3. LATTICES AND BOOLEAN ALGEBRAS

3.1. Lattices and semilattices

Let $\mathbf{A} = \langle A,\sqsubseteq\rangle$ be a partial order. Let $X \subseteq A$

The **supremum of** X, $\sqcup X$, is the unique element of A such that:

1. for every $x \in X$: $x \sqsubseteq \sqcup X$
2. for every $a \in A$: if for every $x \in X$: $x \sqsubseteq a$ then $\sqcup X \sqsubseteq a$

The **infimum of** X, $\sqcap X$, is the unique element of A such that:

1. for every $x \in X$: $\sqcap X \sqsubseteq x$
2. for every $a \in A$: if for every $x \in X$: $a \sqsubseteq x$ then $a \sqsubseteq \sqcap X$

Let $a, b \in A$
The **join of** a and b, $a \sqcup b$, is $\sqcup\{a,b\}$
The **meet of** a and b, $a \sqcap b$, is $\sqcap\{a,b\}$

Hence:
The **join of** a and b, $a \sqcup b$, is the unique element of A such that:

1. $a \sqsubseteq a \sqcup b$ and $b \sqsubseteq a \sqcup b$
2. for every $x \in A$: if $a \sqsubseteq x$ and $b \sqsubseteq x$ then $a \sqcup b \sqsubseteq x$

The **meet of** a and b, $a \sqcap b$, is the unique element of A such that:

1. $a \sqcap b \sqsubseteq a$ and $a \sqcap b \sqsubseteq b$
2. for every $x \in A$: if $x \sqsubseteq a$ and $x \sqsubseteq b$ then $x \sqsubseteq a \sqcap b$

The way I use terminology here, join and meet are two-place partial operations on partial orders, and supremum and infimum are infinitary partial operations on partial orders.
I will not distinguish the two notationally (often one uses a bigger variant of $\sqcup$ and $\sqcap$ for the complete operations, but I won't here.

A **join-semilattice** is a partial order $\mathbf{A} = \langle A, \sqsubseteq\rangle$ which is closed under join:
for every $a,b \in A$: $a \sqcup b \in A$

A **meet-semilattice** is a partial order $\mathbf{A} = \langle A, \sqsubseteq\rangle$ which is closed under meet:
for every $a,b \in A$: $a \sqcap b \in A$

A **lattice** is a partial order $\mathbf{A} = \langle A, \sqsubseteq\rangle$ which is closed under join and meet:
for every $a,b \in A$: $a \sqcup b \in A$ and $a \sqcap b \in A$

A **complete lattice** is a partial order $\mathbf{A} = \langle A, \sqsubseteq\rangle$ which is closed under the complete operations of supremum and infimum: for every $X \subseteq A$: $\sqcup X \in A$ and $\sqcap X \in A$

A **complete^{+}-lattice** is a partial order $\mathbf{A} = \langle A,\sqsubseteq\rangle$ which is closed under the complete operations of supremum and infimum **for non-empty** subsets of A:
for every **non-empty** $X \subseteq A$: $\sqcup X \in A$ and $\sqcap X \in A$

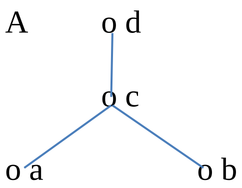
So, as partial orders, lattices are closed under two-place join and meet, which complete lattices are closed under compete join and meet.

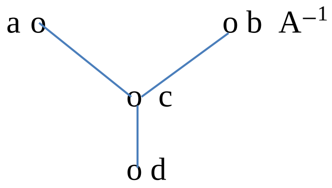
Fact: **A** is a lattice iff for every **finite** subset $X \subseteq A$: $\sqcup(X) \in A$ and $\sqcap(X) \in A$
Proof:
1. If every finite subset of A has a supremum and infimum, then so does every two-element subset, hence **A** is a lattice.
2. Let **A** be a lattice and let $\{x_1,\ldots x_n\} \subseteq A$. Take $(((x_1 \sqcup x_2) \sqcup x_3)\ldots) \sqcup x_n)$ and $(((x_1 \sqcap x_2) \sqcap x_3)\ldots) \sqcap x_n)$
You can prove that $(((x_1 \sqcup x_2) \sqcup x_3)\ldots x_n) = \sqcup\{x_1,\ldots x_n\}$ and $(((x_1 \sqcap x_2) \sqcap x_3)\ldots) \sqcap x_n) = \sqcap\{x_1,\ldots x_n\}$
We will not do that here, because it will follow from the algebraic properties of lattices discussed shortly.

Duality of partial orders:

If **A** = <A, $\sqsubseteq$ > is a partial order then $\mathbf{A}^{-1}$ = <A, $\sqsubseteq^{-1}$> (= <A, $\sqsupseteq$>) is also a partial order, we call it the **dual of A**.

Intuitively, the dual of partial order **A** is the result of turning the partial order upside down:





Obviously, when a partial order is dualized, several other notions dualize with it: minimal elements are turned into maximal elements (a,b are minimal elements in A and maximal in A^{-1}) and maximuns into minimums (d is maximum in A, and a minimum in A^{-1}), joins are turned into meets ($c = a \sqcup_A b$, but $c = a \sqcap_{A^{-1}} b$), and meets into joins, etc…

We see that partial orders are closed under duals.
Many subclasses of partial orders are themselves closed under duals (meaning that the dual of a partial order in class *K* is itself a partial order **in class *K*.**
And duality extends to the notions that dualize with the order.

Thus, the class of lattices is closed under duals in the sense that:

The dual of lattice **A** = <A,$\sqcap_A$,$\sqcup_A$> is lattice $\mathbf{A}^{-1}$ = <A, $\sqcap_A{}^{-1}$, $\sqcup_A{}^{-1}$> where:
$\sqcap_A{}^{-1} = \sqcup_A$ and $\sqcup_A{}^{-1} = \sqcap_A$

Here the lattice operations are paired by duality.
Later we will see more general structions like Boolean algebras that are also closed under duals:

A = <A, $\sqsubseteq_A$, $\neg_A$, $\sqcap_A$, $\sqcup_A$, 0_A, 1_A>

where in the dual structure:

$$\mathbf{A}^{-1} = \langle A, \sqsubseteq_A{}^{-1}, \neg_A{}^{-1}, \sqcap_A{}^{-1}, \sqcup_A{}^{-1}, 0_A{}^{-1}, 1_A{}^{-1} \rangle$$
$$= \langle A, \sqsupseteq_A, \neg_A, \sqcup_A, \sqcap_A, 1_A, 0_A \rangle$$

where the dual of the order is the partial order inverse, meet and join are each other's dual as in lattices, mimum 0 and 1 are reversed (i.e. in the dual structure the 1_A is the minimum), and the operation of complementation – to be introduced below – is **self dual**.

Classes of structures that are closed under duals satisfy the principle of **duality**:

Duality:
Let *K* be a class of structures that is closed under duals and let $\sqsubseteq_1, \ldots \sqsubseteq_n$ be the relations, and $O_1, \ldots O_m$ the operations and $s_1 \ldots s_k$ the special elements that have duals in *K*.

Let φ be a formula possibly containing expressions in: $\sqsubseteq_1, \ldots \sqsubseteq_n$, $O_1, \ldots O_m$, $s_1 \ldots s_k$

Let φ^{-1} be the result of replacing every occurrence of any such symbol by its dual.

Duality: If φ is true on every structure in *K*, then φ^{-1} is also true on every structure in *K*.

Many of the structures we are concerned are closed under duals and hence satisfy duality. This is extremely useful, because it reduces the number of proofs to be given in half.
For instance, suppose we manage to prove that the following formula is true on all lattices:

$$\forall a \forall b \forall c[(a \sqsubseteq c \wedge b \sqsubseteq c) \rightarrow (a \sqcup b \sqsubseteq c)]$$

Then, by duality, the following formula is also true on all lattices:

$$\forall a \forall b \forall c[(c \sqsubseteq a \wedge c \sqsubseteq b) \rightarrow (c \sqsubseteq a \sqcap b)]$$

Thus when we prove a fact about $\sqcup$, we don't have to prove the dual fact about $\sqcap$: its truth follows from duality.

I mention the formulas in the examples expicitly, because you use these over and over in making proofs:

Fact 1: if $a \sqsubseteq x$ and $b \sqsubseteq x$ then $a \sqcup b \sqsubseteq x$
Fact 2: if $x \sqsubseteq a$ and $x \sqsubseteq b$ then $x \sqsubseteq a \sqcap b$

These are actually just consequences of the definition of join and meet.

We have defined lattices as partial orders. We now give a second definition of lattices as **algebras**:

A **lattice** is a an algebra $\mathbf{A} = \langle A, \sqcap, \sqcup \rangle$ where $\sqcap$ and $\sqcup$ are two-place operations satisfying:

1. **Idempotency** $(a \sqcap a) = a$
   $(a \sqcup a) = a$
2. **Commutativity** $(a \sqcap b) = (b \sqcap a)$
   $(a \sqcup b) = (b \sqcup a)$
3. **Associativity** $(a \sqcap (b \sqcap c)) = ((a \sqcap b) \sqcap c)$
   $(a \sqcup (b \sqcup c)) = ((a \sqcup b) \sqcup c)$
4. **Absorption** $(a \sqcap (a \sqcup b)) = a$
   $(a \sqcup (a \sqcap b)) = a$

Theorem: The two concepts of lattices coincide

A. If $\langle A, \sqsubseteq \rangle$ is a lattice then $\langle A, \sqcap, \sqcup \rangle$ is a lattice, where $\sqcap$ and $\sqcup$ are the meet and join operations defined for $\sqsubseteq$

B. If $\langle A, \sqcap, \sqcup \rangle$ is a lattice, then $\langle A, \sqsubseteq \rangle$ is a lattice, where $\sqsubseteq$ is defined as:
$a \sqsubseteq b$ iff $(a \sqcap b) = a$

C. If $\langle A, \sqsubseteq \rangle$ is a lattice, and we form $\langle A, \sqcap, \sqcup \rangle$ by taking join and meet in $\sqsubseteq$ as lattice operations, and from $\langle A, \sqcap, \sqcup \rangle$ we form $\langle A, \sqsubseteq' \rangle$ by defining $a \sqsubseteq' b$ iff $(a \sqcap b) = a$, then $\langle A, \sqsubseteq' \rangle = \langle A, \sqsubseteq \rangle$
Similarly, if $\langle A, \sqcap, \sqcup \rangle$ is a lattice, and we form $\langle A, \sqsubseteq \rangle$ by defining $\sqsubseteq$ as $a \sqsubseteq b$ iff $(a \sqcap b) = a$, and we form $\langle A, \sqcap', \sqcup' \rangle$ where $\sqcap'$ and $\sqcup'$ are meet and join in $\langle A, \sqsubseteq \rangle$, then $\langle A, \sqcap', \sqcup' \rangle = \langle A, \sqcap, \sqcup \rangle$.

Proof of A.
Let $\langle A, \sqsubseteq \rangle$ be a lattice.
Idempotency: $a \sqcup a$ is the smallest element such that $a \sqsubseteq a$ and $a \sqsubseteq a$, which is obviously a
Commutativity: $a \sqcup b$ is the smallest element such that $a \sqsubseteq a \sqcup b$ and $b \sqsubseteq a \sqcup b$, which is obviously also the smallest element such that $b \sqsubseteq a \sqcup b$ and $a \sqsubseteq a \sqcup b$.
Associativity: $(a \sqcup (b \sqcup c)) = ((a \sqcup b) \sqcup c)$
We show: $(a \sqcup (b \sqcup c)) \sqsubseteq ((a \sqcup b) \sqcup c)$ and $((a \sqcup b) \sqcup c) \sqsubseteq (a \sqcup (b \sqcup c))$. Associativity will follow from antisymmetry.
1. Look at $((a \sqcup b) \sqcup c)$.
$a \sqcup b \sqsubseteq ((a \sqcup b) \sqcup c)$ and $c \sqsubseteq ((a \sqcup b) \sqcup c)$ and $a \sqsubseteq a \sqcup b$ and $b \sqsubseteq a \sqcup b$, hence by transitivity: $a \sqsubseteq ((a \sqcup b) \sqcup c)$ and $b \sqsubseteq ((a \sqcup b) \sqcup c)$ and $c \sqsubseteq ((a \sqcup b) \sqcup c)$.
Then by definition of join $(b \sqcup c) \sqsubseteq ((a \sqcup b) \sqcup c)$, and once again by definition of join:
$(a \sqcup (b \sqcup c)) \sqsubseteq ((a \sqcup b) \sqcup c)$.
-The argument for the other side goes in the same way, and the argument for meet follows from duality.
Absorption: $a \sqcup (b \sqcap a) = a$
Obviously, $a \sqsubseteq a \sqcup (b \sqcap a)$, so we only need to show that $a \sqcup (b \sqcap a) \sqsubseteq a$.
This is the case because $a \sqsubseteq a$ and $b \sqcap a \sqsubseteq a$.
-The other absorption law is done in the same way.
□

Before continuing we prove two lemmas:

Lemma 1: $a \sqcap b = a$ iff $a \sqcup b = b$

Proof (algebraic)

(1)	$a \sqcap b = a$	Assumption
(2)	$b \sqcup (b \sqcap a) = b$	Absorption
(3)	$(b \sqcap a) \sqcup b = b$	From (2) by commutativity
(4)	$(a \sqcap b) \sqcup b = b$	From (3) by commutativity
(5)	$a \sqcup b = b$	Use (1) to substitute a for $a \sqcap b$ in (4).

The other side goes similarly, with the other absorption law.

Lemma 2: Define: $a \sqsubseteq b$ iff $a \sqcap b = b$ iff $a \sqcup b = b$ (justified by Lemma 1) Then:
(1) $z \sqsubseteq a \sqcap b$ iff $z \sqsubseteq a$ and $z \sqsubseteq b$
(2) $a \sqcup b \sqsubseteq z$ iff $a \sqsubseteq z$ and $b \sqsubseteq z$

Proof of 2.1. (algebraic)

(1) $z \sqsubseteq a$ and $z \sqsubseteq b$	Assumption
(2) $z \sqcap a = z$ and $z \sqcap b = z$	Definition $\sqsubseteq$
(3) $z = z \sqcap z$	Idempotency
(4) $z = (z \sqcap a) \sqcap (z \sqcap b)$	From (2) and (3)
(5) $z = (z \sqcap z) \sqcap (a \sqcap b)$	From (4) by associativity and commutativity
(6) $z = (a \sqcap b) \sqcap z$	From (5) by idempotency and commutativity
(7) $z \sqsubseteq a \sqcap b$	Definition $\sqsubseteq$ (and commutativity)

(1) $z \sqsubseteq a \sqcap b$	Assumption
(2) $(z \sqcup (a \sqcap b)) = (a \sqcap b)$	From the definition of $\sqsubseteq$
(3) $(z \sqcup (a \sqcap b)) \sqcup a = (a \sqcap b) \sqcup a$	Logic: if x=y then $x \sqcup a = y \sqcup a$
(4) $(z \sqcup (a \sqcap b)) \sqcup a = a$	From (3) and absorption
(5) $(z \sqcup ((a \sqcap b) \sqcup a)) = a$	From (4) and associativity
(6) $z \sqcup a = a$	From (5) and absorption
(7) $z \sqsubseteq a$	From the definition of $\sqsubseteq$

We prove similarly that $z \sqsubseteq b$
2.2 goes by a similar argument.
□

Proof of B

Assume that <A, ⊓, ⊔> is a lattice and define $a \sqsubseteq b$ iff $a \sqcap b = a$ iff $a \sqcup b = b$.

1. $\sqsubseteq$ is reflexive.

(1) $a \sqcap a = a$	Idempotency
(2) $a \sqsubseteq a$	Definition $\sqsubseteq$

2. $\sqsubseteq$ is transitive.

1. $a \sqsubseteq b$ and $b \sqsubseteq c$	Assumption
2. $a \sqcap b = a$ and $b \sqcap c = b$	Definition $\sqsubseteq$
3. $a \sqcap (b \sqcap c) = a$	From (2) by substituting $b \sqcap c$ for b
4. $(a \sqcap b) \sqcap c = a$	From (3) with associativity
5. $a \sqcap c = a$	From (2) and (4) by substituting a for $a \sqcap b$
6. $a \sqsubseteq c$	By definition of $\sqsubseteq$

3. $\sqsubseteq$ is antisymmetric

1. $a \sqsubseteq b$ and $b \sqsubseteq a$	Assumption
2. $a \sqcap b = a$ and $b \sqcap a = b$	Definition $\sqsubseteq$
3. $a \sqcap b = a$ and $a \sqcap b = b$	From (2) by commutativity
4. $a = b$	From (3) by substituting a for $a \sqcap b$

4. $a \sqcap b$ is meet in $\sqsubseteq$ and $a \sqcup b$ is join in $\sqsubseteq$

4.1. $a \sqcap b \sqsubseteq a$ and $a \sqcap b \sqsubseteq b$

(1) $(a \sqcap b) \sqcup a = a$	Absorption
(2) $a \sqcap b \sqsubseteq a$	Definition $\sqsubseteq$

Similarly $a \sqcap b \sqsubseteq b$

4.2. if $z \sqsubseteq a$ and $z \sqsubseteq b$ then $z \sqsubseteq a \sqcap b$

That was lemma 2.

The proof that $a \sqcup b$ is join in $\sqsubseteq$ is similar.

Hence $a \sqcap b$ is meet in $\sqsubseteq$.

$\square$

Proof of C

Let $\langle A, \sqsubseteq \rangle$ be a lattice and form $\langle A, \sqcap, \sqcup \rangle$ by taking meet and join as operations.
By definition of meet and join in $\sqsubseteq$, $a \sqsubseteq b$ iff $a \sqcap b = a$ iff $a \sqcup b = n$
$a \sqcap b \sqsubseteq$. So when next we define $\sqsubseteq'$ as ($a \sqsubseteq' b$) iff $a \sqcap b = a$, it will follow that
$a \sqsubseteq' b$ iff $a \sqsubseteq b$, hence $\sqsubseteq = \sqsubseteq'$.
A similar argument shows the other side.

$\square$

For join and meet semilatiices we have a similar result.
Algebraically we have a **semilattive**:

A **semilattice** is an algebra **A** = $\langle A, \square \rangle$ where $\square$ is a two place operation on A such that:

1. **Idempotency** $a \square a = a$
2. **Commutativity** $a \square b = b \square a$
3. **Associativity** $a \square (b \square c) = (a \square b) \square c$

Theorem: The two concepts of semilattices coincide

A. If $\langle A, \sqsubseteq \rangle$ is a join (meet)-semilattice then $\langle A, \square \rangle$ is a semilattice, where $\square$ is the join (meet) operation defined for $\sqsubseteq$

B. If $\langle A, \square \rangle$ is a semilattice, then $\langle A, \sqsubseteq \rangle$ is a join (meet)-semilattice, where $\sqsubseteq$ is defined as: $a \sqsubseteq b$ iff $(a \sqcup b) = b$ ($a \sqsubseteq b$ iff $(a \sqcap b) = a$).

C. If $\langle A, \sqsubseteq \rangle$ is a join (meet) semilattice, and we form $\langle A, \square \rangle$ by taking join (meet) in $\sqsubseteq$ as semilattice operation, and from $\langle A, \square \rangle$ we form $\langle A, \sqsubseteq' \rangle$ by defining $a \sqsubseteq' b$ iff $(a \sqcup b) = b$ ($a \sqsubseteq' b$ iff $(a \sqcap b) = a$), then $\langle A, \sqsubseteq' \rangle = \langle A, \sqsubseteq \rangle$
Similarly, if $\langle A, \square \rangle$ is a semilattice, and we form $\langle A, \sqsubseteq \rangle$ by defining $\sqsubseteq$ as $a \sqsubseteq b$ iff $(a \sqcup b) = a$ ($a \sqsubseteq b$ iff $(a \sqcap b) = a$), and we form $\langle A, \square' \rangle$ where $\square'$ is join (meet) in $\langle A, \sqsubseteq \rangle$, then $\langle A, \square' \rangle = \langle A, \square \rangle$.

Proof: exercise.

The correspondence results are very important. As partial orders the axioms that define latttices are not positive formulas (for instance, transitivity and antisymmetry are not positive formulas, because they are defined with $\rightarrow$, and hence with $\neg$, if $\rightarrow$ is defined in terms of $\neg$). But the algebraix axioms are identity statements, so they are positive formulas.

This means that the lattice axioms are preserved under homomorphisms. They are in fact preserved under homomorphisms, substructures and direct products. This means that the class of lattices is an equational class of algebras.
And this means that if you have homomorphism from a lattice to another algebra of the same type, it follows that this other structure is not just any algebra of the same type, but in fact a lattice itself. The same for substructures: lattices have only substructures that are themselves lattices.

For lattices as partial orders you do not get these results, the notion of homomorphic image is not constrained the way the algebraic notion is. Since we think of these structures as two sides of the same coin, we will often consider the lattice as an algebra for the same of homomorphisms.

3.2 Bounds and atoms

If (semi) lattice **A** has a minimum we call that 0_A, if **A** has a maximum, we call it 1_A.

A **bounded** lattice is a lattice $\mathbf{A} = <A, \sqsubseteq>$ that has both a minimum 0 and a maximum 1.

A **bounded** lattice is a lattice $\mathbf{A} = <A, \sqcap, \sqcup>$ such that:

Laws of 0 and 1: $a \sqcap 0 = 0$
$a \sqcup 1 = 1$

Since every complete lattice is bounded and every finite lattice is complete, only infinite lattices can be unbounded.

Example 1: A lattice that has a 0 but not a 1:
Let $F = \{X \subseteq \mathbf{N}$: X is finite$\}$. The set of all finite subsets of the natural numbers.
Look at $\mathbf{F} = <F, \cap, \cup>$.
F is a lattice under the operations $\cap$ and $\cup$ (the intersection and union of two finite sets is a finite set). **F** has a 0, namely Ø. But **F** does not have a 1, since there is no largest finite set.
F is, of course, not complete.
Let $S = \{\{n\}: n \in \mathbf{N}\}$, the set of all singletons of natural numbers.
$S \subseteq F$. but $\cup S \notin F$, because $\cup S = \mathbf{N}$, which is infinite.

Example 2: A lattice that has neither a 0 nor a 1:
Let I_0 be the set of non-singleton intervals in R which are bounded **in R** and which contain 0.
Let $\mathbf{I_0} = <I_0, \cap, \cup>$
If $i, j \in I_0$, then i and j overlap and then $i \cup j$ is an interval, namely the interval that starts with whichever lower bound is leftmost and ends with whichever upper bound is right most.
$i \cap j$ is non-empty, since $0 \in i \cap j$, and, in fact, since i and j are not singleton and **R** is dense, $i \cap j$ is a non-singleton interval containing 0, so $i \cap j \in I_0$.
Since **R** is dense, for any interval $i \in I_0$ there is a proper subinterval j of i such that $j \in I_0$.
Hence, I_0 does not contain a minumum.
Since all the intervals in I_0 are bounded in **R**, none of them is bounded by $-\infty$ or ∞, but that means that for any interval $i \in I_0$ there is a proper superinterval j of i such that $j \in I_0$. Hence I_0 does not contain a maximum.
Again, I_0 is not complete: $\cap I_0 = \{0\}$, so $\cap I_0 \notin I_0$; $\cup I_0 = \mathbf{R}$, $\mathbf{R} \notin I_0$

Some observations about completeness:

Lemma: If $\sqcup\emptyset \in A$ then $\sqcup\emptyset = 0_A$
If $\sqcap\emptyset \in A$ then $\sqcap\emptyset = 1_A$

Proof:
Let $\sqcup\emptyset \in A$.

$\sqcup\emptyset$ is the unique element of A such that:

1. *for every $x \in \emptyset$: $x \sqsubseteq \sqcup\emptyset$*
2. for every $a \in A$: if *for every $x \in \emptyset$: $x \sqsubseteq a$* then $\sqcup\emptyset \sqsubseteq a$

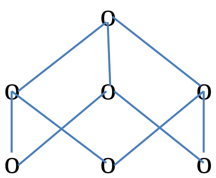
The italicized statements are trivially true, since Ø doesn't have elements. This means that:

$\sqcup\emptyset$ is the unique element of A such that for every $a \in A$: $\sqcup\emptyset \sqsubseteq a$, i.e. the minimum of A, 0_A
The argument that $\sqcap\emptyset = 1_A$ goes by a mirror argument.
□

This means that you have to be careful in defining the notions you want, because:

Lemma: If **A** = <A, ⊑> is a partial order which is closed under complete join, then **A** is a complete lattice.
Proof
Let **A** be a partial order closed under complete join and let $X \subseteq A$.
Define: $LB(X) = \{a \in A\text{: for every } x \in X\text{: } a \sqsubseteq x\}$
Then $LB(X) \subseteq A$. Since **A** is closed under complete join, it follows that $\sqcup(LB(X)) \in A$.
But, of course, $\sqcup(LB(X)) = \sqcap X$, hence $\sqcap X \in A$, and **A** is closed under complete meet.
□

This is what the notion **complete$^+$** is about. A complete$^+$ join semilattive is closed under join for every **non-empty** subset. This means that $\sqcup\emptyset$ is not required to be in A.
Structures can be complete$^+$ join semilattices without being complete lattices. For instance, the following structure is a complete$^+$ join semilattice:



It will be useful here to already introduce two notions that are discussed in a more general setting later.

Let **A** = $<A, \sqsubseteq>$ be a lattice and let $a \in A$

The **ideal generated by** a, (a], is the set of a's $\sqsubseteq$-parts:

$(a] = \{b \in A: b \sqsubseteq a\}$

The **filter generated by** a, [a), is the set of elements that a is $\sqsubseteq$-part of:

$[a) = \{b \in A: a \sqsubseteq b\}$

Note that (a] and [a) are convex sets in A.

I will define the following notions for lattices with 0 and 1. they can be generalized to other structures (lattices with only 0, with only 1, without 0 and 1, etc.):

Let **A** = $<A, \sqsubseteq>$ be a lattice with 0 and 1

Let $a \in A-\{0\}$:

a is an **atom in A** iff for every $b \in A$: if $b \sqsubseteq a$ then $b = a$ or $b = 0$

Let $a \in A-\{1\}$:

a is a **dual atom in A** iff for every $b \in A$: if $a \sqsubseteq b$ then $b = a$ or $b = 1$

A is **atomic** iff for every $b \in A - \{0\}$: there is an atom $a \in A$: $a \sqsubseteq b$

$ATOM_A = \{a \in A$: a is an atom in **A**$\}$

A is **atomless** iff $ATOM_A = \emptyset$

Let $b \in$ **A**: AT_b, the set of atomic parts of b, is:

$AT_b = (b] \cap ATOM_A$

So $AT_b = \{a \in ATOM_A: a \sqsubseteq b\}$

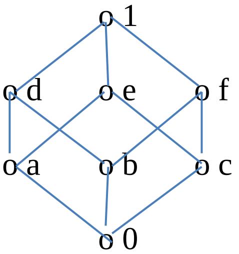
Atoms are 'minimal non-0 parts', elements that have only themselves and 0 as parts.

Let **A** = <A, ⊑ > be a complete lattice.

A is **atomistic** iff for every $b \in A$: $b = \sqcup AT_b$

A and **B** below are atomic lattices, where **B** is not atomistic, while **A** is atomistic:

A



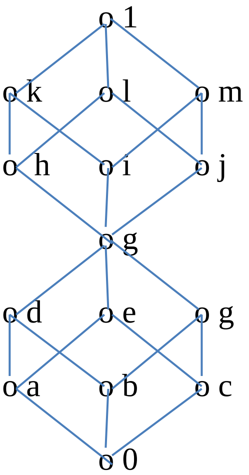
$ATOM_{\mathbf{A}} = \{a,b,c\}$
$AT_1 = \{a,b,c\}$ and $1 = \sqcup\{a,b,c\}$

$AT_d = \{a,b\}$ and $d = a \sqcup b$ $e = a \sqcup c$ $f = b \sqcup c$

$AT_a = \{a\}$ and $a = a \sqcup a$ $b = b \sqcup b$ $c = c \sqcup c$

$AT_0 = \emptyset$ and $\sqcup\emptyset = 0$

B



$ATOM_{\mathbf{B}} = \{a,b,c\}$

$\sqcup\{a,b,c\} = g$
None of the elements of {h,i,j,k,l,m,1} are joins of atoms

Let **A** be a lattice and $x \in A$ and $X \subseteq A$.

x is atomistic, ***atomistic***(x) iff $x = \sqcup AT_x$
X is atomistic, ***atomistic***(X) iff every $x \in X$ is atomistic.

3.3. Modular and distributive lattices

Fact 1: **(1) (2) (3)** and **(4)** are true on all lattices

The distributive inequalities:

(1) $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a \sqcap (b \sqcup c)$

(2) $a \sqcup (b \sqcap c) \sqsubseteq (a \sqcup b) \sqcap (a \sqcup c)$

(3) $(a \sqcap b) \sqcup (b \sqcap c) \sqcup (a \sqcap c) \sqsubseteq (a \sqcup b) \sqcap (b \sqcup c) \sqcap (a \sqcup c)$

The modular inequality:

(4) $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a \sqcap (b \sqcup (a \sqcap c))$

Proof of Fact 1.1: $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a \sqcap (b \sqcup c)$

[1] $a \sqcap b \sqsubseteq a$ and $a \sqcap c \sqsubseteq a$
[2] $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a$ From 1
[3] $a \sqcap b \sqsubseteq b$, hence $a \sqcap b \sqsubseteq b \sqcup c$
$a \sqcap c \sqsubseteq c$, hence $a \sqcap c \sqsubseteq b \sqcup c$
[4] $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq (b \sqcup c)$ From 3
[5] $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a \sqcap (b \sqcup c)$ From 2 and 4

Proof of Fact 1.2: $a \sqcup (b \sqcap c) \sqsubseteq (a \sqcup b) \sqcap (a \sqcup c)$
[1] $a \sqsubseteq a \sqcup b$ and $a \sqsubseteq a \sqcup c$
[2] $a \sqsubseteq (a \sqcup b) \sqcap (a \sqcup c)$ From 1
[3] $b \sqsubseteq a \sqcup b$ hence $b \sqcap c \sqsubseteq a \sqcup b$
$c \sqsubseteq a \sqcup c$ hence $b \sqcap c \sqsubseteq a \sqcup c$
[4] $b \sqcap c \sqsubseteq (a \sqcup b) \sqcap (a \sqcup c)$ From 3
[5] $a \sqcup (b \sqcap c) \sqsubseteq (a \sqcup b) \sqcap (a \sqcup c)$

Proof of Fact 1.3: $(a \sqcap b) \sqcup (b \sqcap c) \sqcup (a \sqcap c) \sqsubseteq (a \sqcup b) \sqcap (b \sqcup c) \sqcap (a \sqcup c)$
[1] $a \sqcap b \sqsubseteq a \sqcup b$ and $a \sqcap b \sqsubseteq b \sqsubseteq b \sqcup c$ and $a \sqcap b \sqsubseteq a \sqsubseteq a \sqcup c$
[2] $a \sqcap b \sqsubseteq \sqsubseteq (a \sqcup b) \sqcap (b \sqcup c) \sqcap (a \sqcup c)$ From 1

[3] $b \sqcap c \sqsubseteq b \sqsubseteq a \sqcup b$ and $b \sqcap c \sqsubseteq b \sqcup c$ and $b \sqcap c \sqsubseteq c \sqsubseteq a \sqcup c$
[4] $b \sqcap c \sqsubseteq (a \sqcup b) \sqcap (b \sqcup c) \sqcap (a \sqcup c)$ From 3

[5] $a \sqcap c \sqsubseteq a \sqsubseteq a \sqcup b$ and $a \sqcap c \sqsubseteq c \sqsubseteq b \sqcup c$ and $a \sqcap c \sqsubseteq a \sqcup c$
[6] $a \sqcap c \sqsubseteq (a \sqcup b) \sqcap (b \sqcup c) \sqcap (a \sqcup c)$ From 5

[7] $(a \sqcap b) \sqcup (b \sqcap c) \sqcup (a \sqcap c) \sqsubseteq (a \sqcup b) \sqcap (b \sqcup c) \sqcap (a \sqcup c)$ From 2,4 and 6

Proof of Fact 1.4: $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a \sqcap (b \sqcup (a \sqcap c))$

[1] $a \sqcap b \sqsubseteq a$ and $a \sqcap c \sqsubseteq a$
[2] $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a$ From 1

[3] $a \sqcap b \sqsubseteq b$
[4] $a \sqcap b \sqsubseteq b \sqcup (a \sqcap c)$ From 3
[5] $a \sqcap c \sqsubseteq a \sqcap c$
[6] $a \sqcap c \sqsubseteq b \sqcup (a \sqcap c)$ From 4
[7] $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a \sqcap (b \sqcup (a \sqcap c))$ From 4 and 6

[8] $(a \sqcap b) \sqcup (a \sqcap c) \sqsubseteq a \sqcap (b \sqcup (a \sqcap c))$ From 2 and 7
$\square$

Fact 2: Modularity: (5) and **(6)** are equivalent within the class of all lattices:

(5) $(a \sqcap b) \sqcup (a \sqcap c) = a \sqcap (b \sqcup (a \sqcap c))$
(6) if $c \sqsubseteq a$ then $a \sqcap (b \sqcup c) = c \sqcup (a \sqcap b)$

A lattice **A** = <A, ⊑> is **modular** iff one of the **modular laws (5)** or **(6)** holds.

Proof of Fact 2:

(5) entails (6)
Assume $c \sqsubseteq a$ and assume **(5)**. $c \sqsubseteq a$ means $a \sqcap c = c$
[1] $(a \sqcap b) \sqcup (a \sqcap c) = a \sqcap (b \sqcup (a \sqcap c))$
[2] $(a \sqcap b) \sqcup c = a \sqcap (b \sqcup c)$ Substituting c for $a \sqcap c$ in [1]

(6) entails (5)
Assume **(6)**: if $c \sqsubseteq a$ then $a \sqcap (b \sqcup c) = c \sqcup (a \sqcap b)$

$a \sqcap c \sqsubseteq a$, so filling in $(a \sqcap c)$ for c in the consequent of **(6)**, we have:

[1] $a \sqcap (b \sqcup \mathbf{(a \sqcap c)}) = \mathbf{(a \sqcap c)} \sqcup (a \sqcap b)$
$\square$

Fact 3: Distributivity (7), (8) (9) are equivalent within the class of all lattices.

(7) $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$
(8) $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap (a \sqcup c)$
(9) $c \sqcap (a \sqcup b) \sqsubseteq a \sqcup (b \sqcap c)$

A lattice **A** = <A, ⊑> is **distributive** iff one of the **distributive laws (7)**, **(8)** or **(9)** holds.

Proof of fact 3.

(7) entails (8)
Assume **(7)**: $a \sqcap (b \sqcup c) = (a \sqcap b) \sqcup (a \sqcap c)$
We prove **(8)** by the following list of equivalences:

[1] $(a \sqcup b) \sqcap \mathbf{(a \sqcup c)}$	
[2] $(a \sqcap (a \sqcup c)) \sqcup (b \sqcap (a \sqcup c))$	Distribute $a \sqcup c$ with **(7)** over a and b
[3] $a \sqcup (b \sqcap (a \sqcup c))$	Absorption of $(a \sqcap (a \sqcup c))$
[4] $a \sqcup ((b \sqcap a) \sqcup (b \sqcap c))$	Distribute b with **(7)** over a and c
[5] $(a \sqcup (b \sqcap a)) \sqcup (b \sqcap c)$	Reorder with associativity
[6] $a \sqcup (b \sqcap c))$	Absorption of $a \sqcup (b \sqcap a))$

(8) entails (7)
Mirror image argument.

(7) entails (9)

[1]	$a \sqcap c \sqsubseteq a \sqsubseteq a \sqcup (b \sqcap c)$	
[2]	$b \sqcap c \sqsubseteq a \sqcup (b \sqcap c)$	
[3]	$(a \sqcap c) \sqcup (b \sqcap c) \sqsubseteq a \sqcup (b \sqcap c)$	
[4]	$c \sqcap (a \sqcup b) \sqsubseteq a \sqcup (b \sqcap c)$	By **(7)**

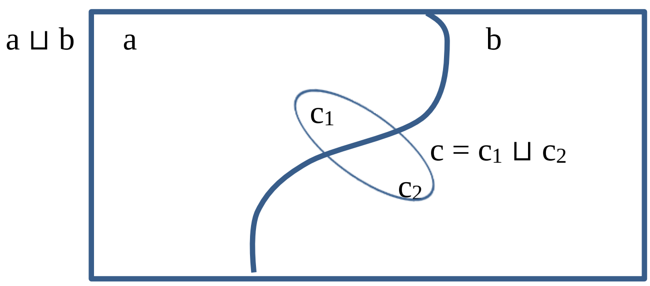
(9) entails (8)
Assume **(9)**. By fact 1.2 we only need to show that:

$$(a \sqcup b) \sqcap (a \sqcup c) \sqsubseteq a \sqcup (b \sqcap c))$$

Because the other half holds in all lattices.
Here goes.

[1]	$z \quad \sqcap (x \sqcup y) \sqsubseteq x \sqcup (y \sqcap \quad z \quad)$	This is **(9)**
[2]	$\mathbf{(a \sqcup c)} \sqcap (a \sqcup b) \sqsubseteq a \sqcup (b \sqcap \mathbf{(a \sqcup c)})$	This is an instance of **(9)**
[2]	$(a \sqcup c) \sqcap (a \sqcup b) \sqsubseteq a \sqcup \mathbf{(b \sqcap (a \sqcup c))}$	
	$z \sqcap (x \sqcup y)) \sqsubseteq x \sqcup (z \sqcap y)$	Another instance of **(9)**
	$\mathbf{a \sqcup (b \sqcap c))}$	
[3]	$(a \sqcup b) \sqcap (a \sqcup c) \sqsubseteq a \sqcup \mathbf{(a \sqcup (b \sqcap c))}$	Apply **(9)** on the bold face in [2]
[4]	$(a \sqcup b) \sqcap (a \sqcup c) \sqsubseteq a \sqcup (b \sqcap c))$	Associativity and Idempotency

A is **distributive** iff for all $a, b, c \in A$
if $c \sqsubseteq a \sqcup b$ then $c \sqsubseteq a$ or $c \sqsubseteq b$ or $\exists c_1 \sqsubseteq a\ \exists c_2 \sqsubseteq b$: $c = c_1 \sqcup c_2$

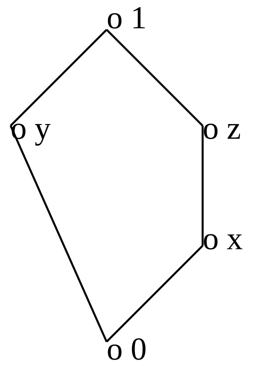


Fact 4: Every distributive lattice is modular.

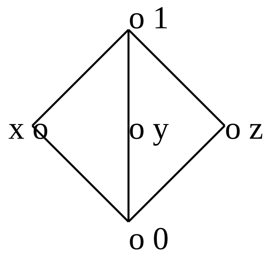
Proof:
Let A be a distributive lattice, let a,b,c ∈ A and let c ⊑ a.

a ⊓ (b ⊔ c) = (a ⊓ b) ⊔ (a ⊓ c) = (a ⊓ b) ⊔ c, since a ⊓ c = c.
◻

The pentagon:



The diamond:



Fact 5: The pentagon is not modular.
Namely: x ⊑ z but x ⊔ (y ⊓ z) = x
z ⊓ (x ⊔ y) = z

Fact 6: The diamond is modular, but not distributive.
That the diamond is modular follows from the theorem below (but can, of course, also be checked).
The diamond is not distributive because:
y ⊔ (x ⊓ z) = y
(y ⊔ x) ⊓ (y ⊔ z) = 1

Theorem (Birkhoff):

1. A lattice **A** is modular iff the pentagon cannot be embedded in **A**
2. A modular lattice **A** is distributive iff the diamind cannot be embedded in **A**.

Hence lattice **A** is distributive iff neither the pentagon nor the diamond can be embedded in **A**.

PROOF:
1a. If the pentagon can be embedded in a lattice, that lattice is not modular.
2a. If the diamond can be embedded in a lattice, that lattice is not distributive.

Proof of 1a: We have seen that the pentagon is not modular:
$x \sqsubseteq y$ but $x \sqcup (y \sqcap z) = x$
$y \sqcap (x \sqcup z) = y$
So, with **(6)**, indeed the pentagon is not modular.

The class of modular lattices is defined by identity **(5)**, hence it is closed under sublattices: every sublattice of a modular lattice is itself a modular lattice.
If the pentagon can be embedded in lattice **A**, it is isomorphic to a sublattice of **A**, and hence **A** has a non-modular sublattice. But then **A** is not modular.
Proof of 1a: similar.

Proof of 1b: If a lattice is not modular, the pentagon can be embedded in it

Let A be a lattice which is not modular.
This means that for some a,b,c: $a \sqsubseteq b$ and $a \sqcup (b \sqcap c) \neq b \sqcap (a \sqcup c)$

Now, we proved above fact 1.2 for all lattices (which is 1-below) and argue:

(1) $a \sqcup (b \sqcap c) \sqsubseteq (a \sqcup b) \sqcap (a \sqcup c)$	[fact 1.2]
(2) $a \sqsubseteq b$	[assumption]
(3) $a \sqcup b = b$	[def on 2]
(4) $a \sqcup (b \sqcap c) \sqsubseteq b \sqcap (a \sqcup c)$	[substitute b for $a \sqcup b$ in 1]

Since $a \sqcup (b \sqcap c) \neq b \sqcap (a \sqcup c)$ and $a \sqcup (b \sqcap c) \sqsubseteq b \sqcap (a \sqcup c)$ it follows that:

$$a \sqcup (b \sqcap c) \sqsubset b \sqcap (a \sqcup c)$$

Fact 1: $a \sqsubset b$:

Namely: if a = b, then $a \sqcup (b \sqcap c) = a \sqcup (a \sqcap b) = a$
and $b \sqcap (a \sqcup c) = a \sqcap (a \sqcup c) = a$. This contradicts the assumption.

Fact 2: $\neg(c \sqsubseteq a)$, $\neg(a \sqsubseteq c)$, $\neg(c \sqsubseteq b)$, $\neg(b \sqsubseteq c)$

If $c \sqsubseteq a$, then $c \sqsubseteq b$ and $a \sqcup (b \sqcap c) = a \sqcup c = a$.
and $b \sqcap (a \sqcup c) = b \sqcap a = a$. This contradicts the assumption.

If $a \sqsubseteq c$, then $b \sqcap (a \sqcup c) = b \sqcap c$.

If a ⊑ c then a ⊑ b ⊓ c and a ⊔ (b ⊓ c) = b ⊓ c. This contradicts the assumption.

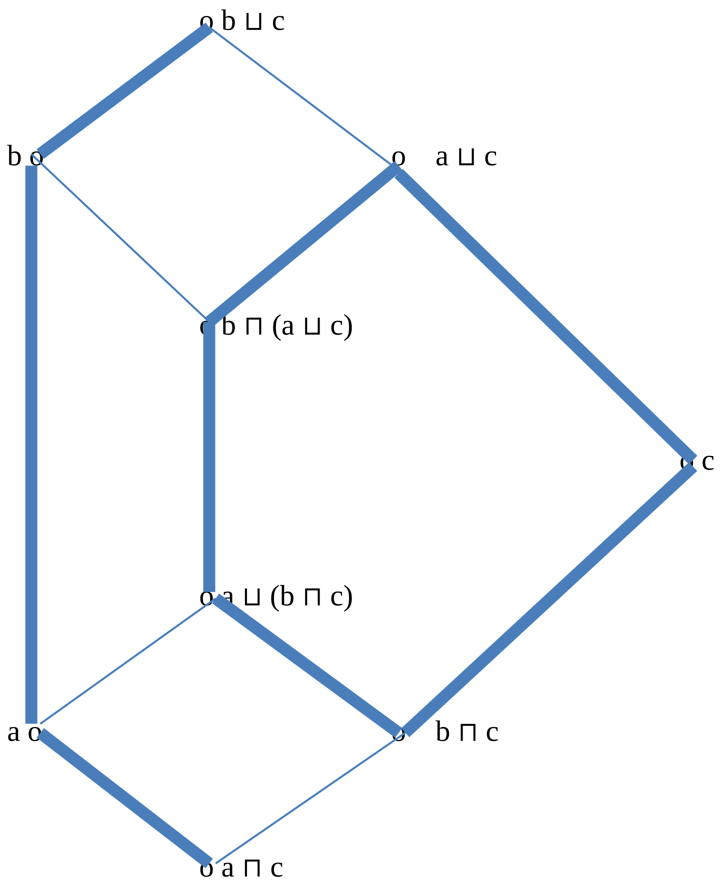
If b ⊑ c, then a ⊔ (b ⊓ c) = a ⊔ b = b
If b ⊑ c, then a ⊑ c and b ⊓ (a ⊔ c) = b ⊓ c = b. This contradicts the assumption.

If c ⊑ b, then a ⊔ (b ⊓ c) = a ⊔ c.
If c ⊑ b then a ⊔ c ⊑ b and b ⊓ (a ⊔ c) = (a ⊔ c). This contradicts the assumption.

It follows straightforwardly from this that c ⊏ a ⊔ c, that b ⊓ c ⊏ c, that b ⊏ b ⊔ c, and that a ⊓ c ⊏ a.
And also that b ⊓ (a ⊔ c) ⊏ a ⊔ c and that b ⊓ c ⊏ a ⊔ (b ⊓ c)

For instance, if b ⊓ c = a ⊔ (b ⊓ c), then a ⊑ b ⊓ c and then a ⊑ c, contradicting the assumption.

Now we can draw a picture of all the relevant elements of A and what we have established:
In the picture, thick lines mean ⊏, thin lines (dotted or not) mean ⊑:

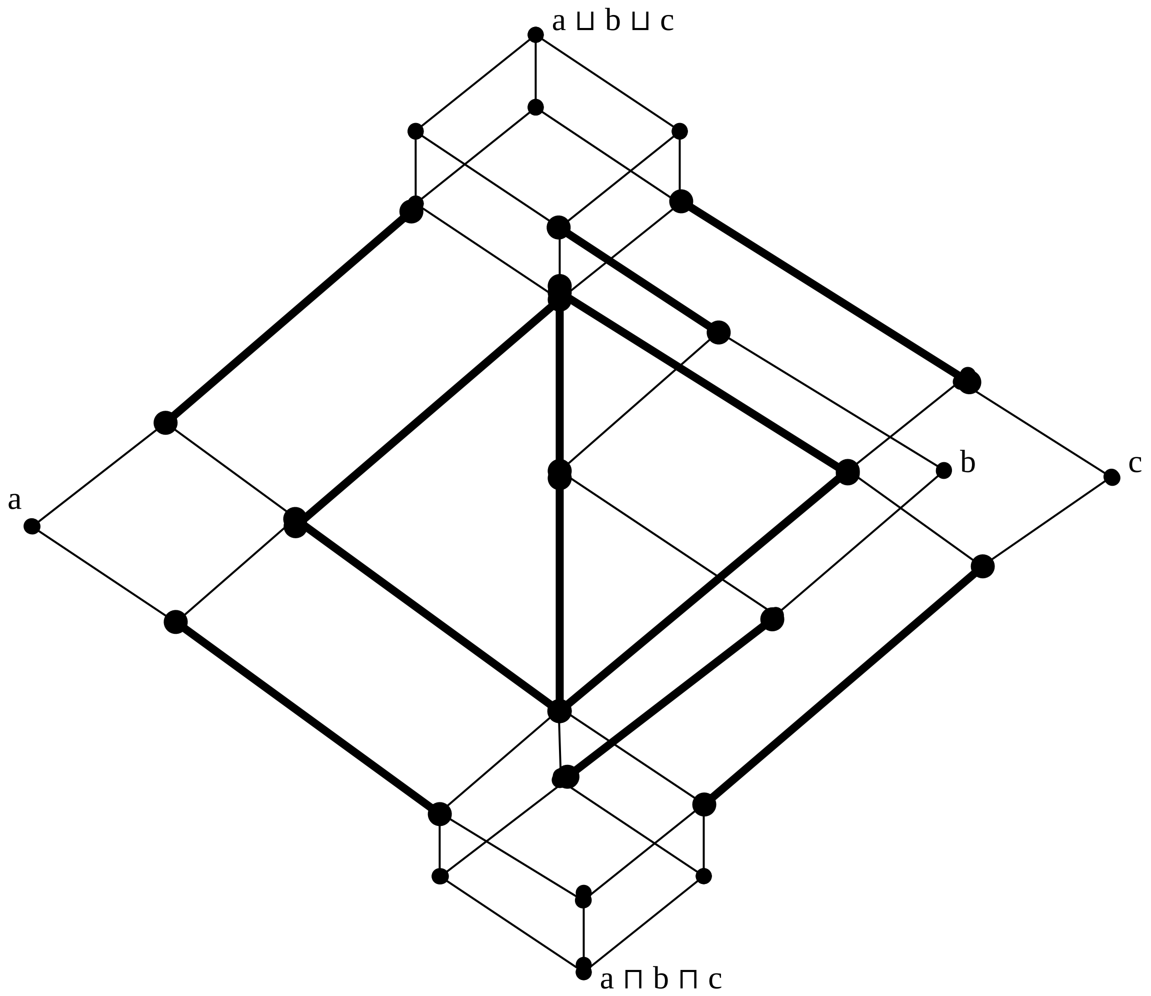


Homomorphisms can contract thin lines, but not thick lines. This means that A contains by necessity the pentagon as a substructure.
□

Proof of 2b: If a lattice is not distributive, the diamond can be embedded in it
Proof omitted, but the gist is this:

2b If a modular lattice is not distributive, the diamond can be embedded in it.

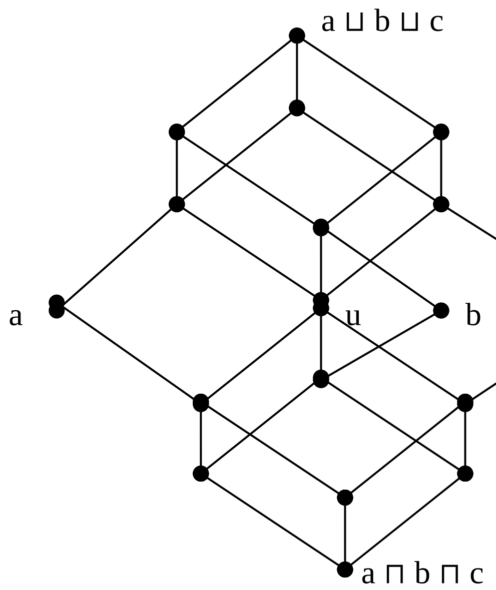
Proof sketch:
The proof goes in the same way as that of 1b, but it is (not surprisingly) much more involved. What we show is that if A is modular, but not distributive, the following substructure can be constructed:



Here too we check which relations are ⊑ , and hence can, in principle be contracted, and which are ⊏ (following from the requirement that the structure be **non-distributive**). ⊏ is indicated in the picture by the thick lines. It it is easy to see that the diamond in the middle cannot be contracted and is the minimal structure left after all contractions.
◻

We make this structure distributive by contracting the diamond in the middle to a single point: this gives the following distributive lattice:

'



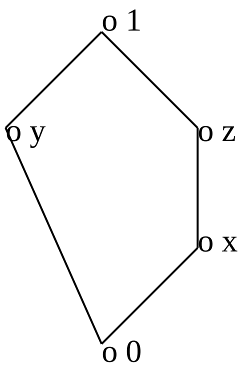
Here point u is the contraction point for the diamond.

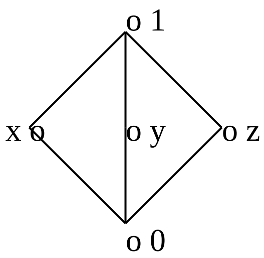
3.4. Complements

let **A** = $\langle A, \sqsubseteq \rangle$ be a bounded lattice and $a,b \in A$

b is a **complement of** a iff $a \sqcap b = 0$ and $a \sqcup b = 1$

You cannot have bounded lattices in which no element has a complement, because 0 and 1 obviously are each other's complement.
In general, in a lattice there may be elements that have more than one complement:
For instance in the pentagon, y has two complements: x and z: x and z each have only one compleement, y.
In the diamond. all the middle elements x, y and z have two complements.





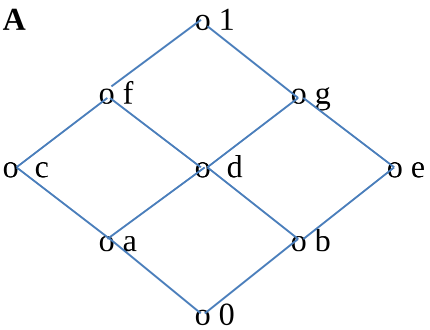
Fact: If **A** is a bounded distributive lattice, every element has at most one complement.
Proof:
Let b_1 and b_2 be complements of a:

$a \sqcap b_1 = a \sqcap b_2 = 0$
$a \sqcup b_1 = a \sqcup b_2 = 1$

$$
\begin{aligned}
b_1 \sqcap b_2 &= b_1 \sqcap b_1 \\
0 \sqcup (b_1 \sqcap b_2) &= 0 \sqcup (b_1 \sqcap b_2) \\
(b_1 \sqcap a) \sqcup (b_1 \sqcap b_2) &= (b_2 \sqcap a) \sqcup (b_1 \sqcap b_2) \\
b_1 \sqcap (a \sqcup b_2) &= b_2 \sqcap (a \sqcup b_1) && \text{By distributivity} \\
b_1 \sqcap 1 &= b_2 \sqcap 1 \\
b_1 &= b_2
\end{aligned}
$$

The lattice below is a distributive lattice:

A



1 and 0 are each other's complement
c and e are each other's compement
none of the other elements has a complement.

If a ∈ **A** has a unique complement we write the complement of a as ¬a.

Fact: If **A** is a bounded distributive lattice and a, ¬a ∈ A then ¬a = ⊔{b ∈ A: a ⊓ b = 0}

A bounded lattice **A** is **complemented** if for every a ∈ A: ¬a ∈ A (i.e. every element a has exactly one complement)

A **Boolean lattice** is a complemented distributive lattice.

A **Boolean algebra** is a structure **B** = <B, ⊑ , ¬ , ⊓ ,⊔, 0, 1> where:

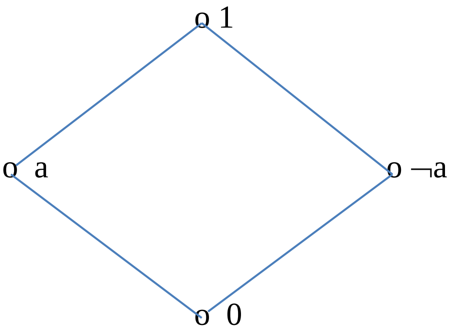
1. <B, ⊑> is a Boolean lattice
2. ¬ is the one place operation that maps every element b ∈ B onto its complement¬b
3. ⊓ and ⊔ are the two place operations of join and meet
4. 0 and 1 are the minimum and maximum

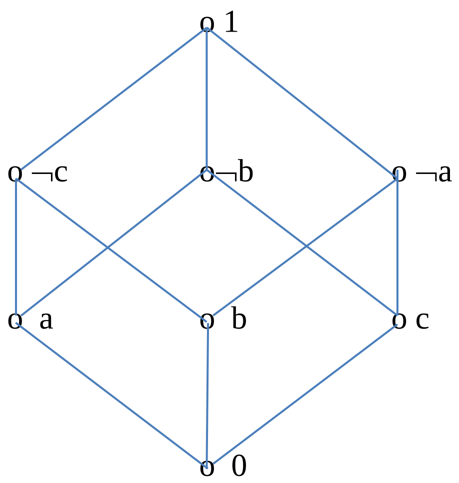
Purely algebraically a Boolean algebra is a structure **B** = <B, ¬, ⊓, ⊔, 0, 1> where

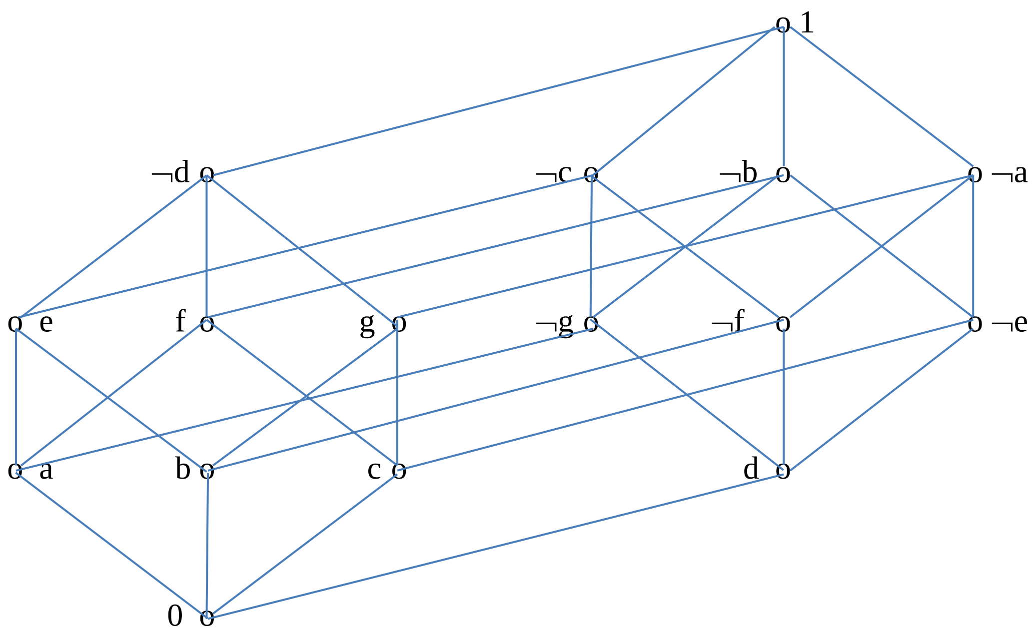
1. <B, ⊓, ⊔> satisfies idempotency, commutativity, associativity, absorption, the laws of 0 and 1 and distributivity
2. **Complementation:** a ⊔ ¬a = 1
   a ⊓ ¬a = 0

It is quite common, in the semantic literature, to actually replace the binary operations ⊓ and ⊔ by the complete operations on subsets. In need of a term, I will call such Boolean algebras c-**Boolean algebra**. c-Boolean algebras are, of course, complete, but should be distinguished from complete Boolean algebras (which have the binary operations ⊓ and ⊔ as operations). Unless explicitly states results for complete structures mentioned refer to Boolean algebras and not necessarily to c-Boolean algebras. Thus, distributivity does not automatically generalize to the complete operations of join and meet and the homomorphism requirement to preserve binary join and meet is not the same as preserving complete join and meet. has the operations of complete join and complete meet as the operations of the algebra.

Boolean algebras with complements indicated:







For complete distributive lattices a weaker notion of complementation suffices to define complete Boolean lattices:

Let **A** = <A, $\sqsubseteq$ > be a complete distributive lattice.
A is **witnessed** iff for every a_1,b $\in$ A$-${0}:
if $a_1 \sqsubset b$ then there is a $a_2 \in A-\{0\}$: $a_2 \sqsubseteq b$ and $a_1 \sqcap a_2 = 0$

When we use bounded structures to model a natural part-of relation, there is a discrepancy between the intuitive notion of *part* and the formal notion. Formally 0 is part of everything, but that is usually not the way the informal notion of *part* is used.
So when we talk about naturalistic parts, we mean **non-zero** parts; when we talk about proper parts, we mean **non-zero** proper parts, etc.
This shows too in the informal notions of disjointness and overlap, which we will define in terms of **non-zero** parts:

a and b **overlap**	**overlap**(a,b) iff $a \sqcap b \neq 0$
a and b **are disjoint**	**disjoint**(a,b) iff $a \sqcap b = 0$

Thus, for a and b to overlap they have to have a **non-zero** part in common.

The weakened condition for complementation says: if a_1 is a proper part of b, then there is some other part a_2 of b which doesn't overlap a_1.
This is a very intuitive notion: if you take something, but not everything away, there is something left, and what is left doesn't overlap what you took away.

Fact: Every witnessed complete distributive lattice is a complete Boolean lattice.

Proof:

Let **A** be a lattice with 0 and $a \in A$.

a^* is a **pseudocomplement** of a iff $a^* \in A$ and $a^* = \sqcup\{b \in A: a \sqcap b = 0\}$

The pseudocomplement of a is the join of all the elements of A that are disjoint from a.

Fact: If A is a bounded distributive lattice and $a^* \in A$ then $a^* = \neg a$.
This was mentioned above.

In general, in a Boolean algebra it is not guaranteed that every element has a pseudo-complement. But this *is* guaranteed if A is a witnessed complete distributive lattice.
We don't need to worry about 0 and 1, so let $a \in A - \{0,1\}$. Since A is witnessed,
$\{b \in A–\{0,1\}: a \sqcap b = 0\}$ is not empty. Since A is complete $\sqcup\{b \in A–\{0,1\}: a \sqcap b = 0\}$.
This is, of course, the same as $\sqcup\{b \in A: a \sqcap b = 0\}$, which is a^*. Hence $a^* \in A$. By the above fact if $a^* \in A$, $a^* = \neg a$, hence A is complemented.

Relative complements

Let **A** be a lattice with 0 and $a,b,c \in A$ and $a \sqsubseteq b$
c is **a relative complement of** a **in** b iff $a \sqcup c = b$ and $a \sqcap c = 0$

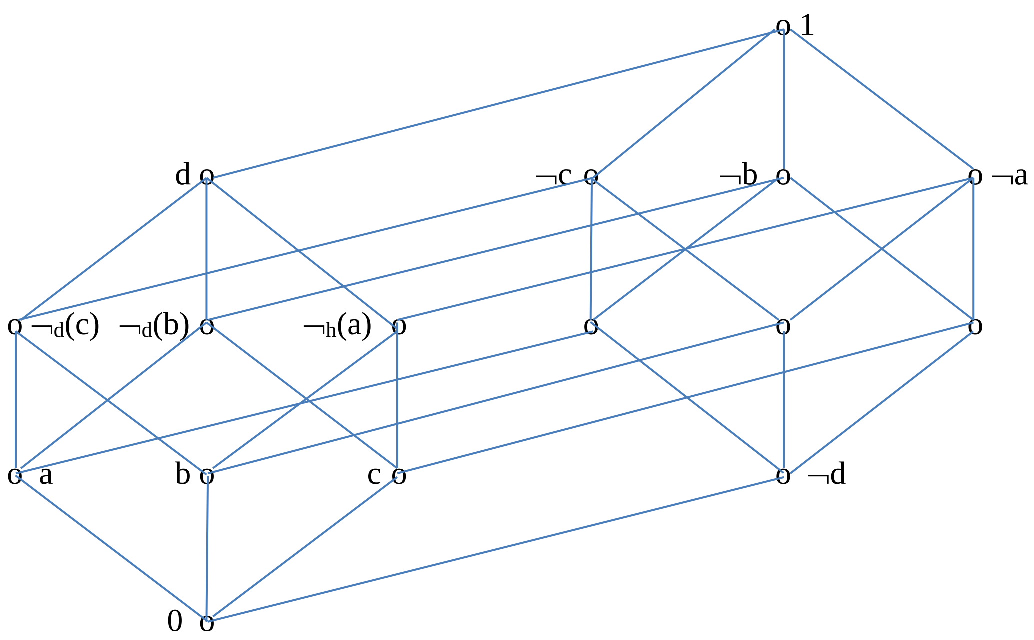
$\neg_b(a)$ is the unique relative complement of a in b (if there is a unique one).

A is **relatively complemented** iff for every $a,b \in A$ such that $a \sqsubseteq b$: $\neg_b a \in A$

Fact: If **A** is a Boolean algebra then **A** is relatively complemented and $\neg_b a = \neg a \sqcap b$

Proof: We assume that $a \sqsubseteq b$.
We need to prove that $a \sqcup (\neg a \sqcap b) = b$ and $a \sqcap (\neg a \sqcap b) = 0$
- $a \sqcap (\neg a \sqcap b) = (a \sqcap \neg a) \sqcap b = 0 \sqcap b = 0$
- $a \sqcup (\neg a \sqcap b) = (a \sqcup \neg a) \sqcap (a \sqcup b) = 1 \sqcap (a \sqcup b) = a \sqcup b = b$ (because $a \sqsubseteq b$)

Note that the ideal generated b, (b] is a convex set in A.
For $a \sqsubseteq b$, $a \in (b]$ and $\neg_b a \in (a]$. We can fruitfully regard $\neg_b a$ as the complement of a **within** convex set (b]. The fact, then, that Boolean algebras are relatively complemented means that each element a has a complement within every convex set (b] with $a \sqsubseteq b$:



$\neg_d(a) = \neg a \sqcap d$
$\neg_d(b) = \neg b \sqcap d$
$\neg_d(c) = \neg c \sqcap d$

$\neg_d(0) = \neg 0 \sqcap d = 1 \sqcap d = d$
$\neg_d(d) = \neg d \sqcap d = 0$

3.4. Appendix: Heyting algebras and de Morgan lattices

Heyting algebras.

Above we introduced the notion of a **pseudo complement**.
I introduce them here again, but in a bit different way:

Let **A** be a lattice with 0 and $a \in A$.

> a^* is a **pseudocomplement** of a iff $a^* \in A$ and $a \sqcap a^* = 0$ and
> for every $b \in A$: if $a \sqcap b = 0$ then $b \sqsubseteq a^*$.

An element can have at most one pseudocomplement: if b and c are both pseudocomplements of a, then $a \sqcap b = 0$ and $a \sqcap c = 0$ and hence $b \sqsubseteq c$ (since c is a pseudocomplement of a) and $c \sqsubseteq b$ (since b is too). Hence $b = c$.

Fact: if $a^* \in A$ then $a^* = \sqcup\{b \in A: a \sqcap b = 0\}$

> **A** is **pseudocomplemented** iff every element of A has a pseudocomplement.

Fact 0: $0^* = 1$ and $1^* = 0$
Proof: $\sqcup\{b \in A: 0 \sqcap b = 0\} = \sqcup A = 1$
$\sqcup\{b \in A: 1 \sqcap b = 0\}= \sqcup\{0\} = 0$

Fact 1: $a \sqsubseteq a^{**}$
Proof: a^{**} is the peudocomplement of a^*. Hence for every $b \in A$ such that $b \sqcap_A a^* = 0$
$b \sqsubseteq a^{**}$. Since $a \sqcap a^* = 0$, $a \sqsubseteq a^{**}$.

Fact 2: if $a \sqsubseteq b$ then $b^* \sqsubseteq a^*$
Proof: Let $a \sqsubseteq b$. Then $a \sqcap b = a$.
$b \sqcap b^* = 0$. Hence $a \sqcap b \sqcap b^* = 0$, hence $a \sqcap b^* = 0$. Hence $b^* \sqsubseteq a^*$

Fact 3: $a^* = a^{***}$
Proof: with fact 1 and 2: $a^{***} \sqsubseteq a^*$.
With fact 1: $a^* \sqsubseteq a^{***}$
Hence: $a^* = a^{***}$

Fact 4: If A is a witnessed bounded lattice then 1 is the only element $x \in A$ such that $x^* = 0$
Proof: Note that $0^* \neq 0$ (since $0 \neq 1$). Suppose $a \neq 1$ and $a \neq 0$. Then $a \sqsubset 1$.
Then, by the witness constraint, for some $b \in A - \{0\}$: $a \sqcap b = 0$
Then $\{b \in A: a \sqcap b = 0\} \neq \emptyset$ and $\{b \in A: a \sqcap b = 0\} \neq \{0\}$
But then $\sqcup\{b \in A: a \sqcap b = 0\} \neq 0$. Hence $a^* \neq 0$.

Fact 5: If **A** is a witnessed complete distributive lattice then **A** is a complete Boolean lattice.
Proof: We prove that $a \sqcup a^* = 1$ and $a \sqcap a^* = 0$

1. $a \sqcup a^* = 1$. We prove that $(a \sqcup a^*)^* = 0$.

(1)	$(a \sqcup a^*)^*$		=
(2)	$\sqcup\{b \in A: (a \sqcup a^*) \sqcap b = 0\}$		=
(3)	$\sqcup\{b \in A: (a \sqcap b) \sqcup (a^* \sqcap b) = 0\}$	By distributivity	=
(4)	$\sqcup\{b \in A: a \sqcap b = 0 \text{ and } a^* \sqcap b = 0\}$	Since the join can't be 0 otherwise	

Now if $a \sqcap b = 0$, $b \in \{b: a \sqcap b=0\}$, hence $b \sqsubseteq a^*$, hence $a^* \sqcap b = 0$ iff $b = 0$, so (4) is identical to (5):

(5) $\sqcup\{b \in A: b = 0\}$ =
(6) $\sqcup\{0\}$
(7) 0
So $(a \sqcup a^*)^* = 0$
Then, by fact 4, $a \sqcup a^* = 1$

2. $a \sqcap a^* = 0$

(1)	$a \sqcap a^*$		=
(2)	$a \sqcap \sqcup\{b \in A: a \sqcap b = 0\}$		=
(3)	$\sqcup\{a \sqcap b: a \sqcap b = 0\}$	By complete distributivity: $a \sqcap \sqcup X = \sqcup\{a \sqcap x: x \in X\}$	=
(4)	$\sqcup\{0\}$		
(4)	0		

Above we defined the notion of *complement* $\neg a$ and then went on to relativize that notion to the notion of *relative complement* by resetting the top element 1 to b: $\neg_b a$.
In Heyting algebras we do something similar with the notion of *pseudocomplement*, except that we reset *not the top element* 1 to b *but the bottom element* 0 to b:

$a^* = \sqcup\{c \in C: a \sqcap c = \mathbf{0}\}$
if $b \sqsubseteq a$ then:
$a \to b = \sqcup\{c \in C: a \sqcap c = \mathbf{b}\}$

Let A be a bounded lattice.

A is **relatively pseudocomplemented** if every element has a relative pseudocomplement.

Relatively pseudocomplemented lattices satisfy the Modus Ponens identity:

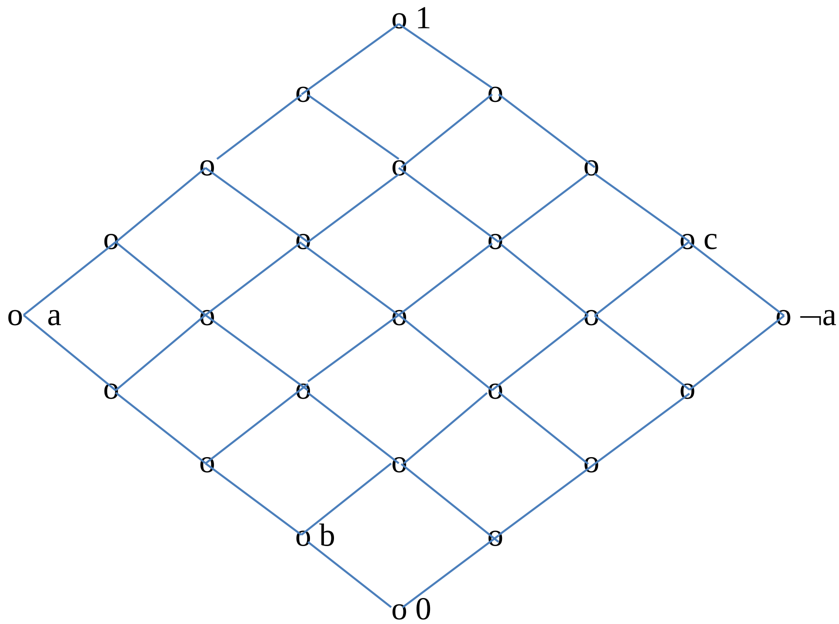
Modus Ponens identity: $(a \to b) \sqcap a = b$

Fact: In a relatively pseudocomplemented lattice the pseudocomplement is defined by: $a^* = a \to 0$

A **Heyting algebra** is a distributive relatively pseudocomplemented lattice.

Just as Boolean algebras are the natural structures associated with Classical Logic, Heyting algebras are the natural structures associated with Intuitionistic Logic.

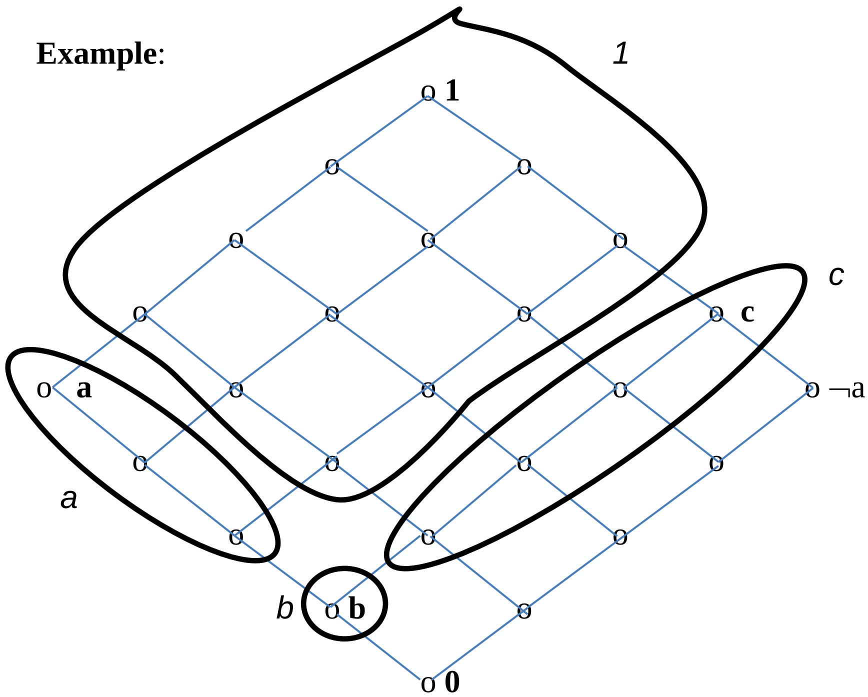
Example: Look at the following 'chess-board' and elements b, a and ¬a.



The elements a, ¬a, are the four elements that have complements.

We will look at the relative pseudcomplements α → b, where b ⊑ α:
We distinguish four regions in the set [b): sets *1, b, a, c*

Example:

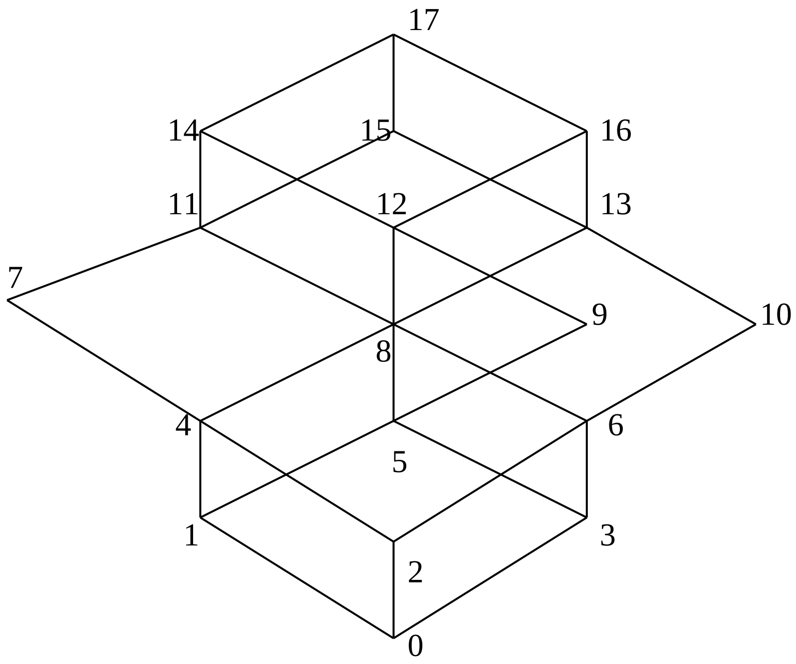


The following facts can be checked:

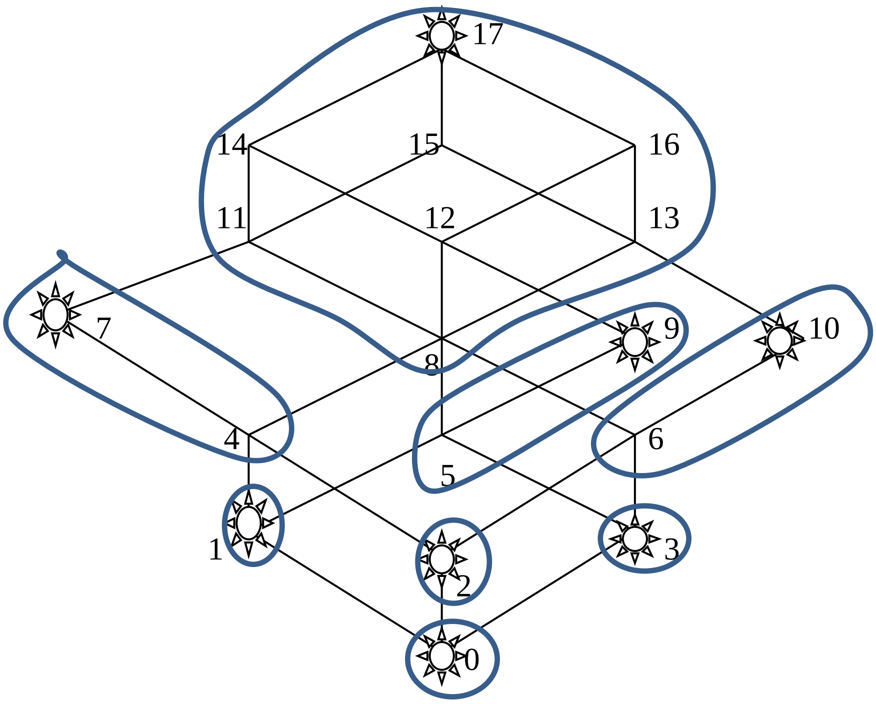
if $\alpha \in$ *1*, then $\alpha \to$ b = b (in logic: $(1 \to b) = b$ i.e. $\top \to \varphi$ is equivalent to φ)
if $\alpha \in$ *b*, then $\alpha \to$ b = 1 (in logic: $(b \to b) = 1$, i.e. $\varphi \to \varphi$ is a tautology)
if $\alpha \in$ *a*, then $\alpha \to$ b = c
if $\alpha \in$ *c*, then $\alpha \to$ b = a

The fact that *1*, *b*, *a*, *c* has four such regions, ordered as indicated, is neither insignificant nor arbitrary.

Distributive lattice D_3 is pseudocomplemented:

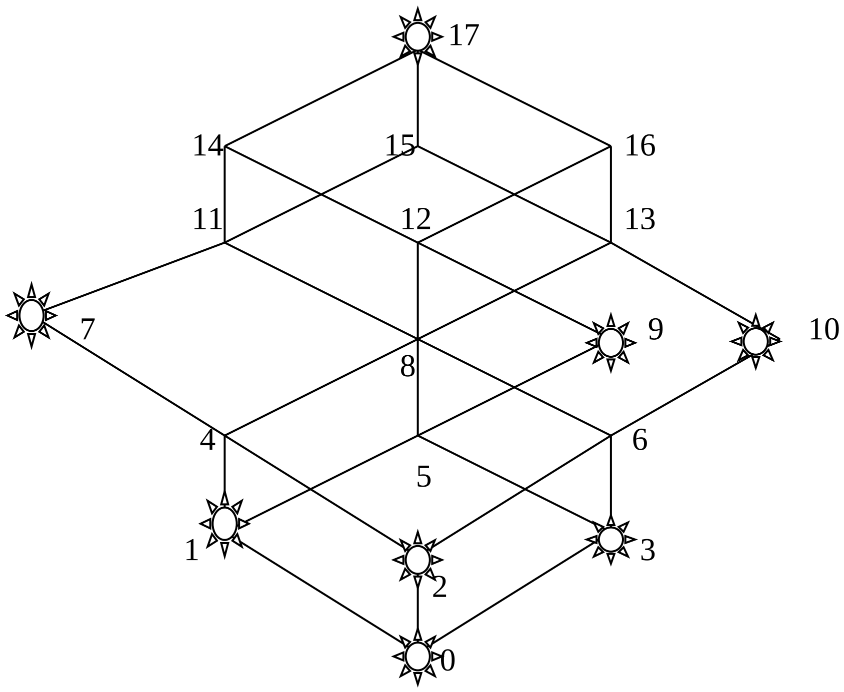


0* = 17	8* = 11* = 12* = 13* = 14* = 15* = 16* = 17* = 0
1* = 10	6* = 10* = 1
2* = 9	5* = 9* = 2
3* = 7	4* = 7* = 3



The pseudocomplements are, of course, also the relative pseudocomplemens α → 0.

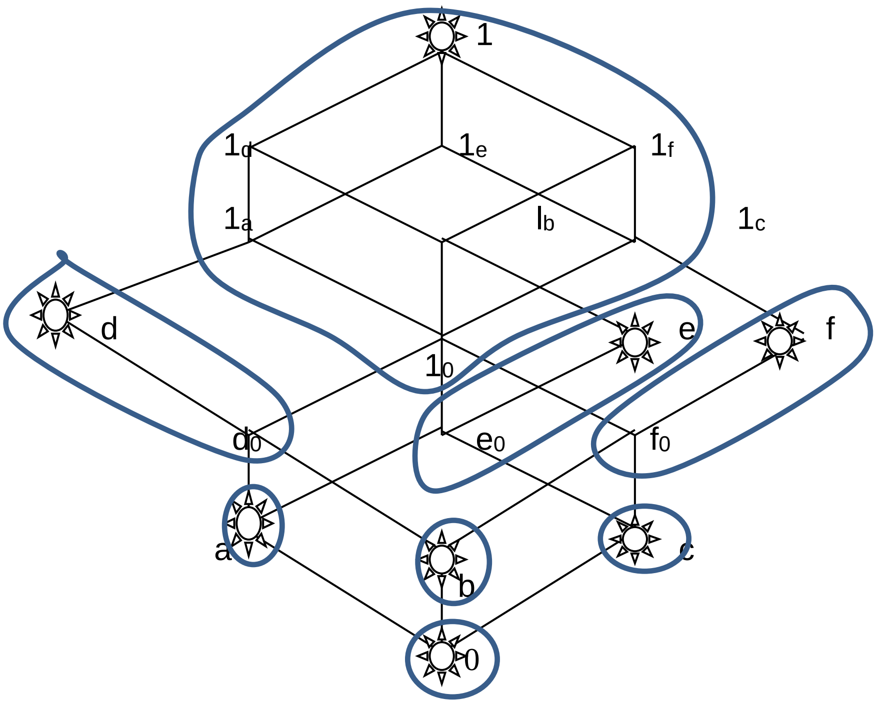
The set of pseudocomplements form a Boolean algebra, but not a sublattice of D_3: while meets are preserved, joins are not:



Fact 1: Every Heyting algebra is pseudocomplemented

Fact 2: Every complete distributive lattice is a Heyting algebra.

(Complete) distributive lattice – Heyting Algebra – represents the logical structure of Intuitionistic Logic.



The pseudo complements are indicated as suns.
The circled elements have the same pseudo complement:

0^* = 1
$1_0^* = 1_a^* = 1_b^* = 1_c^* = 1_d^* = 1_e^* = 1_f^* = 1^*$ = 0

a^* = f
$f_0^* = f^* =$ a

b^* = e
$e_0^* = e^*$ = a

c^* = d
$d_0^* = d^*$ = c

You can verify in the picture that: $a \sqsubseteq a^{**}$, but that not always: $a^{**} \sqsubseteq a$:

For instance: $f_0^* = a$ and $a^* = f$, hence $f_0^{**} = f$, and $f_0 \sqsubseteq f$ but $f_0 \neq f$.

This is precisly where intuitionistic negation differs from classical negation:
φ and $\neg\neg\varphi$ are classically equivalent, but $\neg\neg\varphi$ does not entail φ in intuitionistic logic.

De Morgan Lattices

Let W be a set of worlds.
We define the following set of propositions:

$P_W = \{<p^+,p^->: p^+, p^- \subseteq W \text{ and } p^+ \cap p^- = \emptyset\}$

In classical logic we think of the proposition expressed by φ, ⟦φ⟧ as the set of worlds where φ is true. This gives:

$$⟦\varphi⟧ = \{w \in W: ⟦\varphi⟧_w = 1\}$$

$P_W = \ <pow(P), -, \cup, \cap, W, \emptyset>$

So
$$⟦\neg\varphi⟧ = W - ⟦\varphi⟧$$
$$⟦\varphi \wedge \psi⟧ = ⟦\varphi⟧ \cap ⟦\psi⟧$$
$$⟦\varphi \vee \psi⟧ = ⟦\varphi⟧ \cup ⟦\psi⟧$$

$$⟦\varphi \vee \neg\varphi⟧ = W$$

In strong Kleene three valued setting, φ can be undefined in worlds. Here we think of the proposition expressed by φ as a pair of sets of worlds:

$$⟦\varphi⟧ = \ <⟦\varphi⟧^+,⟦\varphi⟧^->$$
$$⟦\varphi⟧^+ = \{w \in W: ⟦\varphi⟧_w = 1\}$$
$$⟦\varphi⟧^- = \{w \in W: ⟦\varphi⟧_w = 0\}$$

Thus, a proposition p is pair $p = \ <p^+,p^->$ where:
p^+ is the set of worlds where p is true
p^- is the set of worlds where p is false

We set: $p^\perp = W - (p^+ \cup p^-)$
p^- is the set of worlds where p is undefined

Example: Let W = {a,b,c}

Then the **total** propositions are:

<{a,b,c},Ø>

<{a,b},{c}> <{a,c},{b}> <{b,c},{a}>

<{a},{b,c}> <{b},{a,c}> <{c},{a,b}>

<Ø,{a,b,c}>

these are defined in every world.

p = <{a},{b}> and q = <{a},Ø> are partial propositions: p is true in world a, false in b, and undefined in c. q is a partial tautology: true in a, undefined in b and c, but false in no world.

Note the element <Ø,Ø>, the totally partial proposition, true in no world, false in no world.

We form the structure:

$\mathbf{M_A} = \langle M_A, \sqsubseteq, \sim, \sqcap, \sqcup, 0, 1\rangle$

with:

$\langle p^+,p^-\rangle \sqsubseteq \langle q^+,q^-\rangle$ iff $p^+ \subseteq q^+$ and $q^- \subseteq p^-$

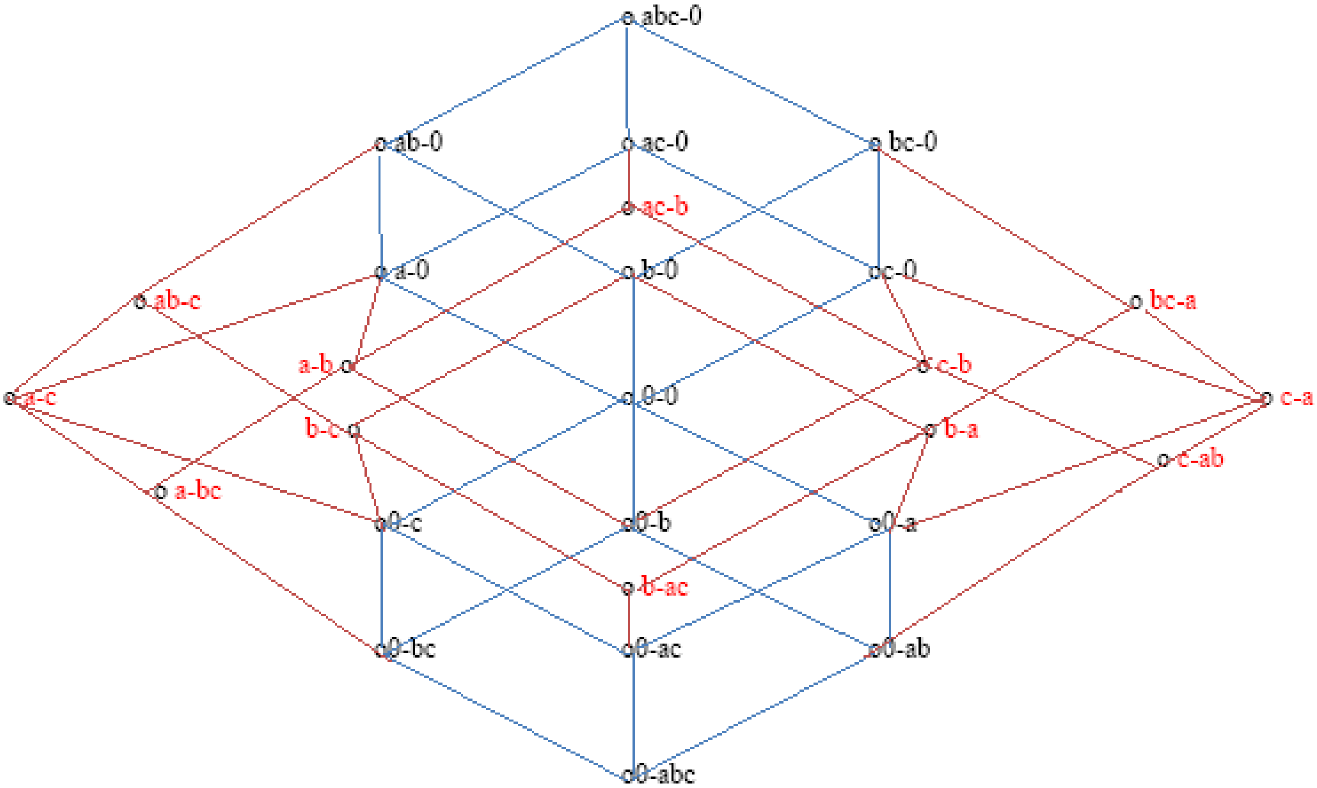
$\sim(\langle p^+,p^-\rangle) = \langle p^-,p^+\rangle$

$\langle p^+,p^-\rangle \sqcap \langle q^+,q^-\rangle = \langle p^+ \cap q^+, q^- \cup q^-\rangle$

$\langle p^+,p^-\rangle \sqcup \langle q^+,q^-\rangle = \langle p^+ \cup q^+, q^- \cap q^-\rangle$

$1 = \langle A,Ø\rangle \qquad 0 = \langle Ø,A\rangle$

This notion of complementation is called *de Morgan-complementation* and the structure is called a (complete) *de Morgan lattice*:



Note that, unlike pseudo-complementation, de Morgan lattices satisfy the laws of double negation and the de Morgan laws:

$\sim\sim p = p$
$\sim(p \sqcup q) = \sim(\sim p \sqcap \sim q)$
$\sim(p \sqcap q) = \sim(\sim p \sqcup \sim q)$

But de-Morgan complements are not complements, what they don't necessarily satisfy is either of the complement laws:

$p \sqcup \sim p = 1$
$p \sqcap \sim p = \mathbf{0}$

Instead, we can define:

$\mathbf{1}_A = \{p \in M_A: p = \langle p^+, \varnothing \rangle \text{ for some } p^+ \subseteq A\}$
$\mathbf{0}_A = \{p \in M_A: p = \langle \varnothing, p^- \rangle \text{ for some } p^- \subseteq A\}$

And the the laws that **do** hold are:

$p \sqcup \sim p \in \mathbf{1}_A$
$p \sqcap \sim p \in \mathbf{0}_A$

The idea is: if p is undefined in world c, then so is ~p, and also $p \sqcup \sim p$.

To give a concrete example. Assume that there are three worlds, a,b and c and there is a unique president in world a and in world b, but not in world c, and the president is smart in world a, and dumb in world b.

Then:

$[\![smart(\sigma(president))]\!] = \langle [\![smart(\sigma(president))]\!]^+, [\![smart(\sigma(president))]\!]^- \rangle = \langle \{a\},\{b\} \rangle$

$[\![smart(\sigma(president) \vee \neg smart(\sigma(president)]\!]^+ =$
$\{w: [\![smart(\sigma(president) \vee \neg smart(\sigma(president)]\!]_w = 1\} =$
$\{w: [\![smart(\sigma(president))]\!]_w = 1\} \cup \{w: [\![\neg smart(\sigma(president)]\!]_w = 1\} =$
$\{w: [\![smart(\sigma(president))]\!]_w = 1\} \cup \{w: [\![smart(\sigma(president)]\!]_w = 0\} = \{a,b\}$

$[\![smart(\sigma(president) \vee \neg smart(\sigma(president)]\!]^- =$
$\{w: [\![smart(\sigma(president) \vee \neg smart(\sigma(president)]\!]_w = 0\} = \varnothing$

So, indeed,

$[\![smart(\sigma(president) \vee \neg smart(\sigma(president)]\!] = \langle \{a,b\},\varnothing \rangle$

3.5 Boolean algebras

Lemma 1: In a distributive lattice an element can have at most one complement.
Proof:
We proved this above.

Lemma 2: In a Boolean lattice: $a = \neg\neg a$
Proof:
Both a and $\neg\neg a$ are the complement of $\neg a$, hence, by lemma 1, $a = \neg\neg a$
□

Lemma 3: In a Boolean lattice: $\neg(a \sqcap b) = \neg a \sqcup \neg b$
Proof:
The following equivalences hold:

$(a \sqcap b) \sqcup (\neg a \sqcup \neg b)$	
$(a \sqcap b) \sqcup \neg a \sqcup \neg b$	Getting rid of some brackets
$((a \sqcup \neg a) \sqcap (b \sqcup \neg a)) \sqcup \neg b$	Distributivity on $(a \sqcap b) \sqcup \neg a$
$(1 \quad \sqcap (b \sqcup \neg a)) \sqcup \neg b$	
$b \sqcup \neg a \quad \sqcup \neg b$	
1	

$(a \sqcap b) \sqcap (\neg a \sqcup \neg b)$
$(a \sqcap b \sqcap \neg a) \sqcup (a \sqcap b \sqcap \neg b)$
$0 \quad \sqcup \quad 0$
0

Hence $\neg(a \sqcap b)$ and $\neg a \sqcup \neg b$ are both the complement of $a \sqcap b$,
hence $\neg(a \sqcap b) = \neg a \sqcup \neg b$.
□

Lemma 4: In a Boolean lattice: $a \sqsubseteq b$ iff $\neg b \sqsubseteq \neg a$.
Proof:
The following equivalences hold:

$a \sqsubseteq b$
$a \sqcap b \;=\; a$
$\neg(a \sqcap b) \;=\; \neg a$
$\neg a \sqcup \neg b \;=\; \neg a$
$\neg b \sqsubseteq \neg a$

Lemma 5: In a Boolean lattice: $a \sqcap b = 0$ iff $a \sqsubseteq \neg b$
Proof:
Assume $a \sqsubseteq \neg b$. Then $a \sqcap \neg b = a$, and $(a \sqcap \neg b) \sqcap b = a \sqcap b$, hence $a \sqcap b = 0$
Assume $a \sqcap b = 0$.
We argue that $a \sqcup \neg b$ is the complement of b
$(a \sqcup \neg b) \sqcup b = 1$
$(a \sqcup \neg b) \sqcap b = (a \sqcap b) \sqcup (\neg b \sqcap b) = 0 \sqcup 0 = 0$
Hence $a \sqcup \neg b = \neg b$, and $a \sqsubseteq \neg b$

Lemma 6: If a is an atom in a Boolean lattice B, then for every $b \in B - \{0\}$:

$a \sqsubseteq b$ or $a \sqsubseteq \neg b$, not both.

Proof:

Let a be an atom and let $b \in B - \{0\}$, and assume $a \not\sqsubseteq b$.

Then $a \sqcap b \neq a$.

Since, $a \sqcap b \sqsubseteq a$, this means that $a \sqcap b = 0$, since a is an atom.

Hence, with lemma 5, $a \sqsubseteq \neg b$.

Not both, because then $a \sqsubseteq b$ and $a \sqsubseteq \neg b$, and hence $a \sqsubseteq b \sqcap \neg b$, and $a = 0$ and not an atom.

Lemma 7: If a is an atom in Boolean lattice B and $b,c \in B$,

then $a \sqsubseteq b \sqcup c$ iff $a \sqsubseteq b$ or $a \sqsubseteq c$.

Proof:

Let a be an atom in B and let $a \sqsubseteq b \sqcup c$ and let $a \not\sqsubseteq b$.

Then, with lemma 6, $a \sqsubseteq \neg b$, and, with lemma 5, $a \sqcap b = 0$.

Since $a \sqsubseteq b \sqcup c$, $a \sqcap (b \sqcup c) = a$.

With distributivity, this means that $(a \sqcap b) \sqcup (a \sqcap c) = a$.

Thus $0 \sqcup (a \sqcap c) = a$, and thus $a = a \sqcap c$, hence $a \sqsubseteq c$.

(The other side is trivial: if $a \sqsubseteq b$ then $a \sqsubseteq b \sqcup c$ and if $a \sqsubseteq c$ then $a \sqsubseteq b \sqcup c$)

◻

We repeat:

$AT_x = \{a \in ATOM_B : a \sqsubseteq x\})$

B is **atomic** iff for every $b \in B - \{0\}$: $AT_b \neq \emptyset$

B is **atomistic** iff for every $b \in B$: $b = \sqcup (AT_b)$

Lemma 8: Let **B** be a complete Boolean lattice. Let $A \subseteq ATOM_B$.

Then $A = AT_{\sqcup A}$

Proof:

-Let $a \in A$. Then $a \in ATOM_B$ and $a \sqsubseteq \sqcup A$. Hence $a \in AT_{\sqcup A}$

So $A \subseteq AT_{\sqcup A}$

-Let $a \in AT_{\sqcup A}$. Then $a \in ATOM_B$ and $a \sqsubseteq \sqcup A$.

Assume $a \notin A$.

Then for every $a_1 \in A$: $a \sqcap a_1 = 0$, since $AT_{\sqcup A} \cup \{a\} \subseteq ATOM_B$.

Then for every $a_1 \in A$: $a_1 \sqsubseteq \neg a$ (by lemma 5).

Hence $\sqcup A \sqsubseteq \neg a$.

Since $a \sqsubseteq \sqcup A$, it follows that $a \sqsubseteq \neg a$.

Hence $a \sqcap \neg a = a$, i.e. $a = 0$. But $a \in ATOM_B$.

Contradiction, so $a \in A$. ◻

Corrollary 9: Let **B** be a complete Boolean lattice and let $A_1, A_2 \subseteq ATOM_B$.

Then $A_1 = A_2$ iff $\sqcup A_1 = \sqcup A_2$.

Proof:

-If $A_1 = A_2$, then obviously $\sqcup A_1 = \sqcup A_2$.

-If $\sqcup A_1 = \sqcup A_2$, then $AT_{\sqcup A1} = AT_{\sqcup A2}$. Then, by lemma 9, $A_1 = A_2$

Corrollary 10: Let **B** be a complete Boolean lattice.
Then: $\{AT_b: b \in B\} = pow(ATOM_B)$.

Proof:

-Obviously $\{AT_b: b \in B\} \subseteq pow(ATOM_B)$.
-Let $A \subseteq ATOM_B$.
Since B is complete, $\sqcup A \in B$.
By lemma 8, $A = AT_{\sqcup A}$
Hence $A \in \{AT_b: b \in B\}$

Theorem 11: Let B be a complete Boolean lattice.
Then B is atomic iff B is atomistic.

1. If B is atomistic, B is obviously atomic.

2. Assume B is atomic. Let $x \in B - \{0\}$.
$\sqcup(AT_x) \sqsubseteq x$, by definition of $\sqcup$.
We want to show: $x \sqsubseteq \sqcup(AT_x)$.
Assume that this doesn't hold. Then $\sqcup(AT_x) \sqsubset x$.

Then for some $y \sqsubseteq x$, $y \neq 0$ and $\sqcup(AT_x) \sqcap y = 0$, by the witness postulate.
Since B is atomic, $AT_y \neq \emptyset$.
Hence, for some $a \in ATOM_y$: $\sqcup AT_x \sqcap a = 0$
$a \sqsubseteq y$ and $y \sqsubseteq x$, so $a \in AT_x$. But then $\sqcup AT_x \sqcap a = 0$, which means that $\sqcup AT_x = 0$.
But that can only be if $AT_x = \emptyset$. But we know that $AT_x \neq \emptyset$, because B is atomic.
Contradiction.

Theorem 12: Let **B** be a complete atomic Boolean algebra.
Then **B** is isomorphic to **pow(ATOM$_B$)**

Proof:
Let h: B $\rightarrow$ **pow**(ATOM$_B$) be the function such that for every $b \in B$: $h(b) = AT_b$.
We prove that h is an isomorphism.

1. h is one-one.
Let h(a) = h(b). Then $AT_a = AT_b$. Then $\sqcup(AT_a) = \sqcup(AT_b)$.
B is atomic, hence, by theorem 11, atomistic. This means that
$a = \sqcup(AT_a)$ and $b = \sqcup(AT_b)$. Hence a = b.

2. h is onto.
Let $A \in$ **pow**(ATOM$_B$). Then, by corrollary 10, $A \in \{AT_b: b \in B\}$.
Hence for some $b \in B$: $A = AT_b$. This means that for some $b \in B$: $A = h(b)$.

3. Let $a \sqsubseteq b$. Since **B** is atomistic, then $\sqcup(AT_a) \sqsubseteq \sqcup(AT_b)$.
Then for every $c \in AT_a$: $c \sqsubseteq \sqcup(AT_b)$.
Hence, again by atomicity, for every $c \in AT_a$: $c \sqsubseteq b$, hence $c \in AT_b$.
So $AT_a \subseteq AT_b$, and hence $h(a) \subseteq h(b)$.

3. $h(\neg a) = AT_{\neg a} = \{c \in ATOM_B: c \sqsubseteq \neg a\}$ = [by lemma 6] $ATOM_B - AT_a = ATOM_B - h(a)$.

4. $h(a \sqcap b) = AT_{a \sqcap b} = \{c \in ATOM_B: c \sqsubseteq a \sqcap b\} = \{c \in ATOM_B: c \sqsubseteq a \text{ and } c \sqsubseteq b\} =$
$\{c \in ATOM_B: c \sqsubseteq a\} \cap \{c \in ATOM_B: c \sqsubseteq b\} = AT_a \cap AT_b = h(a) \cap h(b)$.

5. $h(a \sqcup b) = AT_{a \sqcup b} = \{c \in ATOM_B: c \sqsubseteq a \sqcup b\}$ = [by lemma 7]
$\{c \in ATOM_B: c \sqsubseteq a \text{ or } c \sqsubseteq b\} = \{c \in ATOM_B: c \sqsubseteq a\} \cup \{c \in ATOM_B: c \sqsubseteq b\} =$
$AT_a \cup AT_b = h(a) \cup h(b)$.

6. $h(0) = AT_0 = \emptyset$
$h(1) = AT_1 = ATOM_B$.

Corrollary 13: The complete atomic Boolean algebras are up to isomorphism
the powerset Boolean algebras.

3.6 Some constructions

3.6.1. Product Boolean algebras

Let **A** and **B** be Boolean algebras.

The **product** of **A** and **B, A × B,** is given by:

A × B = <$B_\times, \sqsubseteq_\times, \neg_\times, \sqcap_\times, \sqcup_\times, 0_\times, 1_\times$> where:

1. $B_\times = A \times B$
2. $<a_1,b_1> \sqsubseteq_x <a_2,b_2>$ iff $a_1 \sqsubseteq_A a_2$ and $b_1 \sqsubseteq_B b_2$
3. $\neg_\times(<a,b>) = <\neg_A a, \neg_B b>$
4. $<a_1,b_1> \sqcap_\times <a_2,b_2> = <a_1 \sqcap_A a_2, b_1 \sqcap_B b_2>$
5. $<a_1,b_1> \sqcup_\times <a_2,b_2> = <a_1 \sqcup_A a_2, b_1 \sqcup_B b_2>$
6. $0_\times = <0_A,0_B>$
7. $1_x = <1_A,1_B>$

Fact: A × B is a Boolean algebra.

Proof:
1. $\sqsubseteq_\times$ is a partial order.
-$\sqsubseteq_\times$ is reflexive:
Since for every $a \in A$: $a \sqsubseteq_A a$ and for every $b \in B$: $b \sqsubseteq_B b$,
for every $<a,b> \in A \times B$: $<a,b> \sqsubseteq_x <a,b>$.

-$\sqsubseteq_\times$ is antisymmetric:
Assume $<a_1,b_1> \sqsubseteq_\times <a_2,b_2>$ and $<a_2,b_2> \sqsubseteq_\times <a_1,b_2>$.
Then $a_1 \sqsubseteq_A a_2$ and $b_1 \sqsubseteq_B b_2$ and $a_2 \sqsubseteq_A a_1$ and $b_2 \sqsubseteq_B b_1$,
hence $a_1 = a_2$ and $b_1 = b_2$, hence $<a_1,b_2> = <a_2,b_2>$

-$\sqsubseteq_\times$ is transitive:
Let $<a_1,b_1> \sqsubseteq_\times <a_2,b_2>$ and $<a_2,b_2> \sqsubseteq_\times <a_3,b_3>$.
Then $a_1 \sqsubseteq_A a_2$ and $b_1 \sqsubseteq_B b_2$ and $a_2 \sqsubseteq_A a_3$ and $b_2 \sqsubseteq_B b_3$,
hence $a_1 \sqsubseteq_A a_3$ and $b_1 \sqsubseteq_B b_3$, hence $<a_1,b_1> \sqsubseteq_\times <a_3,b_3>$

2. $\sqcap_\times$ is meet:
-$<a_1,b_1> \sqcap_\times <a_2,b_2> = <a_1 \sqcap_A a_2,b_1 \sqcap_B b_2>$

$a_1 \sqcap_A a_2 \sqsubseteq_A a_1$ and $a_1 \sqcap_A a_2 \sqsubseteq_A a_2$
$b_1 \sqcap_B b_2 \sqsubseteq_B b_1$ and $b_1 \sqcap_B b_2 \sqsubseteq_B b_2$,
hence
$<a_1,b_1> \sqcap_\times <a_2,b_2> \sqsubseteq_\times <a_1,b_1>$ and $<a_1,b_1> \sqcap_\times <a_2,b_2> \sqsubseteq_\times <a_2,b_2>$.

-Let $<a,b> \sqsubseteq_\times <a_1,b_1>$ and $<a,b> \sqsubseteq_\times <a_2,b_2>$.
Then $a \sqsubseteq_A a_1$ and $b \sqsubseteq_B b_1$ and $a \sqsubseteq_A a_2$ and $b \sqsubseteq_B b_2$,
hence $a \sqsubseteq_A a_1 \sqcap_A a_2$ and $b \sqsubseteq_B b_1 \sqcap_B b_2$,
hence $<a,b> \sqsubseteq_\times <a_1,b_1> \sqcap_\times <a_2,b_2>$.

3. We show that $\sqcup_\times$ is join in $\sqsubseteq_\times$ by a similar argument.

4. $0_\times = \langle 0_A,0_B \rangle$.
Since for every $a \in A$ $0_A \sqsubseteq_A a$ and for every $b \in B$ $0_B \sqsubseteq_B b$,
for every $\langle a,b \rangle \in A \times B$ $\langle 0_A,0_B \rangle \sqsubseteq_\times \langle a,b \rangle$.
Hence $0_\times$ is the minumum under $\sqsubseteq_\times$.

Similarly $1_\times$ is the maximum under $\sqsubseteq_A$

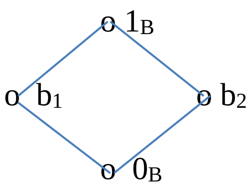
So **A × B** is a bounded lattice.

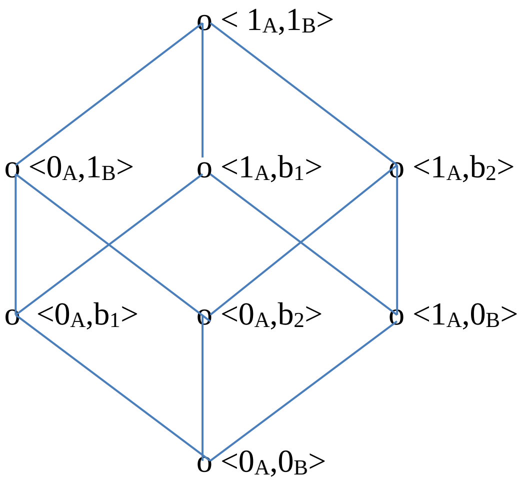
5. Distributivity:
$\langle a_1,b_1 \rangle \sqcap_\times (\langle a_2,b_2 \rangle \sqcup_\times \langle a_3,b_3 \rangle) =$
$\langle a_1 \sqcap_A (a_2 \sqcup_A a_3),b_1 \sqcap_B (b_2 \sqcup_B b_3) \rangle =$
$\langle (a_1 \sqcap_A a_2) \sqcup_A (a_1 \sqcap_A a_3),(b_1 \sqcap_B b_2) \sqcup_B (b_1 \sqcap_B b_3) \rangle =$
$(\langle a_1 \sqcap_A a_2,b_1 \sqcap_B b_2 \rangle \sqcup_\times \langle a_1 \sqcap_A a_3,b_1 \sqcap_B b_3 \rangle =$
$(\langle a_1,b_1 \rangle \sqcap_\times \langle a_2,b_2 \rangle) \sqcup_\times (\langle a_1,b_1 \rangle \sqcap_x \langle a_2,b_2 \rangle)$

6. Complementation:
$\langle a,b \rangle \sqcap_\times \neg_\times(\langle a,b \rangle) =$
$\langle a,b \rangle \sqcap_\times \langle \neg_A a,\neg_B b \rangle =$
$\langle a \sqcap_A \neg_A a,b \sqcap_B \neg_B b \rangle =$
$\langle 0_A,0_B \rangle = 0_\times$.

$\langle a,b \rangle \sqcup_\times \neg_\times(\langle a,b \rangle) =$
$\langle a,b \rangle \sqcup_\times \langle \neg_A a,\neg_B b \rangle =$
$\langle a \sqcup_A \neg_A a,b \sqcup_B \neg_B b \rangle =$
$\langle 1_A,1_B \rangle = 1_\times$.
□

Example





3.6.2. Function space Boolean algebras:

$(A \rightarrow B)$ is the set of all functions from A into B.

Let A be a set and **B** = $<B, \sqsubseteq_B, \neg_B, \sqcap_B, \sqcup_B, 0_B, 1_B>$ a Boolean algebra.

We define:

(A → B) = $<(A \rightarrow B), \sqsubseteq, \neg, \sqcap, \sqcup, 0, 1>$ where:

1. $f \sqsubseteq g$ iff for every $x \in A$: $f(x) \sqsubseteq_B g(x)$
2. $\neg f = \lambda x \in A: \neg_B(x)$
3. $f \sqcap g = \lambda x \in A: f(x) \sqcap_B g(x)$
4. $f \sqcup g = \lambda x \in A: f(x) \sqcup_B g(x)$
5. $0 = \lambda x \in A: 0_B$
6. $1 = \lambda x \in A: 1_B$

Fact 1: (A → B) is a Boolean algebra.

Let $F \subseteq (A \rightarrow B$

$\sqcap F = \lambda x \in A: \sqcap_B(\{f(x): f \in F\}$
$\sqcup F = \lambda x \in A: \sqcup_B(\{f(x): f \in F\}$

Fact 2: (A → B) is complete iff **B** is complete.
Fact 3: (A → B) is atomless iff **B** is atomless.
Fact 4: (A → B) is atomic iff **B** is atomic.

Proof: Similar to the proof for products

$\sqcup X = \{<a, \sqcup_B\{f(a): f \in X\}: a \in A\}$
$\sqcap X = \{<a, \sqcap_B\{f(a): f \in X\}: a \in A\}$

Fact 2: If $\sqcup X \in (A \rightarrow B)$, then $\sqcup X$ is the supremum of X under $\sqsubseteq$.
Similarly, $\sqcap X$ is the infimum of X if it exists.
Proof: Similar to the clauses for binary $\sqcap$ and $\sqcup$

Fact 3: (A → B) is complete iff **B** is complete.
Proof:
1. Assume **B** is complete.
Let $X \subseteq (A \rightarrow B)$.
Then for every $a \in A$: $\sqcup_B\{f(a): f \in X\} \in B$.
Hence $\sqcup X \in (A \rightarrow B)$.

2. Assume **(A → B)** is complete.
Let $X \subseteq B$
Look at $\{\{<a,x>: a \in A\}: x \in X\}$.
Since **(A → B)** is complete, $\sqcup\{\{<a,x>: a \in A\}: x \in X\} \in (A \rightarrow B)$.
$\sqcup\{\{<a,x>: a \in A\}: x \in X\} = \{<a, \sqcup_B\{x: x \in X\}>: a \in A\} =$
$\{<a, \sqcup_B X>: a \in A\}$. Hence $\sqcup_B X \in B$.

A similar argument shows the same for $\sqcap_B X$.
□

Let $a \in A$ and $b \in B$.

$$\mathbf{A_{a,b}} = \{<a,b>\} \cup \{<x, 0_B>: x \in A - \{a\}\}$$

Fact 4: $A_{a,b} \sqsubseteq f$ iff $b \sqsubseteq_B f(a)$
Proof:
Let $A_{a,b} \sqsubseteq f$.
Then for every $x \in A$: $A_{a,b}(x) \sqsubseteq_B f(x)$.
Then $A_{a,b}(a) \sqsubseteq_B f(a)$, hence $b \sqsubseteq_B f(a)$.

Let $f \in (A \rightarrow B)$, $a \in A$, $b \in B$, and $b \sqsubseteq_B f(a)$.
Then for every $x \in A$: $A_{a,b}(x) \sqsubseteq_B f(x)$.
Namely: either $x = a$, and then $A_{a,b}(a) = b$, and $b \sqsubseteq_B f(x)$.
Or $x \neq a$, and then $A_{a,b}(x) = 0_B$ and $0_B \sqsubseteq_B f(x)$.
□

Theorem 5: $AT_{(A \rightarrow B)} = \{A_{a,b}: a \in A \text{ and } b \in ATOM_B\}$.
Proof:
Let $a \in A$, $b \in ATOM_{\mathbf{B}}$. Since $b \neq 0_B$, $A_{a,b} \neq 0$
Assume $g \sqsubseteq A_{a,b}$.
Then for every $x \in A$: $g(x) \sqsubseteq_B A_{a,b}(x)$.
For every $x \in A$: $A_{a,b}(x) = 0_B$ or $A_{a,b}(x) = b$.
Since $b \in ATOM_B$ only $0_B \sqsubseteq_B$ and $b \sqsubseteq_B b$.
Hence, for every $x \in A$: $g(x) = 0_B$ or $g(x) = b$.
This means that $g = 0$ or $g = A_{a,b}$.
Hence $A_{a,b} \in ATOM_{(A \rightarrow B)}$.

Let $f \in ATOM_{(A \rightarrow B)}$.
Then $f \neq 0$. This means that for some $a \in A$, $f(a) \neq 0_B$,
in other words, for some $a \in A$, for some $b \in B - \{0_B\}$: $f(a) = b$.
Look at $A_{a,b}$.
$A_{a,b}(a) = b$ and for every $x \in A - \{a\}$: $A_{a,b}(x) = 0_B$.
So: $A_{a,b} \neq 0$.
$A_{a,b}(a) = f(a)$ and for every $x \in A - \{a\}$: $A_{a,b}(x) \sqsubseteq_B f(x)$.
Hence $A_{a,b} \sqsubseteq f$.
Since f is an atom in $\mathbf{(A \rightarrow B)}$, this means that $f = A_{a,b}$.

Suppose that $b \notin ATOM_B$. Then for some y: $0_B \sqsubset_B y \sqsubset_B b$.
Then $0 \sqsubset A_{a,y} \sqsubset A_{a,b}$, hence, again f is not an atom in $\mathbf{(A \rightarrow B)}$. Contradiction.
Hence $b \in ATOM_B$.
□

Corrollary 6: **$(A \to B)$** is atomless iff **B** is atomless.
Proof:
$\{A_{a,b}$: $a \in A$ and $b \in ATOM_B\} = Ø$ iff $ATOM_B = Ø$. ◻
Fact 7: **$(A \to B)$** is atomic iff **B** is atomic.

Proof:
Let **B** be atomic.
Let $f \in (A \to B) - \{0\}$.
Then for some $a \in A$: $f(a) \neq 0_B$.
Since B is atomic, then for some $b \in ATOM_B$: $b \sqsubseteq_B f(a)$.
Hence $A_{a,b} \sqsubseteq f$. Since $A_{a,b} \in ATOM_{(A \to B)}$, **$(A \to B)$** is atomic.

Let **$(A \to B)$** be atomic. Let $y \in B$
Let $a \in A$ and look at $A_{a,y} \in (A \to B)$.
Since **$(A \to B)$** is atomic, for some $f \in AT_{(A \to B)}$: $f \sqsubseteq_{\to} A_{a,y}$.
Hence for every $x \in A$: $f(x) \sqsubseteq_B A_{a,y}(x)$.
Since for every $x \in A - \{a\}$: $A_{a,y}(x) = 0_B$, this means that
for every $x \in A - \{a\}$: $f(x) = 0_B$.
Since $f \in ATOM_{(A \to B)}$, this means that $f(a) \in ATOM_B$.
Since $f(a) \sqsubseteq_B y$, it follows that **B** is atomic.
◻

3.6.3. Generated ideal Boolean algebras

Let **B** = $<B,\sqsubseteq,\neg,\sqcap,\sqcup,0,1>$ be a Boolean algebra and $c \in B$.

The **ideal generated by** c, (c], is:
$(c] = \{b \in B: b \sqsubseteq c\}$
The **filter generated by** c, [c), is:
$[c) = \{b \in B: c \sqsubseteq b\}$

The **ideal-relativization of** c, **(c]**, is the structure:
(c] = $<$**(c]**$,\sqsubseteq_{(c]},\neg_{(c]},\sqcap_{(c]},\sqcup_{(c]},0_{(c]},1_{(c]}>$, where:

1. $\sqsubseteq_{(c]} = \sqsubseteq\restriction(c]$
3. $\neg_{(c]} = \neg_c$ $\qquad \neg_{(c]}(a) = c \sqcap \neg a$
4. $\sqcap_{(c]} = \sqcap\restriction_{(c]}$
5. $\sqcup_{(c]} = \sqcup\restriction_{(c]}$
6. $0_{(c]} = 0$
7. $1_{(c]} = c$

Fact 1a If $c \neq 0$, then **(c]** is a Boolean algebra.

Proof:
(c] has minimum 0 and maximum c and is closed under $\sqcap$ and $\sqcup$.
We saw above that (c] is closed under relative complement: $\neg_c a = \neg a \sqcap c$
◻

As Boolean algebras, **(c]** is not a sub-Boolean algebra of B, because 1 is not preserved. It is a Boolean algebra on a subset of B.

Of course, this theorem dualizes to **[c)**, the **filter-relativization of** c this too forms a Boolean algebra with maximum 1 and minumum c, and the operation of complementation. appropriately dualized. We will actually be interested in **[¬c)** as a Boolean algebra, so directly define that:

The **filter-relativization of** ¬c, **[¬c)**, is the structure:

$\mathbf{[\neg c)} = <\mathbf{[\neg c)}, \sqsubseteq_{[\neg c)}, \neg_{[\neg c)}, \sqcap_{[\neg c)]}, \sqcup_{[\neg c)}, 0_{[\neg c)}, 1_{[\neg c)}>$, where:

1. $\sqsubseteq_{[\neg c)} = \sqsubseteq \restriction \mathbf{[\neg c)}$
3. $\neg_{[\neg c)} = \neg(\neg_c)$ $\quad \neg_{[\neg c)}(a) = \neg c \sqcup a$
4. $\sqcap_{[\neg c)} = \sqcap \restriction_{[\neg c)}$
5. $\sqcup_{[\neg c)} = \sqcup \restriction_{[\neg c)}$
6. $0_{[\neg c)} = \neg c$
7. $1_{(\neg c]} = 1$

Fact 1b: If $\neg c \neq 1$ then **(¬c]** is a Boolean algebra.

Let **B** = $<B, \sqsubseteq, \neg, \sqcap, \sqcup, 0, 1>$ be a Boolean algebra and $c \in B - \{0\}$.

Theorem 2: (c] and **[¬c)** are isomorphic.

Proof:

If c =1, then (c] = [¬c) = B. So clearly they are isomorphic.
So let $c \neq 1$.

We define: h: (c] → [¬c) by:
for every $x \in$ (c]: $h(x) = x \sqcup \neg c$.

1. For every $x \in$ (c]: $\neg c \sqsubseteq x \sqcup c$ (because this holds in B generally).
Hence every $x \in$ (c]: $h(x) \in$ [¬c). So h is a function from (c] into [¬c).

2. Let $y \in$ [¬c). Then $\neg c \sqsubseteq y$. Then $\neg y \sqsubseteq c$, hence $\neg y \in$ (c].
Take the relative complement of ¬y in (c]: $\neg_c(\neg y) = \neg\neg y \sqcap c = y \sqcap c$.
Hence $y \sqcap c \in$ (c].

We calculate: $h(y \sqcap c) = (y \sqcap c) \sqcup \neg c = (y \sqcup \neg c) \sqcap (c \sqcup \neg c) = (y \sqcup \neg c) \sqcap 1 = y \sqcup \neg c = y$.
Hence h is onto.

3. Let $h(x_1) = h(x_2)$.
Then $x_1 \sqcup \neg c = x_2 \sqcup \neg c$. Then $\neg(x_1 \sqcup \neg c) = \neg(x_2 \sqcup \neg c)$. Then $\neg x_1 \sqcap c = \neg x_2 \sqcap c$.
Since these are the relative complements of x_1 and x_2 in Boolean algebra (c],
it follows that $x_1 = x_2$. Hence h is one-one.

4. $h(0) = 0 \sqcup \neg c = \neg c$.
$h(c) = c \sqcup \neg c = 1$
$h(a \sqcap b) = (a \sqcap b) \sqcup \neg c = (a \sqcup \neg c) \sqcap (b \sqcup \neg c) = h(a) \sqcap h(b)$
$h(a \sqcup b) = (a \sqcup b) \sqcup \neg c = (a \sqcup \neg c) \sqcup (b \sqcup \neg c) = h(a) \sqcup h(b)$
$h(\neg_{(c]}(a)) = h(\neg a \sqcap c) = (\neg a \sqcap c) \sqcup \neg c = \neg(a \sqcup \neg c) \sqcup \neg c = \neg_{[\neg c)}(a \sqcup \neg c) = \neg_{[\neg c)}(h(a))$.
◻

Fact 3: If $c \neq 1$ then $(c] \cap [\neg c) = \emptyset$
Proof:
Let $x \in (c] \cap [\neg c)$. Then $x \sqsubseteq c$ and $\neg c \sqsubseteq x$. Then $x \sqsubseteq c$ and and $\neg x \sqsubseteq c$.
Then $x \sqcup \neg x \sqsubseteq c$, hence $c = 1$.
◻

Fact 4: If a is an **atom** in B, then $(\neg a] \cup [a) = B$.
Proof:
We proved that if a is an atom in B then for every $b \in B - \{0\}$: $a \sqsubseteq b$ or $a \sqsubseteq \neg b$.
Let $a \in ATOM_B$ and let $b \in B$ and $b \notin [a)$.
Then $a \not\sqsubseteq b$. Then $a \sqsubseteq \neg b$. We proved that then $b \sqsubseteq \neg a$, hence $b \in (\neg a]$.
◻

With this, we have proved a decomposition theorem:

Decomposition Theorem:
If **B** is a Boolean algebra with atom $a \in B$, then **B** can be partitioned into two non-overlapping isomorphic Boolean algebras **(¬a]** and **[a)**.

The other way round we have a **composition theorem**.

Let **A** and **B** be non-overlappping isomorphic Boolean algebras and let h be an isomorphism between them.
We define:

The **product of A and B under h**, $\mathbf{B^h_{A+B}}$:

$\mathbf{B^h_{A+B}}$ = < B^h_{A+B}, ⊑, ¬, ⊓, ⊔, 0, 1 > where:

1. $B^h_{A+B} = A \cup B$
2. $\sqsubseteq\ = \sqsubseteq_A \cup \sqsubseteq_B \cup \{<a,b>: h(a) \sqsubseteq_B b\}$
3. ¬ is defined by:

$$\text{For all } x \in A \cup B\text{: } \neg x = \begin{cases} \neg_B(h(x)) & \text{if } x \in B \\ \neg_A(h^{-1}(x)) & \text{if } x \in A \end{cases}$$

.

4. ⊓ is defined by:

$$\text{For all } x,y \in B\text{: } x \sqcap y = \begin{cases} x \sqcap_A y & \text{if } x,y \in A \\ x \sqcap_B y & \text{if } x,y \in B \\ x \sqcap_A h^{-1}(y) & \text{if } x \in A \text{ and } y \in B \end{cases}$$

5. ⊔ is defined by:

$$\text{For all } x,y \in B\text{: } x \sqcup y = \begin{cases} x \sqcup_A y & \text{if } x,y \in A \\ x \sqcup_B y & \text{if } x,y \in B \\ h(x) \sqcup_B y & \text{if } x \in A \text{ and } y \in B \end{cases}$$

6. $0 = 0_A$.
7. $1 = 1_B$.

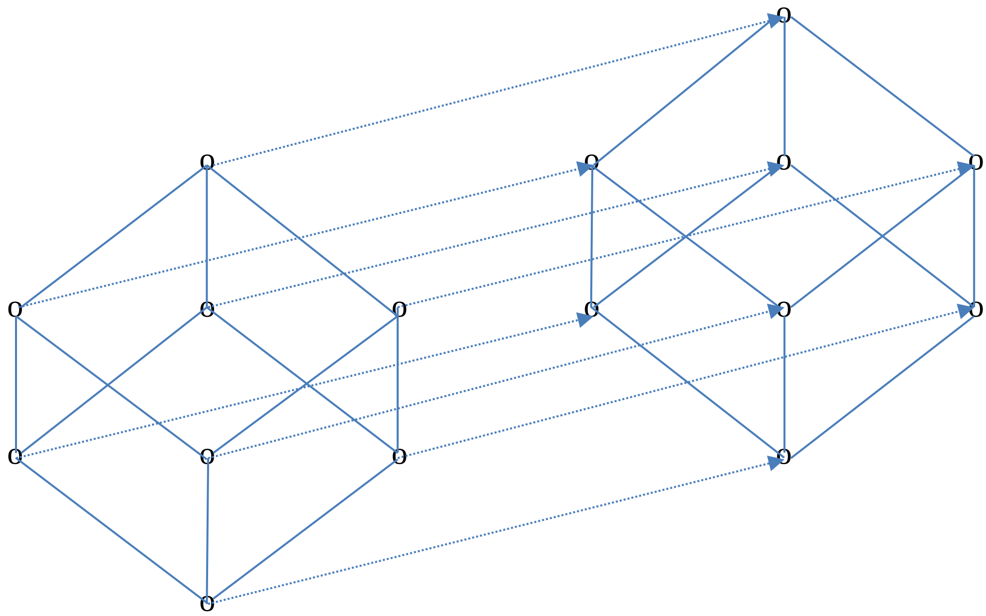
Construction Theorem: $\mathbf{B^h_{A+B}}$ is a Boolean algebra.

Proof: Similar to the proof for the product.
We show here only that $\mathbf{B^h_{A+B}}$ is complemented.

-If $a \in A$ then
$a \sqcap \neg a = a \sqcap_A h^{-1}(\neg a) = a \sqcap_A h^{-1}(\neg_B(h(a))) = a \sqcap_A h^{-1}(h(\neg_B a)) = a \sqcap_A \neg_A a = 0_A = 0$
$a \sqcup \neg a = h(a) \sqcup_B \neg a = h(a) \sqcup_B \neg_B(h(a)) = 1_B = 1.$

-If $a \in B$ then
$a \sqcap \neg a = h^{-1}(a) \sqcap_A \neg a = h^{-1}(a) \sqcap_A \neg_A(h^{-1}(a)) = 0_A = 0$
$a \sqcup \neg a = a \sqcup_B h(\neg a) = a \sqcup_B h(\neg_A(h^{-1}(a)) = a \sqcup_B \neg_B(h(h^{-1}(a)) = a \sqcup_B \neg_B a = 1_B = 1.$

In a picture:



Note that the construction turns 0_B into a new atom, to which, of course, the decomposition applies.

We already know that the complete atomic Boolean algebras are, up to isomorphism the powerset Boolean algebras. Since all finite Boolean algebras are complete and atomic, the finite Boolean algebras are exactly the powerset Boolean algebras.
The construction theorem gives us an algorithm for constructing them:
Take two finite non-overlapping Boolean algebras A and B of cardinality 2^n.
They are isomorphic. Take isomorphism h.
Then $B^h{}_{A+B}$ is the Boolean algebra of cardinality 2^{n+1}.

3.6. Atomless Boolean algebras 1: Boolean algebras +generated by chains

Let **B** = <B,$\sqsubseteq$> be a partial order and a,b $\in$ B

a **covers** b iff a $\sqsubset$ b and there is no c $\in$ B: a $\sqsubset$ c $\sqsubset$ b.

Example: 0 covers every atom, and every dual atom covers 1.

The **symmetric difference** of a and b, a + b, is:
$a + b = (a \sqcap \neg b) \sqcup (\neg a \sqcap b)$

Fact 1: If $a \sqsubseteq b$ then $a + b = \neg_b(a)$
Proof:
$a + b = (\neg a \sqcap b) \sqcup (\neg b \sqcap a)$
If $a \sqsubseteq b$ then $\neg b \sqcap a = 0$.
Thus, $a + b = (\neg a \sqcap b) \sqcup 0 = \neg a \sqcap b = \neg_b(a)$
□

Let **B** be a Boolean algebra, let a,b $\in$ B and $a \sqsubseteq b$.

Lemma 2: a **covers** b iff $a + b \in ATOM_{\mathbf{B}}$
Proof:
1. Let a cover b.
This means that a is a dual atom in **(b]**.
But that means that $\neg_b(a)$ is an atom in **(b]**.
Since $ATOM_{(b]} = ATOM_B \cap$ **(b]**, it follows that $\neg_b(a) \in ATOM_B$.
Since $a \sqsubseteq b$, $a + b = \neg_b(a)$, hence $a + b \in ATOM_B$.

2. Let $a + b \in ATOM_B$.
Since $a \sqsubseteq b$, $a + b = \neg_b(a)$. Hence $\neg_b(a) \in ATOM_B$.
Since $\neg_b(a) = \neg a \sqcap b$, $\neg_b(a) \in$ **(b]**, hence $\neg_b(a) \in AT_{(b]}$.
Then $\neg_b(\neg_b(a))$ is a dual atom in **(b]**, which means that $\neg_b(\neg_b(a))$ covers b.
But $\neg_b(\neg_b(a))$ =
$\neg_b(\neg a \sqcap b)$ =
$\neg(\neg a \sqcap b) \sqcap b$ =
$(\neg\neg a \sqcup \neg b) \sqcap b$ =
$(a \sqcup \neg b) \sqcap b$ =
$(a \sqcap b) \sqcup (\neg b \sqcap b)$ =
$(a \sqcap b) \sqcup 0$ =
$a \sqcap b$.
Since $a \sqsubseteq b$, $a \sqcap b = a$.

Hence $\neg_b(\neg_b(a)) = a$.
Hence a covers b.
□

Corrollary 3:

If **B** is an atomless Boolean algebra,
then every maximal chain in **B** is a dense linear order with endpoints.

Proof:
If **B** is atomless, then for no a,b ∈ B: a covers b (else a + b ∈ $ATOM_B$).
Hence indeed every maximal chain is dense.
The endpoints of any maximal chain are, of course, 0 and 1.
◻

Let **B** be a Boolean algebra and let C ⊆ B be a chain in **B** with 0 ∈ C.

C **+generates B** iff
for every b ∈ B: there are some $c_1,...,c_n$ ∈ C such that: b = c_1+ ... + c_n.

So chain C +generates **B** iff every element of B can be generated as the symmetric difference of (finitely many) elements of C.

Theorem 4: If B_1 is +-generated by C_1 and B_2 is +-generated by C_2 and C_1 and C_2 are isomorphic, then B_1 and B_2 are isomorphic.

i.e. Boolean algebras +-generated by isomorphic chains are isomorphic.

Theorem 5: Every countable Boolean algebra is +generated by a maximal chain.

Corrollary 6: Up to isomorphism there is exactly one countable atomless Boolean algebra.

Proof: Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be countable atomless Boolean algebra. Each of them is generated by a maximal chain, by theorem 5. By corrollary 3, any such maximal chain is a dense linear order with end points 0 and 1. By Cantor's theorem, all such maximal chains are isomorphic. Hence by theorem 4, $\mathbf{B}_1$ and $\mathbf{B}_2$ are isomorphic.

Let **B** be a Boolean algebra.

B is **homogenous** iff for every b ∈ B−{0}: **B** is isomorphic to **(b]**.

Corrollary 7: The countable atomless Boolean algebra is homogenous.
Proof:
Let **B** be the countable atomless Boolean algebra, and let b ∈ B.
Then **(b]** is also a countable atomless Boolean algebra (namely, any atom in **(b]** is also an atom in **B**), and by Corrollary 6, **(b]** and **B** are isomorphic.
◻

3.6. Atomless Boolean algebras 2: Agebras of regular open sets

I will define here the countable atomless Boolean algebra and what can be called the standard model for complete atomless Boolean algebras. It will be defined in terms of intervals of real numbers.

Let $I_{\mathbb{R}}$ be the set of all intervals in $\mathbb{R}$.
We assume all the notions we have defined before, plus:

Let $i,j \in I_{\mathbb{R}}$.
i and j are ***apart*** iff either $\vee i < \wedge j$ or $\vee j < \wedge j i$

Intervals that are not apart overlap or are separated by a single point:
e.g. the intervals $(0,\pi)$ (= $\{r \in \mathbb{R}: 0 < r < \pi\}$) and $(\pi,4)$ (= $\{r \in \mathbb{R}: \pi < r < 4\}$) are not apart, since $\vee(0,\pi) = \wedge(\pi,4)$:

0(————————)π(————————)4

Note that intervals in $\mathbb{R}$ that are apart are separated by uncountably many points.
Following Givant and Halmos 2009, I will call a single point separating two intervals a ***crack*** in the union of those two intervals. Let $x \subseteq \mathbb{R}$.

The set M_x of *maximal subintervals of* x, is the set:
$M_x = \{i \in I_{\mathbb{R}}: i \subseteq x \text{ and } \forall j \in I_{\mathbb{R}}: \text{if } j \subseteq x \text{ and } i \subseteq j \text{ then } j = i\}$

x is an ***open set*** iff every $i \in M_x$ is an open interval.
x is a ***regular set*** iff for every $i, j \in M_x$ if $i \neq j$ then i and j are apart.

RO is the set of all *regular open sets* in $\mathbb{R}$.

[Note for the specialists: the standard definition of regular open set is topological: sets that are identical to the interior of their closure. On the interval structures given here those are precisely the open regular sets as defined here.]

Let $x \in$ **RO**
x is a finite regular open set iff M_x is finite.

Finite regular open sets are regular open sets that have only finitely many maximal subintervals.

FRO is the set of all finite regular open sets.

We define the following operation ~ on subsets x ⊆ ℝ:

$$\sim x = (\mathbb{R} ¡ x) ¡ \{r \in \mathbb{R}: \exists i \in M_{(\mathbb{R} ¡ x)}: r = \wedge i \text{ or } r = \vee i\}$$

~ does the following:

1. ~ takes the set-theoretic complement of x in ℝ: ℝ ¡ x
2. ~ deletes any number from ℝ−x which is either the lower bound of a maximal subinterval of ℝ−x or the upper bound of a maximal subinterval of ℝ−x.

Example:

x 0(————)π 4(————)10
ℝ−x −∞(————0] [π————4] [10————)+∞
~x −∞(————0) (π————4) (10————)+∞

The picture shows that if x is a regular open set, so is ~x. Moreover, by reading the picture bottom up, it is easy to see that ~~x = x.

~~x is a crack-repairing operation: it turns an open set that contains cracks into a regular open set by filling in the cracks:

Example:

x 0(————)π(————)4
~x −∞(————0) (4————)+∞
(single point closed interval [π] gets eliminated by ~)
~~x 0(————————)4

We notice that Ø, ℝ are themselves regular open set.

Let α ∈ {**RO, FRO**}
We define the following relation and operations for s,t ∈ ***α*** and X ⊆ ***α***:

1. s $\sqsubseteq_{\alpha}$ t iff s ⊆ t
2. 0_{α} = Ø; 1_{α} = ℝ
3. $\neg_{\alpha}$ s = ~s
4. s $\sqcap_{\alpha}$ t = s ∩ t
5. s $\sqcup_{\alpha}$ t = ~~(s ∪ t)
6. $\sqcap_{\alpha}$ X = ~~∩X
7. $\sqcup_{\alpha}$ X = ~~∪X

Theorem 1: <***α***, $\sqsubseteq_{\alpha}$, $\neg_{\alpha}$, $\sqcap_{\alpha}$, $\sqcup_{\alpha}$, 0_{α}, 1_{α} > is a an atomless Boolean algebra.

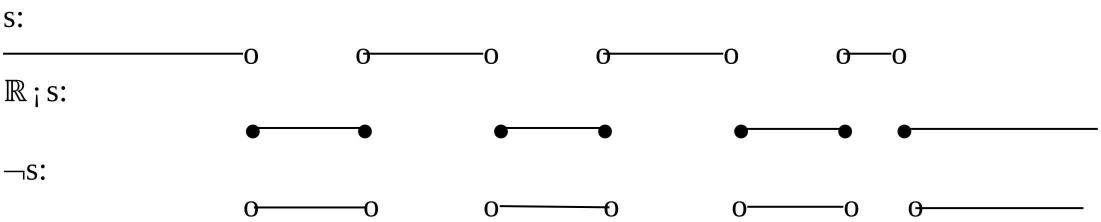
Theorem 2a: if α = **FRO**: then α is not complete and countable, hence the countable atomless Boolean algebra.
Theorem 2b: **RO** is complete: **RO** is the completion of **FRO**, the smallest complete atomless Boolean algebra containing **FRO** (**RO** has the cardinality of ℝ).

The facts about cardinality follow from the construction.
If x ∈ **RO**¸M_x is countable, because between any two intervals in M_x there is a rational number, because the rational numbers lie dense in the real numbers. Each interval in M_x can be seen as a pair of two real numbers, and real numbers can be constructed as countable sets of rationational numbers. This means *de facto* that **RO** has the same cardinality as the set of all countable subsets of a countable set, which is $2^{\aleph_0}$, the cardinality of ℝ.
By the same argument **FRO** corresponds to the set of all ***finite*** subsets of a countable set, and that set can be proved to be countable.

Proof of Theorem 1.
Step 1: $\sqsubseteq_\alpha$ is a partial order on α, 0_α and $1_\alpha \in \alpha$ and α is closed under $\neg_{\alpha,}$ $\sqcap_\alpha$ and $\sqcup_\alpha$.
That $\sqsubseteq_\alpha$ is a partial order is trivial because ⊆ is. Ø, ℝ ∈ **FRO**, and hence in **RO**

2. α is closed under ¬

Look at the picture:



-Let s ∈ **RO**. The intervals in M_s are open and apart.
Consequently, the intervals in ℝ ¡ s are closed (or half-closed with an open bound not in ℝ). and apart.
Removing the bounds gives a set ¬s such that in $M_{\neg s}$ the intervals are open and apart, hence ¬s ∈ **RO**.
-Let s ∈ **FRO**. $|M_{\neg s}|$ is the same as the number of maximal gaps between the intervals in M_s.
This means that $|M_{\neg s}|$ is one less, the same, or one more than $|M_s|$, hence ¬s ∈ **FRO**.

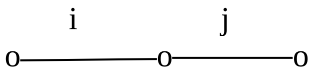
3. α is closed under ⊓.
-Assume s, t ∈ **RO**.
We want to prove that s $\sqcap_{RO}$ t ∈ **RO**. So, we need to prove that any two intervals in $M_{s \cap t}$ are open and apart.
-s ∩ t introduces no new bounds in $M_{s \cap t}$ with respect to the bounds in $M_s \cup M_t$: any new bound that wasn't in s, was in t and vice versa. So all bounds in $M_{s \cap t}$ are open.
Similarly, since the intervals in $M_{s \sqcap t}$ are gotten by nibbling away parts of intervals in M_s and parts of intervals in M_t, and the intervals in M_s and M_t were already apart, the intervals in $M_{s \sqcap t}$ are as apart or more apart.
-Let s, t ∈ **FRO**. Obviously $|M_{s \sqcap t}| \leq |M_s|$ and $|M_{s \sqcap t}| \leq |M_t|$, so s ⊓ t ∈ **FRO**.

4. α is closed under $\sqcup_\alpha$.
s $\sqcup_{\boldsymbol{\alpha}}$ t = ~~(s ∪ t)

s ∪ t at most takes away bounds from s and from t, hence the remaining bounds in s ∪ t are open. Similarly, ~~ at most takes away bounds, so the bounds in ~~(s ∪ t) are open.

~~ is there to deal with apartness.

Given the argument for openness: if s, t ∈ **RO**, and all intervals in $M_{s \cup t}$ are apart, then s ∪ t ∈ **RO.** Now if i ∩ j ≠ Ø, then i ∉ $M_{s \cup t}$ and j ∉ $M_{s \cup t}$, because there is a maximal subinterval that contains both. This means that the only case we need to be concerned with is the case where s, t ∈ **RO** and s ∪ t ∉ **RO** because for some i,j ∈ $M_{s \cup t}$: i and j are a single point apart:



Look at what ~~(s ∪ t) does with i and j:

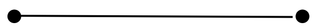
~(s ∪ t):

Step 1: take complements:

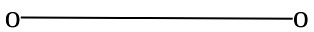
Step 2: remove bounds:

~~(s_1 ∪ s_2):

Step 3: take complement:



Step 4: remove bounds:



As Givant and Halmos express it: the intuition about solids, regular open sets, is that they don't have cracks in them.

A crack in a set is a single point missing (i.e. open bounds touching): fill in all the cracks in the set, and you get an interval.

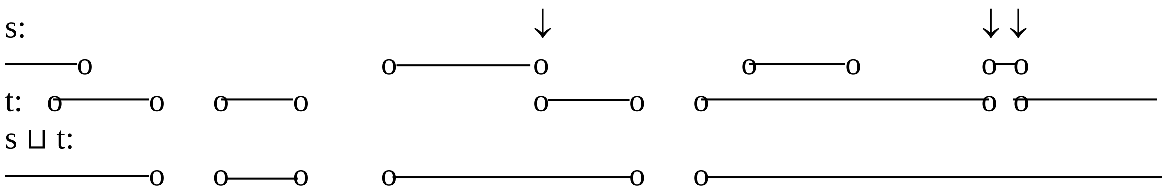
s_1 ∪ s_2 may have such cracks.

They are removed in ~~(s_1 ∪ s_2).

So clearly, ~~(s_1 ∪ s_2) ∈ **RO**.

-For **FRO** the situation is similar to the case before: if s and t are finite regular open sets , so is ~~(s ∪ t).

In a picture: (↓ indicates a crack in s ∪ t)



The definitions of the complete operations require a comment:

7. $\sqcap X \quad := \quad \sim\sim\cap X$
8. $\sqcup X \quad := \quad \sim\sim\cup X$

The definition (8) is not surprising, since it generalizes definition (4).
In (7), we see an asymmetry with the finite operation $\sqcap$: while $\sqcap$ is just intersection, the complete meet operation takes the interior of the closure of the intersection.

The reason is that, while the intersection of two open sets will be an open set, the complete intersection of a set op open sets is not necessarily an open set.

Think about the construction of real numbers through infinite sets of nested intervals of rational numbers: if we define a single real number this way, then obviously, when we take the intersection of an infinite set of nested intervals within the real numbers, we get the singleton set containing the number these intervals approximate:
i.e. $\cap\{(1/2, 1\ 1/2). (2/3, 1\ 1/3), (3/4, 1\ 1/4), (4/5, 1\ 1/5),\ldots\} = \{1\}$

In general, if X is a set of open intervals, $M_{\cap X}$ is going to be a set of open intervals and single isolated points (i.e. intervals [r,r] for $r \in \mathbb{R}$).
~~ removes these single isolated points and hence $\sim\sim\cap X$ is an open set.

As expressed above $\sqcap$ is the generalization of binary $\sqcap$, because on solids, the operation ($\sqcap$) and ~~($\sqcap$) are the same operation.

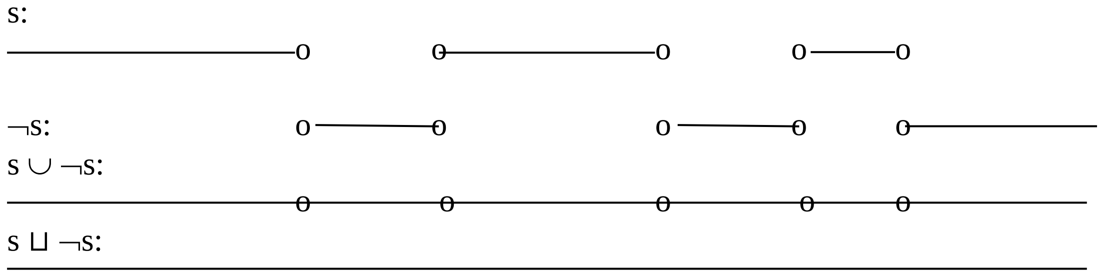
Fact: RO is closed under the complete operations $\sqcap$ and $\sqcup$.
The arguments are the same as in the binary case.
Note: FRO is not closed under the complete operations $\sqcap$ and $\sqcup$ (this follows from general concerns below).

We have shown that **α** is closed under the Boolean operations. We still need to show that the operations are indeed the Boolean operations.

1. $\sqcap_\alpha$ is meet for $\sqsubseteq_\alpha$. This is obvious, because $\sqcap = \cap$ and $\sqsubseteq = \subseteq$.
2. $\sqcup$ is join for $\subseteq$.
Let $v \in \alpha$ and $s \subseteq v$ and $t \subseteq v$. Then $s \cup t \subseteq v$.
But obviously, no superset of $s \cup t$ is going to be in α, unless you fill up the cracks in $s \cup t$., the elements that make intervals in $M_{s \cup t}$ not apart.
That means that $s \cup t \subseteq s \sqcup t \subseteq v$.
Hence $s \sqcup t$ is join for $\sqsubseteq_\alpha$.
3. $\neg$ is complementation.
$s \sqcap \neg s = s \cap \sim s = \emptyset$

$s \sqcup \neg s = \sim\sim(s \cup \sim s)$.
In $s \cup \sim s$ every bound is a crack.
~~ fills in the cracks,
hence $\sim\sim(s \cup \sim s) = \mathbb{R}$.

In a picture:



4. ⊓ and ⊔ satisfy distributivity.

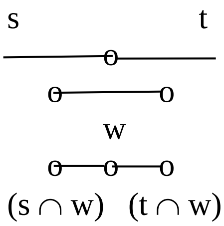
$(s \sqcup t) \sqcap w$
$= \sim\sim(s \cup t) \cap w$
$= (s \cup t \cup CRACK_{s \cup t}) \cap w$, where $CRACK_{s \cup t}$ is the set of cracks in $s \cup t$
$= ((s \cap w) \cup (t \cap w)) \cup (CRACK_{s \cup t} \cap w)$
$= ((s \cap w) \cup (t \cap w)) \cup CRACK_{((s \cap w) \cup (t \cap w))}$
$= \sim\sim((s \cap w) \cup (t \cap w))$
$= (s \sqcap w) \sqcup (t \sqcap w)$

The central step in this argument is:

$$CRACK_{s \cup t} \cap w = CRACK_{((s \cap w) \cup (t \cap w))}$$

1. If $r \in CRACK_{s \cup t}$ and $r \in w$ then $r \in CRACK_{((s \cap w) \cup (t \cap w))}$

This is easily seen in a picture:



i.e. $r \notin s \cap w$ and $r \notin t \cap w$,
hence $r \notin (s \cap w) \cup (t \cup w)$,
so $r \in CRACK_{((s \cap w) \cup (t \cap w))}$

2. Let $r \in CRACK_{(s \cap w) \cup (t \cap w)}$.
Then $r \in w$ and $r \in CRACK_{s \cup t}$
Namely: if $r \notin w$ then $r \in CRACK_w$, but w is a solid, so $CRACK_w = Ø$.
Hence $r \in w$.
But then r can only be a crack by being a crack in $s \cup t$.
□

Finally we prove: α is atomless

Let $s \in \alpha$ and let $i \in M_s$, i is a maximal subinterval in s.
Let $j \in I_{\mathbb{R}}$ be a non-empty open proper subinterval of i, i.e. $j \neq \emptyset$, $j \in I_{\mathbb{R}}$ and $j \subset i$.
Then $s - (i - j) \subset s$ and $s - (i - j) \in \alpha$.
Namely: $s - (i - j)$ has the same maximal subintervals as s, except that i is replaced by the smaller j, which is itself open.
Clearly all maximal subintervals in $s - (i - j)$ are open and apart, so indeed $s - (i - j) \in \alpha^+$.
What this proves is: $\forall s \in \alpha \exists t \in \alpha$: $t \sqsubset s$.
Since this holds for all non-empty regular open sets, none of those can be an atom.
The argument of course doesn't depend on whether or not s is finite or not.

We have now proved indeed that **FRO** is a countable atomless Boolean algebra and that **RO** is a complete atonmless Boolean algebra. it follows that:

FRO is the countable atomless Boolean algebra.
Now we can see that **FRO** is not complete: It is generated by a dense countable order with end points 0,1. We know that the bit of the chain between 0 and 1 is isomorphic to $\mathbb{Q}$, and hence to $\mathbb{Q}_1 + \mathbb{Q}_2$ (two copies of $\mathbb{Q}$ from bottom to top), But, as we know $\sqcup(\mathbb{Q}_1)$ does not exist.

RO is the completion of **FRO:** the relation between **FRO** and **RO** is in a fundamental way similar to that between $\mathbb{Q}$ and $\mathbb{R}$.

3.7 Filters and ideals, Stone's Theorems.

Let **A** be a lattice.

A **filter** in **A** is a non-empty subset $F \subseteq A$ such that:

1. if $a \in F$ and $a \sqsubseteq b$ then $b \in F$.
2. if $a,b \in F$ then $a \sqcap b \in F$.

An **ideal** in **A** is a non-empty subset $I \subseteq A$ such that:

1. if $b \in I$ and $a \sqsubseteq b$ then $a \in I$
2. if $a,b \in I$ then $a \sqcup b \in I$

We will prove things for filters, but they dualize to ideals, of course.

Lemma 1_F: F is a filter in **A** iff F is a non-empty subset of A such that:
$a,b \in F$ iff $a \sqcap b \in F$.

Proof:
Let F be a filter and let $a \sqcap b \in F$. Then $a,b \in F$ because $a \sqcap b \sqsubseteq a$ and $a \sqcap b \sqsubseteq b$

Let F be a non-empty set such that $a,b \in F$ iff $a \sqcap b \in F$. Let $a \in F$ and $a \sqsubseteq b$.
Then $a = a \sqcap b$, hence $a \sqcap b \in F$, hence $a,b \in F$. □

Lemma2_F: F is a filter in **A** iff F is a sublattice of **A** such that:
if $a \in F$ and $b \in A$, then $a \sqcup b \in F$.

Proof:
Let F be a filter.
If $a,b \in F$ then $a \sqcap b \in F$, by definition.
If $a,b \in F$, then $a \sqcup b \in F$, since $a \sqsubseteq a \sqcup b$. Hence F is a sublattice of A.
Let $a \in F$ and $b \in A$. Again $a \sqsubseteq a \sqcup b$, hence $a \sqcup b \in F$.

Let F be a sublattice of A such that if $a \in F$ and $b \in A$ then $a \sqcup b \in F$.
Since F is a sublattiCe of A, if $a,b \in F$ $a \sqcap b \in F$.
Let $a \in F$ and $a \sqsubseteq b$. Then $a \sqcup b \in F$. Since $b = a \sqcup b$, then $b \in F$. □

Lemma 3_F: Let X be a set of filters in **A**.
If $\cap X$ is not empty, then $\cap X$ is a filter in **A**.

Proof:
Let $a,b \in \cap X$. Then for every $F \in X$, $a,b \in F$, hence for every $F \in X$:
$a \sqcap b \in F$, hence $a \sqcap b \in \cap X$. Let $a \in \cap X$ and $a \sqsubseteq b$. Then for every $F \in X$: $a \in F$, hence for every $F \in X$, $b \in F$. Then $b \in \cap X$. □

Let X be a non-empty subset of A.

The **filter generated by X, [X)**, is the intersection of all filters extending X
$[X) = \cap\{F \subseteq A: X \subseteq F \text{ and } F \text{ is a filter in } A\}$

The **ideal generated by X, (X]**, is the intersection of all ideals extending X
$(X] = \cap\{I \subseteq A: X \subseteq I \text{ and } I \text{ is an ideal in } A\}$

In case X = {a}, we write **[a)** for [{a}).
It is simple to prove that:

[a) = {b $\in$ A: a $\sqsubseteq$ b}
A filter F is **principal** iff $\sqcap$F $\in$ F.

Obviously, F is principal iff F = [$\sqcap$F).

We call [a) the **principal filter generated by** a.

Similarly, ideal I is **principal** iff $\sqcup$I $\in$ I.
(a] = {b $\in$ A: a $\sqsubseteq$ b}.
(a] is the **principal ideal generated by** a.

Lemma 4$_F$: [X) = {a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a}
Proof:
1. Suppose a $\in$ {a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a}.
Then for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a.
Since X $\subseteq$ [X) and [X) is a filter, $x_1 \sqcap ... \sqcap x_n \in$ [X). Hence a $\in$ [X).
Hence{a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a} $\subseteq$ [X).

2. We will show that {a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a} is a filter extending X.
a. Let x $\in$ X. Since x = x $\sqcap$ x, x $\in$ {a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a}
So X $\subseteq$ {a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a}.
b. Suppose for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a
and for some $y_1,...,y_m \in$ X: $y_1 \sqcap ... \sqcap y_m \sqsubseteq$ b
Then $(x_1 \sqcap ... \sqcap x_n) \sqcap (y_1 \sqcap ... \sqcap y_m) \sqsubseteq$ a $\sqcap$ b, hence:
a $\sqcap$ b $\in$ {a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a}.
c. Suppose for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a, and suppose a $\sqsubseteq$ b.
Then $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ b. Hence b $\in$ {a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a}

So {a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a} is indeed a filter extending X.
This means that [X) $\subseteq$ {a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a}.
Hence [X) = {a $\in$ A: for some $x_1,...,x_n \in$ X: $x_1 \sqcap ... \sqcap x_n \sqsubseteq$ a}. ◻

F is a **proper filter** in **A** iff F is a filter in **A** and F $\neq$ A.
I is a **proper ideal** in **A** iff I is an ideal in **A** and I $\neq$ A

Let **A** be a lattice with 0.

Fact 5: F is a proper filter in **A** iff F is a filter in **A** and 0 $\notin$ F.

Lemma 6: [X) is proper iff for every $x_1,...,x_n \in X$: $x_1 \sqcap ... \sqcap x_n \neq 0$
Proof:
If [X) is not proper, then $0 \in [X)$. Then, by lemma 4,
for some $x_1,...,x_n \in X$: $x_1 \sqcap ... \sqcap x_n \sqsubseteq 0$.
Hence for some $x_1,...,x_n \in X$: $x_1 \sqcap ... \sqcap x_n = 0$.
If for some $x_1,...,x_n \in X$: $x_1 \sqcap ... \sqcap x_n = 0$, then, again by the lemma, $0 \in [X)$,
hence [X) is not proper. □

F is a **maximally proper filter** in **A** iff F is a proper filter in **A**, and the only two filters in **A** containing F are F and A.

F is an **ultrafilter** in **A** iff F is a proper filter in **A** and
for every $a \in A$: $a \in F$ or for some $b \in F$: $b \sqcap a = 0$

F is a **prime filter** in **A** iff F is a proper filter in **A** and
for every $a,b \in A$: if $a \sqcup b \in F$ then $a \in F$ or $b \in F$.

Theorem 7: F is a maximally proper filter in **A** iff F is an ultrafilter in **A**.
Proof:
Let F be an ultrafilter.
Assume that F is not a maximally proper filter.
Then for some proper filter G: $F \subseteq G$ and $F \neq G$. Say, $a \notin F$ and $a \in G$.
Since F is an ultrafilter, for some $b \in F$: $b \sqcap a = 0$.
Since $F \subseteq G$, $b \in G$. So $a,b \in G$, hence, since G is a filter, $b \sqcap a \in G$. Hence $0 \in G$.
But G is proper. Contradiction. Hence F is a maximally proper filter.

Let F be a maximally proper filter.
Assume F is not an ultrafilter. Let $a \notin F$.
Assume: for every $b \in F$: $b \sqcap a \neq 0$.
Look at $F \cup \{a\}$. For every $x_1,...,x_n \in F \cup \{a\}$: $x_1 \sqcap .... \sqcap x_n \neq 0$.
Namely, if $x_1, ... x_n \in F$, then $x_1 \sqcap ... \sqcap x_n \neq 0$, because F is a proper filter.
If $x_1,...,x_{n-1} \in F$ and $x_n = a$, then $x_1 \sqcap ... \sqcap x_{n-1} \in F$, since F is a filter, and hence, by assumption, $(x_1 \sqcap ... \sqcap x_{n-1}) \sqcap a \neq 0$. Hence $x_1 \sqcap ... \sqcap x_n \neq 0$.
This means that $[F \cup \{a\})$ is a proper filter.
Since $F \subseteq [F \cup \{a\})$, this means that F is not a maximally proper filter. Contradiction.
Hence F is an ultrafilter. □

Theorem 8: In a distributive lattice with 0, every ultrafilter is a prime filter.
Proof:
Let **A** be a distributive lattice, and let U be an ultrafilter in **A**.
Let $a \sqcup b \in U$, and assume $a \notin U$.
Since U is an ultrafilter, for some $x \in U$: $x \sqcap a = 0$.
Since U is a filter, $x \sqcap (a \sqcup b) \in U$.
Since **A** is distributive, $(x \sqcap a) \sqcup (x \sqcap b) \in U$. Hence $0 \sqcup (x \sqcap b) \in U$,
hence $x \sqcap b \in U$, and, again, because U is a filter, $b \in U$.
So U is a prime filter. □

Theorem 9: In a Boolean lattice, every prime filter is an ultrafilter.
Proof:
Let **A** be a Boolean lattice, and let F be a prime filter in **A**.
This means that F is a proper filter.
Since F is non-empty and a filter, $1 \in F$.
For every $a \in A$: $1 = a \sqcup \neg a$. Hence, for every $a \in A$: $a \sqcup \neg a \in F$.
Since F is prime, this means that for every $a \in A$: $a \in F$ or $\neg a \in F$
(not both, since F is proper).
Since $a \sqcap \neg a = 0$, this means that for every $a \in A$: $a \in F$ or for some $b \in F$: $a \sqcap b = 0$.
Hence F is an ultrafilter. □

Corrollary 10: In a Boolean lattice, F is a maximally proper filter iff F is an ultrafilter iff F is a prime filter.

The next theorems require the axiom of choice (AC) of set theory.
We use the following principle which is provably equivalent to the axiom of choice:

The **Maximal Chain Principle** (= AC):
Every chain in a partial order can be extended to a maximal chain.

Theorem 11 [AC]: (Stone)
Let **A** be a lattice with 0, F a filter in **A** and I an ideal in **A** and assume $F \cap I = Ø$.
F can be extended to a filter M_F, which is maximal in the set of filters extending F and not overlapping with I.

Proof:
Let **A** be a lattice with 0, F a filter in **A**, I an ideal in **A** and $F \cap I = Ø$.
Let $E_F = \{G \subseteq A: F \subseteq G$ and G is a filter in **A** and $G \cap I = Ø\}$.
$\mathbf{E_F} = \langle E_F, \subseteq \rangle$. What we want to prove is that $\mathbf{E_F}$ has a maximal element M_F.

$\{F\}$ is a chain in $\mathbf{E_F}$, hence, by the Maximal Chain Principle, $\{F\} \subseteq M$, where M is a maximal chain in $\mathbf{E_F}$.

Let $M_F = \cup M$.
1. $F \subseteq \cup M$. This is obvious, since $F \in M$.
Hence $F \subseteq M_F$
2. a. Let $a,b \in \cup M$. Then for some $G_1 \in M$: $a \in G_1$, and for some $G_2 \in M$: $b \in G_2$.
Since M is a chain, $G_1 \subseteq G_2$ or $G_2 \subseteq G_1$, say the first. Then $a,b \in G_2$, hence, since G_2 is a filter, $a \sqcap b \in G_2$. Since $G_2 \in M$, then $a \sqcap b \in \cup M$.
2. b. Let $a \in \cup M$ and $a \sqsubseteq b$. Then for some $G \in M$: $a \in G$, hence, since G is a filter, $b \in G$. Since $G \in M$, then $b \in \cup M$.
Hence, M_F is a filter.

3. Assume $\cup M \cap I \neq Ø$, say, $x \in \cup M \cap I$. Then $x \in \cup M$, hence for some $G \in M$, $x \in G$. Then $G \cap I \neq Ø$. Contradiction. Hence $\cup M \cap I = Ø$.
Hence $M_F \cap I = Ø$.

So $M_F \in E_F$.

4. Let $K \in E_F$ and $\cup M \subseteq K$. Then for every $G \in M$: $G \subseteq K$. Hence $M \cup \{K\}$ is a chain. Since M is a maximal chain, then $M \cup \{K\} = M$. Hence $K \in M$, and thus $K \subseteq \cup M$. So $K = \cup M$.

So M_F is a maximal element in $\mathbf{E_F}$. □

Stone's Theorem [AC] 12: Let **A** be a distributive lattice with 0.
Every proper filter in **A** disjoint with some ideal I in **A** can be extended to a prime filter in **A** disjoint with I.

Proof:
Let **A** be a distributive lattice with 0.
Let F be a filter in **A**, and I and ideal in **A**, and let $F \cap I = \emptyset$.
Then, by theorem 12, there is a filter M_F in **A** extending F, maximally disjoint with I (a maximal element in $\mathbf{E_F}$**).**

Claim: M_F is a prime filter.

Suppose M_F is not prime.
Since $M_F \cap I = \emptyset$, and $0 \in I$, M_F is a proper filter.
So assume that for some $a,b \in A$: $a \sqcup b \in M_F$, but $a \notin M_F$ and $b \notin M_F$.
Look at $[M_F \cup \{a\})$ and $[M_F \cup \{b\})$.

Since M_F is a maximal element in $\mathbf{E_F}$, $[M_F \cup \{a\}) \cap I \neq \emptyset$,
say, $[M_F \cup \{a\}) \cap I = k$.
Hence for some $x_1,...,x_n \in M_F \cup \{a\}$, $x_1 \sqcap ... \sqcap x_n \sqsubseteq k$.
Since I is an ideal, and $k \in I$, this means that $x_1 \sqcap ... \sqcap x_n \in I$.
Since $M_F \cap I = \emptyset$, this means that for some $x_1,...,x_{n-1} \in M_F$:
$x_1 \sqcap ... \sqcap x_{n-1} \sqcap a \in I$. Since M_F is a filter, $x_1 \sqcap ... \sqcap x_{n-1} \in M_F$.
So we conclude: for some $p \in M_F$: $p \sqcap a \in I$.

Similarly, since $[M_F \cup \{b\}) \cap I \neq \emptyset$.
we conclude: for some $q \in M_F$: $q \sqcap b \in I$.

Since I is an ideal, $(p \sqcap a) \sqcup (q \sqcap b) \in I$.

Since **A** is distributive:
$(p \sqcap a) \sqcup (q \sqcap b) = ((p \sqcap a) \sqcup q) \sqcap ((p \sqcap a) \sqcup b) =$
$(p \sqcup q) \sqcap (a \sqcup q) \sqcap (p \sqcup b) \sqcap (a \sqcup b)$.

Now, $p,q, (a \sqcup b) \in M_F$.
Hence: $p \sqcup q \in M_F$, $a \sqcup q \in M_F$, $p \sqcup b \in M_F$, $a \sqcup b \in M_F$.
Hence, since M_F is a filter:
$(p \sqcup q) \sqcap (a \sqcup q) \sqcap (p \sqcup b) \sqcap (a \sqcup b) \in M_F$.
Hence $M_F \cap I \neq \emptyset$. Contradiction.

Hence M_F is a prime filter. □

Corrollary 13: Let **A** be a distributive lattice with 0.

Every proper filter in **A** can be extended to a prime filter in **A**.

Proof:

{0} is an ideal in **A** and for every proper filter F in **A**, F ∩ {0} = Ø.
With Stone's theorem, there is a prime filter extending F disjoint with {0}, which is just a prime filter extending F. □

Corrollary 14: Let **A** be a distributive lattice with 0. Let a ∈ A.

Every filter not containing a can be extended to a prime filter not containing a.

Proof:

If a ∉ F, then (a] is an ideal in **A** such that F ∩ (a] = Ø.
With Stone's theorem, there is a prime filter extending F and disjoint with (a].
This is a prime filter extending F not containing a. □

Corrollary 15: Let **A** be a distributive lattice with 0.

If a,b ∈ A and a ≠ b, then there is a prime filter in **A** containing exactly one of a and b.

Proof:

If a ≠ b, then a ⋢ b or b ⋢ a. Say, the first.
Then (b] ∩ [a) = Ø.
Then, with Stone's theorem, there is a prime filter extending [a), disjoint with (b].
This is a prime filter containing a but not containing b. □

Corrollary 16: Let **A** be a distributive lattice with 0.

Every filter F in **A** is the intersection of all prime filters in **A** extending F.

Proof:

Let F' be the intersection of all prime filters extending F.
F ⊆ F', since F is a subset of every prime filter extending F.
Suppose F ≠ F'. Then for some a ∈ A: a ∉ F and a ∈ F'.
Then there is a prime filter P extending F such that a ∉ P. But then ¬(F' ⊆ P). Contradiction.
Hence F = F'. □

Let X be a set.

A **ring of sets in pow(X)** is a set R ⊆ **pow**(X) which is closed under ∩ and ∪.

A **field of sets in pow(X)** is a set F ⊆ **pow**(X), containing Ø and 1_F ⊆ X which is closed under ∩, ∪, and λx ∈ F.1_F − x.

R is a ring of sets in **pow**(X) iff <R, ∩, ∪ > is a sublattice of <**pow**(X), ∩, ∪ >.
F is a field of sets in **pow**(X) iff <F, ∩, ∪ > is a sublattice of <**pow**(X), ∩, ∪ > and <F, ⊆, λx∈ F.1_F − x, ∩, ∪, Ø, 1_F > is a Boolean algebra.

Stone's Representation Theorem 17:

Every Boolean algebra **B** is isomorphic to a field of sets in **pow(pow(B))**.

Proof:

Let **B** be a Boolean algebra.

$P_B = \{P \subseteq B$: P is a prime filter in **B**$\}$

For every $b \in B$: $P_b = \{P \in P_B: b \in P\}$

$\Pi_B = \{P_b: b \in B\}$

Π_B is a field of sets in **pow**(**pow**(B)):

$<\Pi_B,\cap,\cup>$ is a sublattice of <**pow**(**pow**(B)), $\cap$, $\cup$> and

$\mathbf{\Pi_B} = <\Pi_B, \subseteq, \lambda x \in \Pi_B.P_B - x, \cap, \cup, \emptyset, P_B>$ is a Boolean algebra.

Claim: B is isomorphic to $\mathbf{\Pi_B}$.

Let $h: B \rightarrow \Pi_B$ be given by: for every $b \in B$: $h(b) = P_b$.

1. h is onto. By definition.

2. h is one-one.
Let $a,b \in B$ and $a \neq b$.
Then, by corrollary 15, there is a prime filter containing exactly one of a and b, say, $a \in P$ and $b \notin P$.
Then $P \in P_a$ and $P \notin P_b$. Then $P_a \neq P_b$, hence $h(a) \neq h(b)$.

3. Assume $a \sqsubseteq b$. Let $P \in P_a$. Then $b \in P$, hence $P \in P_b$.
So $P_a \subseteq P_b$. Hence $h(a) \subseteq h(b)$.

4. $h(\neg a) = P_{\neg a}$ =
$\{P \in P_B: \neg a \in P\}$ = [ultrafilter]
$\{P \in P_B: a \notin P\}$ =
$P_B - \{P \in P_B: a \in P\}$ =
$P_B - P_a$ =
$P_B - h(a)$.

5. $h(a \sqcap b)$ =
$P_{a \sqcap b}$ =
$\{P \in P_B: a \sqcap b \in P\}$ = [filter]
$\{P \in P_B: a \in P$ and $b \in P\}$ =
$\{P \in P_B: a \in P\} \cap \{P \in P_B: b \in P\}$ =
$P_a \cap P_b$ =
$h(a) \cap h(b)$.

6. $h(a \sqcup b) =$
$P_{a \sqcup b} =$
$\{P \in P_B: a \sqcup b \in P\}$ = [prime filter]
$\{P \in P_B: a \in P \text{ or } b \in P\} =$
$\{P \in P_B: a \in P\} \cup \{P \in P_B: b \in P\} =$
$P_a \cup P_b =$
$h(a) \cup h(b)$.

7. $h(0) =$
$P_0 =$
$\{P \in P_B: 0 \in P\}$ = [proper filter]
Ø

8. $h(1) =$
$P_1 =$
$\{P \in P_B: 1 \in P\}$ = [filter]
P_B. ◻

Corrollary 18:
Every distributive lattice **B** is isomorphic to a ring of sets in **pow(pow(A)).**
Proof:
Let **B** be a distributive lattice.
Ignore clauses 6,7, and 8 in the proof of theorem 17.
The proof provides an isomorphism between **B** and $<\Pi_B, \cap, \cup>$. ◻

This tells us that in a very fundamental sense the **pow-power set** Boolean algebras are the most general Boolean algebra: For every Boolean algebra can be embedded (as a lattice) into a **pow-power set** Boolean algebra.

Let **K** be an equational class of lattices.
A lattice **A** ∈ **K** that has the property that every lattice of cardinality smaller or equal to that of **A** can be embedded in **A** is called a **free lattice in K.**

The pow-powerset Boolean algebras are the free Boolean algebras.

3.8 Generated lattices and free lattices

Let **A** = <A, ⊓, ⊔ > be a lattice and let ***B*** be a non-empty set of sublattices of **A.**
Since ∩***B*** is closed under ⊓ and ⊔, ∩***B*** is itself a sublattice of **A**, if it is non-empty.
generated lattices, sets of generators, freely generated lattices.

Let X ⊆ A be a non-empty subset of **A**.

The sublattice of A generated by X is the structure:
[X] = < [X], $⊓_{[X]}$, $⊔_{[X]}$ > where:

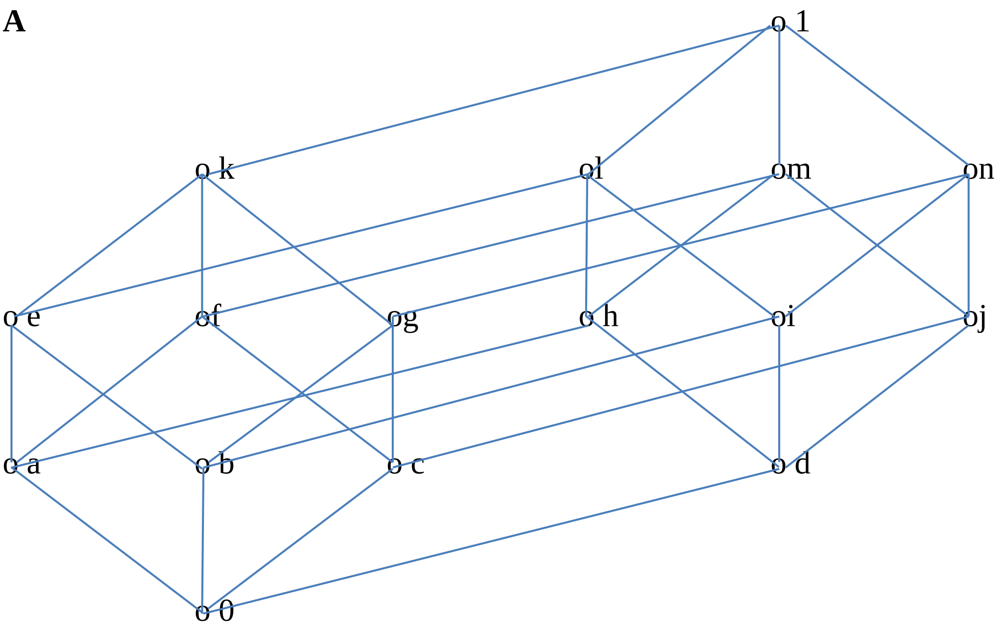
[X] = ∩{B: **B** = <B, $⊓_B$, $⊔_B$> is a sublattice of **A** and X ⊆ B}

Since X is non-empty, [X] is non-empty, and **[X]** is indeed a sublattice of **A**.
[X] is the smallest sublattice of **A** that contains X.
We defined **[X]** from the outside by closing in on sublattices containing X.
From the inside, in algebra **A**, [X] is the result of closing X under the operations of **A.**
-For lattices, you close X under ⊓ and ⊔;
-For c-lattices, you close X unde complete ⊔ and complete ⊓,
-For Boolean algebras, you close X under ⊓ and ⊔ and ¬.
etc.

X is a **set of generators for** lattice A, X generates **A** iff [X] = A
A set of generators X for **A** is **minimal or independent** iff
1. [X] = A
2. For every x ∈ A: [X−{x}] ⊂ A

Example: Look at the following lattice:

A



[{a,b,c}] = {0,a,b,c, e,f,g, k}

$0 = a \sqcap b$ $e = a \sqcup b$ $f = a \sqcup c$ $g = b \sqcup c$ $k = e \sqcup f$

{a,b,c,d} is a set of generators for **A**.

Sets of generators are rarely unique, since if X generates **A** and $X \subseteq Y \subseteq A$, then Y generates **A**

{a,b,c,d} is a set of minimal generators for **A** (under ⊓ and ⊔), and so is {k,l,m,n}.

If we regard **A** as a Boolean algebra, matters are different. Now we consider generation under the operations ⊓, ⊔ and ¬. We need less elements to generate **A**.

In fact:

{e,f} generates **A**

$i = \neg e$	$j = \neg f$
$a = e \sqcap f$	$k = e \sqcup f$
$n = \neg a$	$d = \neg k$
$0 = e \sqcap \neg e$	$1 = e \sqcup \neg e$
$h = a \sqcup d$	$g = \neg h$
$b = e \sqcap g$	$c = f \sqcap j$
$m = \neg b$	$l = \neg c$

Let ***K*** be an equational class of lattices and X a non-empty set.

$F_{\boldsymbol{K}}(X)$ is a **free lattice** in ***K*** generated by X iff

2. $X \subseteq F_{\boldsymbol{K}}(X)$ and for every $x_1, x_2 \in X$: $x_1 \not\sqsubseteq x_2$ and $x_2 \not\sqsubseteq x_1$
3. For any lattice $\mathbf{L} \in \boldsymbol{K}$ and function $f: X \to L$:
   f can be extended to a homomorphism $h: F_{\boldsymbol{K}}(X) \to L$

Some facts about free lattices without proofs.

Fact: Let $F_{\boldsymbol{K}}(X)$ be a **free lattice** in ***K*** generated by X and $\mathbf{L} \in \boldsymbol{K}$ and $f: X \to L$
Then there is exactly one homomorphism extending f.

Corrollary: If $F_{\boldsymbol{K}}(X)$ and $F_{\boldsymbol{K}}(X)$’ are **free lattices** in ***K*** generated by X then $F_{\boldsymbol{K}}(X)$ and $F_{\boldsymbol{K}}(X)$’ are isomorphic

The Fundamental Theorem for Free Lattices:
$F_{\boldsymbol{K}}(X)$ exists iff there is a lattice **L** in ***K*** such that
$X \subseteq \mathbf{L}$ and for every $x_1, x_2 \in X$: $x_1 \not\sqsubseteq x_2$ and $x_2 \not\sqsubseteq x_1$

Corrollary: For every equational class of lattices ***K*** and every n, there is, up to isomorphism, exactly one free lattice in ***K*** with n generators.

The free lattice **F** on n-generators in equational class ***K*** is the most general lattice in *K* on n generators, all other lattices in ***K*** on n or less generators can be gotten from the free lattice **F** by definining a homomorphism on **F** that contracts elements of **F.**

The proof of the Stone representation theory (see chapter 6 and 7)tells us that:

Fact: The free c-Boolean algebras are up to isomorphism the **powpower** set Boolean algebras.

Application to functional type theory.

In functional type theory, we have sets of types TYPE and BOOL:

TYPE is the smallest set such that:

1. e,t $\in$ TYPE
2. If a,b $\in$ TYPE then $<a,b> \in$ TYPE

BOOL is the smallest set such that:

1. t $\in$ BOOL
2. If a $\in$ TYPE and b $\in$ BOOL then $<a,b> \in$ TYPE

Given a domain of individuals D, we associate with each type τ a domain of type τ:

$D_e = D \qquad D_t = \{0,1\} \qquad D_{<a,b>} = (D_a \to D_b)$

Fact 1: All Boolean types are complete atomic Boolean algebras

This follows from the lifting theorem.
Hence, all Boolean types are isomorphic to powerset Boolean algebras.
Some connections:

$D_{<e,t>} \simeq$ **pow**(D)
$D_{<e,<e,t>>} \simeq$ **pow**$(D \times D)$
$D_{<<e,t>,t>} \simeq$ **pow(pow**(D))

Since for each cardinality for which there exists a complete atomic Boolean algebra of that cardinality, there is up to isomorphism, only one complete atomic Boolean algebra, it suffices to make a cardinality argument:

$|D_{<e,t>}| = |(D \to \{0,1\})| = 2^{|D|} = |\mathbf{pow}(D)|$
$|D_{<e,<e,t>>}| = |(D \to D_{<e,t>})| = (2^{|D|})^{|D|} = 2^{|D|\times|D|} = |\mathbf{pow}(D \times D)|$
$|D_{<<e,t>,t>}| = |((D \to \{0,1\}) \to \{0,1\})| = 2^{(2^{|D|})} = |\mathbf{pow}(\mathbf{pow}(D))|$

This means that $D_{<<e,t>,t>}$ is the free Boolean algebra on |D| generators.

In type logic we have for each type τ a set CON_τ of constants of type τ and a countable set VAR_τ of variables of type τ, and the expressions of type logic are defined by:

EXP_τ is the smallest set such that:

1. $CON_\tau \cup VAR_\tau \subseteq EXP_\tau$
2. If $\tau = <a,b>$ and $\alpha \in EXP_\tau$ and $\beta \in EXP_a$ then $(\alpha(\beta)) \in EXP_b$
3. If $\tau = <a,b>$ and $x \in VAR_a$ and $\beta \in EXP_b$ then $\lambda x\beta \in EXP_\tau$
4. If $\alpha, \beta \in EXP_\tau$ then $(\alpha = \beta) \in EXP_t$

A model for type logic is a pair $M = \langle D_M, F_M \rangle$ where
D_M is a non-empty domain of individuals and
for every type $\tau \in$ TYPE: F_M: $CON_\tau \rightarrow D_{M,\tau}$

An assignment function on model M is a function g such that
for every type $\tau \in$ TYPE g: $VAR_\tau \rightarrow D_{m,\tau}$

The interpretation of type logic defines $[\![\alpha]\!]_{M,g}$ the interpretation of α in M relative to g:

1. If $c \in CON_\tau$ then $[\![c]\!]_{M,g} = F_M(c)$
   If $x \in VAR_\tau$ then $[\![x]\!]_{M,g} = g(x)$
2. If $\alpha \in EXP_{<a,b>}$ and $\beta \in EXP_a$
   then $[\![(\alpha(\beta))]\!]_{M,g} = [\![\alpha]\!]_{M,g}([\![\beta]\!]_{M,g})$
3. If $x \in VAR_a$ and $\beta \in EXP_b$
   then $[\![(\lambda x\beta]\!]_{M,g} = h{:}D_a \rightarrow D_b$ such that for all $d \in D_a$ $h(d) = [\![\beta]\!]_{M,g[x:d]}$

where g[x:d] is the assignment function that at most differs from g in that g[x:d](x)=d

In type logic we define a lifting operation which maps expressions of type e onto expressions of type <<e,t>,t>:

LIFT: $EXP_e \rightarrow EXP_{<<e,t>,t>}$
For every $\alpha \in EXP_e$: $LIFT[\alpha] = \lambda P(P(\alpha))$

where $P \in VAR_{<e,t>}$ and P does not occur in α

Let M be a model for type logic and g an assignment on M.

Fact 2: If $g(x_1) \neq g(x_2)$ then $[\![\lambda P(P(x_1))]\!]_{M,g} \neq [\![\lambda P(P(x_1))]\!]_{M,g}$
Proof:
If $g(x_1) \neq g(x_2)$ then $[\![\lambda x.x{=}x_1]\!]_{M,g} \in [\![\lambda P(P(x_1))]\!]_{M,g} - [\![\lambda P(P(x_2))]\!]_{M,g}$

Thus, lifting expressions from type e to type <<e,t>t> is interpreted as an **injection** from D_M into **powpow(D_M))**.

Fact 3: $\{[\![\lambda P(P(x))]\!]_{M,g[x,d]}: d \in D_M\}$ is an independent set of c-generators for $D_{<<e,t><t>}$.
Proof (sketch):
We start with individuals:$\{[\![\lambda P(P(x))]\!]_{M,g[x,d]}: d \in D_M\}$,
for all d the set of properties that d has (= the set of all sets that d is in).

We add their complements: $\{\neg[\![\lambda P(P(x))]\!]_{M,g[x,d]}: d \in D_M\}$ =
$\{[\![\lambda P\neg(P(x))]\!]_{M,g[x,d]}: d \in D_M\}$, for all d the set of properties that d doesn't have.

Of the resulting set we can take any subset and add its complete join and complete meet.
With this we can **define** for each $X \subseteq D_M$:
The set of properties that all $x \in X$ have, and all $x \in D_M - X$ don't have.
Arguably this gives us **all** sets of properties in $D_{<<e,t>,t>}$.

In type logic we define a lifting operation which maps expressions of type <e,<e,t>> onto expressions of type <<<e,t>,t>,<e,t>>: (we can call it O-LIFT, for *object lift*):

O-LIFT: $EXP_{<e,<e,t>>} \rightarrow EXP_{<<<e,t>,t>,<e,t>>}$
For every $\alpha \in EXP_e$: O-LIFT[α] = $\lambda T\lambda x.(T(\lambda y\alpha(x,y)))$

where $\alpha(x,y) = ((\alpha(y))(x))$ and $x,y \in VAR_e$ and $T \in VAR_{<<e,t>,t>}$
and x,y,T do not occur in α

Now $[\![\alpha]\!]_{M,g} \in D_{<e,<e,t>>}$ and $[\![$ O-LIFT$(\alpha)]\!]_{M,g} \in EXP_{<<<e,t>,t>,<e,t>>}$
so $[\![\alpha]\!]_{M,g}: D_e \rightarrow D_{<e,t>}$
$[\![$ O-LIFT$(\alpha)]\!]_{M,g}: D_{<<e,t>,t>} \rightarrow D_{<e,t>}$

We define for $R \in D_{<<e,t>,t>}$: R* as follows:
R* is the function that maps the correlates of individuals inside $D_{<<e,t>,t>}$ onto properties of individuals with the following constraint:
R*: $\{[\![LIFT(x)]\!]_{M,g[x,d]}: d \in D_M\} \rightarrow D_{<e,t>}$ and
for all $d \in D_M$: R(d) = R*($[\![\lambda P(P(x))]\!]_{M,g[x:d]}$)
Obviously, $D_{<e,<e,t>>}$ and $\{R^*: R \in D_{<e,<e,t>>}\}$ are in one-one correspondence.
For $R \in D_{<e,<e,t>>}$, R* is a function that maps a subset of $D_{<<e,t>,t>}$ onto $D_{<e,t>}$.
Since $D_{<<e,t>,t>}$ is the free Boolean algebra on |D| generators and $D_{<e,t>}$ is isomorphic to a subalgebra of $D_{<<e,t>,t>}$, the facts about free algebras mentioned above tell us that there is a unique homomorphism from $D_{<<e,t><t>}$ into $D_{<e,t>}$ extending R*.

Fact: $[\![$O-LIFT$[\alpha]]\!]_{m.g}$ is the unique homomorpishm extending $[\![\alpha]\!]_{M,g}$*
Proof:
1. The following λ-conversions are valid:
(O-LIFT[α]($\lambda P(P(x_1))$ =
$(\lambda T\lambda x.(T(\lambda y\alpha(x,y)))\ (\lambda P(P(x_1)))$ =
$\lambda x.(\lambda P(P(x_1))(\lambda y\alpha(x,y)))$ =
$\lambda x(\lambda y\alpha(x,y)(x_1))$ =
$\lambda x.\alpha(x,x_1)$ =
$\alpha(x_1)$
So indeed $[\![$O-LIFT$[\alpha]]\!]_{M,g}$ extends $[\![\alpha]\!]_{M,g}$*

2. It is easy to check that $[\![$O-LIFT$[\alpha]]\!]_{M,g}$ is a c-homomorphism
Example:
(O-LIFT[α]($\neg_{<<e,t>,t>}\lambda P(P(x_1))$ =
(O-LIFT[α]($\lambda P\neg(P(x_1))$ =
$(\lambda T\lambda x.(T(\lambda y\alpha(x,y)))\ (\lambda P\neg(P(x_1)))$ =
$\lambda x.(\lambda P\neg(P(x_1))(\lambda y\alpha(x,y)))$ =
$\lambda x(\neg(\lambda y\alpha(x,y)(x_1)))$ =
$\lambda x.\neg\alpha(x,x_1)$ =
$\neg_{<e,t>}(\alpha(x_1))$