 0$$
ODE of the order 3

$y: \mathbb{R} \to \mathbb{R}$, $y(t) = ?$

Scalar equation of the form
$$y^{(n)} = F(t, y, \dot{y}, \dots, y^{(n-1)})$$
can always be equivalently rewritten as a system of normal form ODEs

$y^{(k)} = x_k$ $\quad y = x_0$

$$\begin{cases} \dot{x}_0 = x_1 \\ \dot{x}_1 = x_2 \\ \dots \\ \dot{x}_{n-2} = x_{n-1} \\ \dot{x}_{n-1} = F(t, x_0, x_1, \dots, x_{n-1}) \end{cases} \Longleftrightarrow \quad y^{(n)} = F(t, y, \dots, y^{(n-1)})$$

$\boxed{\text{one-to-one correspondence}}$

Example:

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order 2 $\quad \dot{y}^2 - \ddot{y}^3 + t = 0 \iff \begin{cases} \dot{x}_0 = x_1 \\ \cancel{\dot{x}_1 = x_2} \\ x_0^2 - \dot{x}_1^3 + t = 0 \end{cases}$

it is not the normal form

order 3 $\quad \dddot{y} - y^3 t + \dot{y}^5 = 0 \iff \begin{cases} \dot{x}_0 = x_1 \\ \dot{x}_1 = x_2 \\ \dot{x}_2 = x_0^3 t - x_2^5 \end{cases}$

normal form

$\dot{y} = y(y(t))$ not a DE

An ODE is called autonomous if it ~~do~~ explicitly contains the time variable (can be denoted by a variable different from t)

Example

$\cos(t\ddot{y}) - \ln(y^7 + 1) = 0$

nonautonomous ODE of the order 2

$\cos(y^2\ddot{y}) + \dot{y}\sin y^{11} = 0$

autonomous of the order 2

ODE is called linear if it linearly depends on the unknown function and its derivatives.

$\begin{cases} y_1' = y_2 \sin(x \cos x^2) + x^3 \\ y_2' = y_1 + x^{11} y_2 \end{cases}$ linear ODE normal form non-autonomous