QUANTIFICATION AND MODALITY

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revised Oct 2021

PART 1: QUANTIFICATION

INTRODUCTION

I. SEMANTIC MEANING/PRAGMATIC MEANING

Recommendation letter: I only write

He has beautiful azure eyes

Pragmatic implication: Don't take the guy.

Gricean reasoning. speechcontext, etc.

knowledge about what one is supposed to write in a recommendation letter.

Semantic implications:

-He has azure eyes

-He has eyes etc.

Depends only on the speaker/hearer knowledge of the language $\rightarrow$ semantic competence

So semantic facts are –it seems- much more boring than pragmatic facts.
But even stupid facts like the above are interesting because they are part of **patterns** that are interesting.

II ADJECTIVES

Intersectivity: A. An *azure eye* is an *eye*

B. An *azure eye* is *azure*

Many adjectives are intersective.

Some adjectives do not quite look intersective, but are what is called **subsective.** These are typically degree adjectives:

A. A *small elephant* is an *elephant*

X B. A *small elephant* is *small*

It is not clear that subsective adjectives aren't really intersective.

Assumption 1:

Degree adjectives have an interpretation aspect which is not lexicalized , a **comparison class.**

Assumption 2: Pragmatics of comparison class

1. Prenominal/attributive adjectives:

Out of the blue the comparison class is the denotation of the noun:

small [rel. C] elephant C = elephant

$\rightarrow$ *small [rel elephant] elephant*

2. Predicative adjectives:

Out of the blue the comparison class is contextual:

small [rel C] C = set of contextual objects

Now look at:

A small elephant is small

interpretation:

A small [rel C_1] elephant is small [rel C_2]

Intersectivity **only** says that the following should be true:

A small [rel C_1] elephant is small [rel $\boldsymbol{C_1}$*]*

And this is uncontroversial: if Jumbo is a small elephant, then Jumbo is small for an elephant.

But the pragmatics of comparison class gives you out of the blue:

A small [rel elephant] elephant is small [rel set of contextual objects].

This is, of course, not necessarily true, on anybody's theory.
This means that we can maintain intersectivity, despite the seeming evidence to the contrary.

Intersectivity claims that the inference in (1) is true:

(1) a. Jumbo is a small elephant.
 b. Jumbo is an elephant and small (in comparison with the other elephants).

Intersectivity does not claim that the inference in (2) is true:

(2) a. Jumbo is a small elephant
 b. Jumbo is an elephant and small (in comparison with the other animals in the zoo)

Evidence for comparison class:
Even for attributive adjectives, the comparison class can be contextually determined:
Kamp & Partee

My three year old
The college team
built a huge snowman

$C_{huge} \neq$ Snowmen

$C_{1,huge}$ = Snowmen built by 3 year olds
$C_{2,huge}$ = Snowmen built by college teams

EXCURSUS
One could speculate –but this is more tentative – that an argument for intersectivity even applies to adjectives like *dead* and *fake*.

1	A	A *dead poet* is a *poet*
	B	A *dead poet* is *dead*
2	A	A *fake Rembrandt* is a *Rembrandt*
	B	A *fake Rembrandt* is *fake*

also *a fake diamond*
Here the inference to the adjective is valid, but the inference to the noun is not.
The idea would be that the pragmatics of *dead*/*fake* allows for 'temporary widening of the denotation of the noun.

2A would be ambiguous:

A_{wide}	A *fake Rembrandt*^{WIDE} is a *Rembrandt*^{WIDE}	True
B_{narrow}	A *fake Rembrand*^{WIDE}*t* is a *Rembrandt*^{NARROW}	False

cf: *Most Rembrandts are fake.*

However, whether this line is ultimately fruitful is questionable.
The reason is that there is ultimately a big difference between (3a) and (3b):

(3) a. A *small* elephant is ***not*** small.
b. A *fake* Rembrandt is ***not*** a Rembrandt.

While the truth of (3a) (with stress indicated) is dependent on the context, (3b) seems to be true absolutely.

END OF EXCURSUS

We see that intersectivity applies to a wide class of adjectives.
But not to all: **Temporal and modal adjectives**.

Temporal:

A A *former friend* is a *friend* FALSE
B A *former friend* is *former* INFELICITOUS
Similarly *future wife*, etc…

-Not subsective (A is false)
-Most intensional adjectives (= temporal or modal can not be use predicatively.

Modal:

A. A *potential counterexample* is a *counterexample* FALSE
B. A *potential counterexample* is *potential* INFELICITOUS

So the 'stupid facts' actually form part of a semantic classification of adjectives in terms of **intersective** versus **intensional.**

And this generalizes.
I. We find the same distribution for **adverbials**:

Intersective adverbials: for example manner adverbials:
Elegantly, Sasha jumped
A: Sasha's jumping was jumping
i.e. Sasha jumped
B: Sasha's jumping was elegant
i.e. Something that happened was elegant

Intensional adverbials:
Potentially, Sasha will jump
A: Sasha will dance FALSE
B: Something that will happen is potential INFELICITOUS

III GENERALIZATIONS

There is a different kind of generalization that we are particularly interested in.

THREE KINDS OF SEMANTIC MEANING

1. WORD MEANING	[Lexicography]
2. SENTENCE MEANING	[Logic]
3. CONSTITUENT MEANING	[Semantics]

Sentence meaning: We use judgements of native speakers about inference and felicity as **data**. These judgements involve sentence meanings.

Constituent meaning: Semantic generalizations are most often best stated **neither** at the level of word meaning, **nor** at the level of sentence meaning, **but** at an intermediate level of constituent meaning.

Example.

He has *beautiful azure* eyes *which shine in the dark with black eye lashes*

Adjectives, relative clause, prepositional phrase.

Facts:

A *Azure eyes* are *eyes*
B *Azure eyes* are *azure*

A *Eyes with black eye lashes* are *eyes*
B *Eyes with black eye lashes* have *black eye lashes*

A *Eyes which shine in the dark* are *eyes*
B *Eyes which shine in the dark shine in the dark*

Observation: Intersectivity is a principle that concerns **not just** adjectives, but also prepositional phrases and relative clauses.

This means that intersectivity is **not** a lexical property of the meanings of certain words (like adjectives), but **of the meanings of classes of PHRASES**.
More precisiely, it is a meaning constraint on how the meanings of ADJUNCTS like APs, PPs, CPs combine with the meanings nouns, verbs.

But this means that we need a theory of **constituent meanings** and a theory of **the meaning of adjunction** in order to even state the generalization.
This is what semantics is about.

Generalization:
Syntactic adjuncts come in two kinds:

A Those derived from predicates

B Those not derived from predicates (intensional)

The semantic interpretation of adjunction for class A is predicate **intersection**.

IV ABOUTNESS AND SEMANTIC COMPETENCE.

A core part of what we call meaning concerns the relation between linguistic expressions and non-linguistic entities, or 'the world' as our semantic system assumes it to be, the world as structured by our semantic system.

Some think about semantics in a realist way: semantics concerns the relation between language and the world.
Others think about semantics in a more conceptual, or if you want idealistic way: semantics concerns the relation between language and an intersubjective level of shared information, a conceptualization of the world, the world as we jointly structure it. Emmon Bach: Natural Language Metaphysics.
Both agree that semantics is a theory of **interpretation** of linguistic expressions: semantics concerns the relation between linguistic expressions and what those expressions are about. Both agree that important semantic generalizations are to be captured by paying attention to what expressions are about, and important semantic generalizations are missed when we don't pay attention to that.

But semantics concerns semantic **competence**. Semantic competence does not concern **what** expressions happen to be about, but **how** they happen to be about them.

Native speakers obviously do not have to know what, say, a name happens to stand for in a certain situation, or what the truth value of a sentence happens to be in a certain situation. That is not necessarily part of their semantic competence. What is part of their semantic competence is **reference conditions**, **truth conditions**:

Take the Dutch sentence:

Er is geen pen onder de tafel. *Er is geen pen onder mijn hand*

A Dutch speaker can use that sentence to distinguish situation one [pen under the table] from situation two [pen above the table].
In which do you think is the sentence true?
Well, what Dutch speakers know is that *g-* is a negative morpheme in Dutch, so it is situation two. So: the Dutch speaker can use this sentence to distinguish these two types of situation, while you can't. This is not because the dutch are more intelligent than you are, but **only** because the Dutch speakers have something that you don't have: semantic compentence in Dutch.
Note that it is **not** part of the Dutch speakers competence to know whether the sentence is true of false (that is the business of detectives and scientists). What **is** part of your semantic competence is that, in principle, you're able to distinguish situations where that sentence is true, from situations where it is false, i.e. that you know **what it takes** for a possible situation to be the kind of situation in which that string of words, that sentence, is true, and what it takes for a situation to be the kind of situation where that sentence is false.

Note too that we are talking about **linguisic competence**: my cat too can classify situations in terms of situations where there is a cockroach in the house, and where there isn't. But he cannot use language to do that classification, and we can.

The first thing to stress is: semantics is not interested in truth; semantics is interested in truth conditions.
From this it follows too that we're not interested in truth conditions per se, but in **truthconditions relative to contextual parameters**.

Take the sentence: *I was behind the table one minute ago*. The truth of this sentence depends on who the speaker is, when it is said, what the facts in the particular situation are like. But we're not interested in the truth of this sentence, hence we're not interested in who is the speaker, when it was said, and what the facts are like.

What we're interested in is the following: **given** a certain situation (any situation) at a certain time where a certain speaker (any speaker) utters the above sentence, and certain facts obtain in that situation (any combination of facts): do we judge the sentence true or false under those circumstantial conditions?

A semantic theory assumes that when we have set such contextual parameters, native speakers have the capacity to judge the truth or falsity of a sentence in virtue of the meanings of the expressions involved, i.e. in virtue of their semantic competence. And that is what we're interested in.

Semantic competence involves recognizing how truth values of sentences of your native language change, when you vary aspects of evaluation situations.

- -vary the facts: make my green t-shirt yellow.
- -vary the time: go to a point where I am 23.
- -vary the speaker: go to a speaker who now is 23.
- -vary the person pointed at: *she* has azure eyes.

Some of these aspects are linguistically creative, they get linguistically encoded in many languages, and classes of expressions, often cross-linguistically, are sensitive to this aspect, others are not.

i.e. Facts are less linguistically creative than time is:
Changing the color of my shirt is not going affect the truth value of sentence that are not about me, but varying the time is. Languages evaluate relative to time and have time-operations, but they do no evaluate relative to fred-shirt-color, and they do not have fred-shirt-color operations.

To summarize: a semantic theory contains a theory of aboutness and this will include a theory of truth conditions.

Given the above, when I say **truth**, I really mean, **truth relative to settings of contextual parameters**.

Furthermore, given what I said before about realistic vs. idealistic interpretations of the domain of non-linguistic entities that the expressions are about, you should not necessarily think of truth in an absolute or realistic way: that depends on your ontological assumptions. If you think that semantics is directly about the real world as it is in itself, then **truth** means **truth in a real situation**. If you think that what we're actually talking about is a level of shared information about the 'real' world, then situations are shared conceptualizations, structurings of the real world, and **truth** means **truth in a situation which is a structuring of reality**. This difference has very few practical consequences for most actual semantic work: it concerns the interpretation of the truth definition rather than its formulation.
This is a gross overstatement, but for all the phenomena that we will be concerned with in this course, this is true enough.

Specifying a precise theory of truth conditions, makes our semantic theory **testable**. We have a general procedure for defining a notion of **entailment** in terms of truth conditions. Once we have formulated a theory of the truth conditions of sentences containing the linguistic expressions whose semantics we are studying, our semantic theory gives a theory of what entailments we should expect for such sentences. Those predictions we can compare with our judgments, the intuitions concerning the entailments that such sentences actually have.

This may sound trivial, but it isn't really. We will mention later Aristotle's theory of the Syllogism, a theory which dominated logical thought for 2000 years, but which patently fails to makes any predictions at all about large classes of data that it is concerned with. [i.e. it is easy for interesting theories to be nevertheless inadequate]

Even for good ideas, it is easy for interesting theories to go wrong [even if at heart they are good, fruitful theories], and we will need to think about how to make them go right.

Example: Event Theory

Event theory proposes that verbs have an event argument. The theory allows for insightful analyses of the semantics of adverbials, tense and aspect.
A simple minded version of the theory is based on the following paraphrases:

Sasha chased Fido

Analysis: There is a chasing event in the past with Sasha as chaser and Fido as chasee.

Sasha chased Fido quickly

Analysis: There is a chasing event in the past with Sasha as chaser and Fido as chasee and that event was done in a quick manner.

Some cat chased Fido

Analysis: There is a chasing event in the past with some cat as chaser and Fido as chasee.

Some cat chased some dog

Analysis: There is a chasing event in the past with some cat as chaser and some dog as chasee.

Based on this, we would expect the following analysis:

Some cat chased no dog

Analysis: There is a chasing event in the past with some cat as chaser and no dog as chasee.

But this analysis derives the wrong meaning: it says that some cat chased a non-dog, which is, of course not what the sentence means.

It is easy to see what the most natural reading that the sentence *does* have should be:

Some cat chased no dog

Analysis: There is a cat for which there isn't a chasing event in the past with that cat as chaser and a dog as chasee.

There are *bona fide* versions of grammars using event theory that block the wrong readings and get the right readings. But in order to get this and maintain the advantages that were the rationale for introducing event theory in the first place requires a subtle balance and requires subtle thinking about the syntax-semantics relation.
The task of formulating elegant semantic theories that get the facts right is highly non-trivial, challenging (and fun!).

End of Example.

David Lewis' Practical Guide:

Do not ask what a meaning is, but what a meaning does, and find something that does that.

→ Intension of φ: function from situations to truthvalues
Intension of φ does (by and large) what we want a meaning to do.

This is not yet a theory: we need to specify what we put in situations (which distinctions are linguistically relevant

Facts, time, speaker, events,…

3When we fix that we have a precise theory of objects that do what we want meanings to do, a theory that makes predictions about entailments which can be checked with the facts.

If you tell me: 'but that's not what meanings are", I will ask you: ' Well, what more do you want meanings to **do**?

-If you want meanings to do the dishes, intensions won't

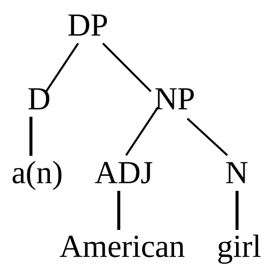
-Possibly you find the particular notion of intention used not finegrained enough. In that case, I will try to make my situations more finegrained.

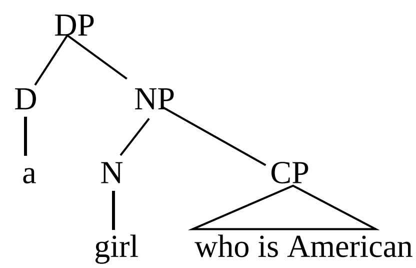
But the fact it that practically speaking the theories that **have** been developed are succesful in dealing with a large number of phenomena, and in stating important generalizations.

V. COMPOSITIONALITY.

The interpretation of a complex expression is a function of the interpretations of its parts and the way these parts are put together.

Semantic theories differ of course in what semantic entities are assumed to be the interpretations of syntactic expressions. They share the general format of a compositional interpretation theory. Let us assume that we have certain syntactic structures, say, the following trees:





In a compositional theory of interpretation, we choose semantic entities as the interpretations, meanings of the parts. This means that we start with meanings for the lexical items:

1. **m**(*a(n)*) **m**(*American*) **m**(*girl*)

What these are will depend, of course, on your semantics theory.

We assume that corresponding to the build up rules in the syntax, there are corresponding semanic interpretation rules.

Fbirst we make the standard assumption that the little trees projected from the lexicon have as their meaning the meanings of the lexical items:

m(D) = **m**(*a*)	**m**(ADJ) = **m**(*American*)	**m**(N) = **m**(*girl*)
|	|	|
a	*American*	*girl*

Next, we assume that, corresponding to the syntactic operation of adjunction forming a noun phrase out of an adjective and a noun(phrase), there is a semantic operation forming the meaning of the complex noun phrase as a function of the meaning of the adjective and the meaning of the noun. And the same for the operation forming a determiner phrase out of a determiner and a noun phrase.

Moreover, we have argued above that the semantic operation that combines an adjective with a noun and the semantic operation that combines a relative clause with a noun should be the same **intersective** semantic operation. This is a generalisation that we want to express in the grammar:

DET + NP $\Rightarrow$ DP
m(DP) = $\mathbf{OP_1}$[**m**(D), **m**(NP)]

NP_1 + ADJUNCT $\Rightarrow$ NP_2
m(NP_2) = $\mathbf{OP_2}$[**m**(ADJUNCT), **m**(NP_1)]

Our grammatical assumtion is that the same semantic operation corresponds in adjunction in both trees.

Let us now make a specific assumption to illustrate compositionality.
We assume that the adjective *American* and the relative clause *Who is American* have the same meaning.

Note, we do not have to make that assumption, but let us assume here that we are dealing with a notion of meaning for which that is reasonable.

•

Assumption:

m([$_{CP}$ *who is American*] = **m**(*American*)

In that case, the principle of compositionality tells that **in this grammar** the two trees derived have **the same meaning**: in both cases we derive for the whole tree:

$\mathbf{OP_1}$[**m**(*a*), $\mathbf{OP_2}$[**m**(*American*), **m**(*girl*)]]

It is easy to see that the principle of Compositionality of Meaning entails a principle of Substitution of Meaning:

Look at (1) and (2)

(1) Fred dances with an American girl
(2) Fred dances with a girl who is American

(2) is the result of replacing adjoined constituent *american* by adjoined constituent *who is american* at the same place in the semantic structure. If we assume that the semantic composition operations in the derivation of (1) and (2) are the same (as we do) and we that m(*American*) = m(*who is American*), then it follows from compositionality that m(1) = m(2)

If you substitue in an expression a sub-expression β by an expression γ with the same meaning, the meaning of the whole stays the same.

ARGUMENTS FOR COMPOSITIONALITY

1. *A priori* arguments.
Compositionality is semantic recursiveness. Frege 1918 *Der Gedanke* gives in essence the same argument for semantics as Chomsky for syntax later:
We understand sentences that we have never heared before. Sentence comprehension cannot be a creative exercise because we do it fast, on-line. It is not clear how this could possible work without assuming compositionality.
[recursion: simplest form: modifiers like adjective: input is of the same type as the output)

2. *Practical* arguments.
2a. The meaning of a complex expression is a network of interacting factors: i.e. interesting phenomena on the intersection of aspect, quantification, mass-noun distinctions, plurality, etc. etc.
Compositionality is **analysis**, it separates the semantic contributions of the parts and the contribution of the semantic **glue.** So it helps you in telling in a complex of interacting factors which bits or meaning are contributed by what.

2b. Compositional analyses work better than non-compositional ones.
Argument by Intimidation

3. *Theoretical* arguments.
The compositional analysis in 2 allows you formulate your semantic generalizations at the appropriate level of constituent meaning.
For instance, **intersectivity** is a semantic correlate of the **adjunction operation**.

Compostionality leads you to correct generalizations, while non-compositional lead away from those.

I. SET THEORY (Cantor, Boole)

Set Theory is based on the **element-of** relation $\in$.
The fundamental properties of sets and the element-of relation are given by the following principles:

Separation: Given a domain D and a property P, we can form the set of all objects in D that have property P: $\{x \in D: P(x)\}$.

-We write $\{a,b,c\}$ for the set $\{x \in D: x = a \text{ or } x = b \text{ or } x = c\}$.

Extensionality: sets are only determined by their elements:

$$A = B \text{ iff for every } a \in D\text{: } a \in A \text{ iff } a \in B$$

-It follows from extensionality that $\{c,b,a,c\} = \{a,b,c\}$
-It follows from Separation that, if there is a domain D, there is an empty set, a set with no elements (because we can define the set of all elements of D that have the property of being non-identical to itself).
-It follows from Extensionality that there is only one empty set (because any two empty sets have the same elements, and hence are identical):

Empty set: The **empty set**, $Ø = \{x \in D: x \neq x\}$
($\neq$: 'is not identical to')

From now on we write A,B,C for sets of objects in domain D.

Subset relation: A is a **subset of** B, $A \subseteq B$, iff for every $a \in D$: if $a \in A$ then $a \in B$.

FACTS about $\subseteq$:

- -For every set A: $Ø \subseteq A$
- -For every set A: $A \subseteq A$ (reflexivity)
- -For every sets A,B,C: if $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$ (transitivity)
- -For every sets A,B: if $A \subseteq B$ and $B \subseteq A$ then $A=B$ (anti-symmetry)

Union: The **union** of A and B, $A \cup B = \{x \in D: x \in A \text{ or } x \in B\}$

FACTS about $\cup$ and $\subseteq$:

- -for every A: $A \cup A = A$ (idempotency)
- -for every A,B: $A \cup B = B \cup A$ (commutativity)
- -for every A,B,C: $A \cup (B \cup C) = (A \cup B) \cup C$ (associativity)
- -for every A, B: $A \cup B$ is the smallest set of elements of D such that $A \subseteq A \cup B$ and $B \subseteq A \cup B$ (the join of A and B in D)

Intersection: The **intersection** of A and B, $A \cap B = \{x \in D: x \in A \text{ and } x \in B\}$

FACTS about $\cap$ and $\subseteq$:

- for every A: $A \cap A = A$ (idempotency)
- for every A, B: $A \cap B = B \cap A$ (commutativity)
- for every A,B,C: $A \cap (B \cap C) = (A \cap B) \cap C$ (associativity)
- for every A,B: $A \cap B$ is the biggest set of elements of D such that $A \cap B \subseteq A$ and $A \cap B \subseteq B$ (the meet of A and B in D)

FACTS about $\cup$, $\cap$ and $\subseteq$:

- for every A,B: $A \cap (B \cup A) = A$ (absorption)
- for every A,B: $A \cup (B \cap A) = A$ (absorption)
- for every A,B,C: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (distributivity)
- for every A,B,C: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ (distributivity)

Complement: The **complement** of B in A**,** $A - B = \{a \in D: a \in A \text{ and } a \notin B\}$
($\notin$: 'is not an element of')

The **complement** of B, $-B = D - B$

FACTS about $\neg$:

- $-\varnothing = D$ (laws of 0 and 1)
- $-D = \varnothing$ (")
- for every A: $A \cup -A = D$ (")
- for every A: $A \cap -A = \varnothing$ (")
- for every A: $--A = A$ (double negation)
- for every A,B: $-(A \cap B) = (-A \cup -B)$ (de Morgan laws)
- for every A,B: $-(A \cup B) = (-A \cap -B)$ (de Morgan laws)

Cardinality: The **cardinality** of A, |A| is the number of elements of A.

Powerset: The **powerset** of A, $pow(A) = \{B: B \subseteq A\}$

FACT about pow:

- If A has n elements, pow(A) has 2^n elements.
- $pow(\varnothing) = \{\varnothing\}$
- $pow(\{a\}) = \{\varnothing, \{a\}\}$
- $pow(\{a,b\}) = \{\varnothing, \{a\}, \{b\}, \{a,b\}\}$
- $pow((\{a,b,c\}) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

Ordered pairs:
A set with one element we call a **singleton** set.
A set with two elements we call an **unordered pair.**
Unordered means that $\{a,b\} = \{b,a\}$.

The **ordered pair** of a and b, $\langle a,b \rangle$ differs from the unordered pair in that the order of the elements is fixed. Ordered pairs satisfy the following condition:

$\langle a_1,a_2 \rangle = \langle b_1,b_2 \rangle$ iff $a_1=b_1$ and $a_2=b_2$.

We understand the notion of ordered pair such that while $\{a,a\} = a$, $\langle a,a \rangle \neq a$.

FACT: -if $a \neq b$, then $\langle a,b\rangle \neq \langle b,a\rangle$

Similarly, we call $\langle a,b,c\rangle$ an **ordered triple**. We use quadruple, quintuple, sextuple, etc. The general case we call an ordered n-tuple:

$\langle a_1,...,a_n\rangle$ with n a number is an **ordered n-tuple**.

Cartesian product: The **cartesian product** of A and B,

$$A \times B = \{\langle a,b\rangle : a \in A \text{ and } b \in B\}$$

Similarly, the cartesian product of A, B and C is:

$$A \times B \times C = \{\langle a,b,c\rangle : a \in A \text{ and } b \in B \text{ and } c \in C\}$$

Given this, $A \times A = \{\langle a,b\rangle : a, b \in A\}$. We also write A^2 for $A \times A$

Similarly, $A^3 = A \times A \times A = \{\langle a,b,c\rangle : a,b,c \in A\}$

FACT: -if $|A| = n$ and $|B|=m$ then $|A \times B| = n \times m$

-Hence $|A^2| = |A|^2$, $|A^3| = |A|^3$, etc.

-$\{a,b\}\times\{c,d,e\} = \{\langle a,c\rangle,\langle a,d\rangle,\langle a,e\rangle,\langle b,c\rangle,\langle b,d\rangle,\langle b,e\rangle\}$

-$\{a,b\}^2 = \{a,b\}\times\{a,b\} = \{\langle a,a\rangle,\langle a,b\rangle,\langle b,a\rangle,\langle b,b\rangle\}$

Relations: R is a (two-place) **relation** between A and B iff $R \subseteq A \times B$

Hence: the set of all (two-place) relations between A and B is $pow(A \times B)$.

R is a (two place) relation on A iff $R \subseteq A \times A$.

Hence $pow(A^2)$ is the set of all (two-place) relations on A.

Similarly, the set of all three-place relations on A, B and C is $pow(A \times B \times C)$, the set of all three-place relations on A is $pow(A^3)$, and the set of all n-place relations on A is the set: $pow(A^n)$.

Note: We sometimes make the notational convention: $\langle a\rangle = a$. If we do that, we can write A^1 for A. On this notation $pow(A) = pow(A^1)$, the set of all one-place relations on A. Thus the set of all one-place relations on A , also called properties, is the set of all subsets of A.

Domain and range:

Let R be a two-place relation between A and B, $R \subseteq A \times B$.

The **domain** of R, $dom(R) = \{a \in A: \text{for some } b \in B: \langle a,b\rangle \in R\}$

The **range** of R, $ran(R) = \{b \in B: \text{for some } a \in A: \langle a,b\rangle \in R\}$

Let $A = \{a,b,c\}$, $B = \{a, c,d,e\}$, $R = \{\langle a,a\rangle, \langle a,c\rangle, \langle b,d\rangle\}$.

Then $dom(R) = \{a,b\}$, $ran(R) = \{a,c,d\}$.

Converse relation, total relation, empty relation:

Let $R \subseteq A \times B$ be a relation between A and B.

The **converse relation** of R, $R^c = \{\langle b,a\rangle : \langle a,b\rangle \in R\}$

$A \times B$ is itself a relation between A and B, we call it the **total** relation (everything relates to everything else).

Ø is also a relation between A and B, we call it the **empty** relation (nothing relates to anything).

Functions: f is a (one-place, total) **function** from A into B, $f: A \to B$ iff:

1. f is a relation between A and B: $f \subseteq A \times B$.
2. dom(f) = A and ran(f) $\subseteq$ B
   i.e. for every $a \in A$ there is a $b \in B$ such that $\langle a,b\rangle \in f$.
3. for every $a \in A$, $b_1,b_2 \in B$: if $\langle a,b_1\rangle \in f$ and $\langle a,b_2\rangle \in f$ then $b_1 = b_2$.

If dom(f) $\subseteq$ A and the other conditions hold we call f a **partial** (one-place) function from A into B.
When I say function, I mean total function unless I tell you differently explicitly.

The intuition is: a function from A into B takes each element of A and maps it onto an element of B.

Arguments and values:
We call the elements of the domain of f the **arguments** of f, and the elements of the range of f the **values** of f.
A function maps each argument in its domain on one and only one value in its range.
So: each argument has a value, and no argument has more than one value.
(But note, different arguments may have the same value.)

We write: $f(a)=b$ for $\langle a,b\rangle \in f$.

Example: Let A = {a,b,c} and B = {0,1}.
f = {<a,1>,<b,1>,<c,0>} is a function from A into B.
We also use the following notation for f:

$$f: \begin{pmatrix} a \to 1 \\ b \to 1 \\ c \to 0 \end{pmatrix}$$

n place operations:
If $f: A \to A$ we call f a (one-place) **operation** on A.

We call a function $f: A \times B \to C$ a two-place function from A and B into C.
If $f: A \times A \to A$, we call f a two-place operation on A.
Similarly, $f: A^n \to A$ is an n-place operation on A.

Function space: The **function space** of A and B: $(A \to B) = \{f: f:A \to B\}$
The function space of A and B is the set of all functions from A into B.
This is also notated as B^A.

FACTS: - $|(A \to B)| = |B|^{|A|}$
- $(\{a,b,c\} \to \{0,1\}) = \{f_1,f_2,f_3,f_4,f_5,f_6,f_7,f_8\}$ where:

f_1 $a \to 1$
$b \to 1$
$c \to 1$

f_2: $a \to 1$	f_3: $a \to 1$	f_4: $a \to 0$
$b \to 1$	$b \to 0$	$b \to 1$
$c \to 0$	$c \to 1$	$c \to 1$
f_5: $a \to 1$	f_6: $a \to 0$	f_7: $a \to 0$
$b \to 0$	$b \to 1$	$b \to 0$
$c \to 0$	$c \to 0$	$c \to 1$

f_8: $a \to 0$
$b \to 0$
$c \to 0$

Note that indeed $|\{0,1\}|^{|\{a,b,c\}|} = 2^3 = 8$

Injections, surjections, bijections:
Let f: $A \to B$ be a function from A into B.

f is a **injection** from A into B, a **one-one** function from A into B iff
for every $a_1,a_2 \in A$: if $f(a_1) = f(a_2)$ then $a_1=a_2$.
i.e. no two arguments have the same value.

f is a **surjection** from A into B, a function from A **onto** B iff
for every $b \in B$ there is an $a \in A$ such that $f(a)=b$.
i.e. every element of b is the value of some argument in A.

f is a **bijection** from A into B iff f is an injection and a surjection from A into B.

Inverse function:
If f: $A \to B$ is an injection from A into B, f is a bijection from A into ran(f).
In this case, f^c, the converse relation of f, is itself a function from ran(f) into A (and in fact, also a bijection). We call this the **inverse function** and write f^{-1} for f^c.

Identity function on A:
The **identity function** on A, id_A is the function id_A: $A \to A$ such that
for every $a \in A$: $id_A(a)=a$.
(the function that maps every element onto itself).

Constant functions:
A function $f:A \to B$ is **constant** iff for every $a_1,a_2 \in A$: $f(a_1)=f(a_2)$.

If f is a constant function and the value is b, we call f the constant function on b (and write c_b).

Composition of functions:
Let f: $A \rightarrow B$ and g: $B \rightarrow C$ be functions.
Then the **composition** of f and g, g o f, (g over f, or g after f), is the following function from A into C:

g o f: $A \rightarrow C$ is the function such that:
for every $a \in A$: g o f(a) = g(f(a))

Intuitively, the composition takes you in one step where the functions f and g take you in two steps.

Let MOTHER: IND $\rightarrow$ IND be the function which maps every individual on its mother, and FATHER: IND $\rightarrow$ IND the function which maps every individual on its father.
Then MOTHER o FATHER is the paternal grandmother function, mapping every individual onto the mother of its father.
Similarly, MOTHER o MOTHER is the maternal grandmother function, mapping every individual onto the mother of its mother.

Similarly, if we take a function INT: LIVING-IND $\rightarrow$ TIME INTERVALS
which mapes every individual alive now onto the maximal time interval that it has been alive in up to now, and we take a function
TIME: TIME INTERVALS $\rightarrow$ NUMBERS which assigns to every time interval a length measured in terms of years (so, intervals smaller than a year are assigned 0, etc.), then the function
AGE: LIVING-IND $\rightarrow$ NUMBERS defined by:

AGE = TIME o INT

assigns to every living individual its current age measured in years.

Characteristic functions:

Let $B \subseteq A$
The **characteristic function** of B in A is the function:
ch_B: $A \rightarrow \{0,1\}$ defined by:
for every $a \in A$: $ch_B(a) = 1$ if $a \in B$
$ch_B(a) = 0$ if $a \notin B$

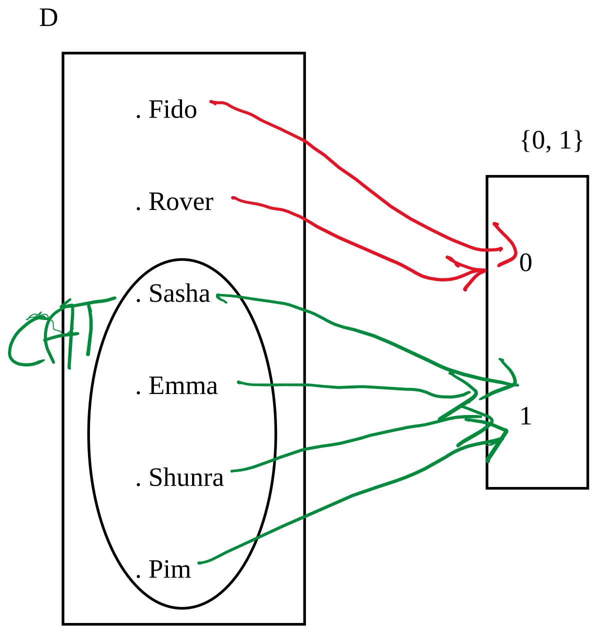
Let f:$A \rightarrow \{0,1\}$ be a function from A into {0,1}
The **subset** of A **characterized by** f, $ch_f = \{a \in A: f(a)=1\}$.

FACT: The elements of pow(A) (the subsets of A) and the elements of ($A \rightarrow \{0,1\}$) (the functions from A into {0,1}) are in one-one correspondence:
-each function in ($A \rightarrow \{0,1\}$) uniquely characterizes a subset of A.
-each subset of A has a unique characteristic function in ($A \rightarrow \{0,1\}$).

Characteristic functions and sets are two sides of the same coin: if you know the domain and the set, you know the characteristic function and if you know the characteristic function, you know the set characterized.

This means that if we assume that *walk* is interpreted as a set, the set of walkers, this is for all purposes **the same** as saying that *walk* is interpreted as the function mapping each individual onto 1 if that individual is a walker, and onto 0 if that individual isn't.

It also means that if we identify the intension of a sentence as the function which maps each situation onto 1 if the sentence is true in it, and onto 0 otherwise, this is for all purposes **the same** as saying that the intension of that sentence is identical to the set of all situations where it is true.



CAT ⊆ D
CAT = {Sasha, Emma, Shunra, Pim}

$ch_{CAT}: D \rightarrow \{0,1\}$

ch_{CAT} = {<Fido,0> , <Rover,0>, <Sasha,1>, <Emma,1>, <Shunra,1>, <Pim,1>}

II. L_1, A LANGUAGE WITHOUT VARIABLES (Frege, Boole)

SYNTAX OF L_1

1. Lexicon of L_1

NAME = {SASHA,SHUNRA, FIDO,...}	The set of names.
$PRED^1$ = {PURR, MEOUW, CAT, DOG,...}	The set of one-place predicates.
$PRED^2$ = {CHASE, HUG,...}	The set of two-place predicates.
NEG = $\{\neg\}$	"not"
CONN = $\{\wedge,\vee,\rightarrow\}$	"and", "or", "if...then..."

LEX = NAME $\cup$ $PRED^1$ $\cup$ $PRED^2$ $\cup$ NEG $\cup$ CONN

2. Sentences of L_1

FORM, the set of all formulas of L_1 is the smallest set such that:

1. If $P \in PRED^1$ and $\alpha \in$ NAME, then $P(\alpha) \in$ FORM.
2. If $R \in PRED^2$ and $\alpha,\beta \in$ NAME, then $R(\alpha,\beta) \in$ FORM.
3. If $\varphi \in$ FORM, then $\neg\varphi \in$ FORM.
4. If $\varphi,\psi \in$ FORM, then $(\varphi \wedge \psi) \in$ FORM.
5. If $\varphi,\psi \in$ FORM, then $(\varphi \vee \psi) \in$ FORM.
6. If $\varphi,\psi \in$ FORM, then $(\varphi \rightarrow \psi) \in$ FORM.

SEMANTICS FOR L_1

1. Models for L_1 (evaluation situations)

A **Model for** L_1 is a pair $M = \langle D_M, F_M\rangle$, where:

1. D_M is a (non-empty) set, the **domain of** M.
2. F_M, the **interpretation function for the lexical items**, is a function such that:
   a. F_M is a function from **names** to **individuals** in D_M.
      F_M: NAME $\rightarrow D_M$
      i.e. for every $\alpha \in$ NAME: $F_M(\alpha) \in D_M$.
   b. F_M is a function from **one-place predicates** to **sets of individuals**:
      F_M: $PRED^1 \rightarrow pow(D_M)$
      i.e. for every $P \in PRED^1$: $F_M(P) \subseteq D_M$.
   c. F_M is a function from **two-place predicates** to **sets of pairs of individuals (two-place relations)**:
      F_M: $PRED^2 \rightarrow pow(D_M \times D_M)$
      i.e. for every $R \in PRED^2$: $F_M(R) \subseteq D_M \times D_M$.
   d. $F_M(\neg)$: $\{0,1\} \rightarrow \{0,1\}$
      $F_M(\neg) = \begin{pmatrix} 0 \rightarrow 1 \\ 1 \rightarrow 0 \end{pmatrix}$
      $F_M(\neg)$ is a **one-place truth function**: a function from truth values to truth values.

e. $F_M(\wedge)$: $\{0,1\} \times \{0,1\} \to \{0,1\}$

$$F_M(\wedge) = \begin{pmatrix} <1,1> \to 1 \\ <1,0> \to 0 \\ <0,1> \to 0 \\ <0,0> \to 0 \end{pmatrix}$$

f. $F_M(\vee)$: $\{0,1\} \times \{0,1\} \to \{0,1\}$

$$F_M(\vee) = \begin{pmatrix} <1,1> \to 1 \\ <1,0> \to 1 \\ <0,1> \to 1 \\ <0,0> \to 0 \end{pmatrix}$$

g. $F_M(\to)$: $\{0,1\} \times \{0,1\} \to \{0,1\}$

$$F_M(\to) = \begin{pmatrix} <1,1> \to 1 \\ <1,0> \to 0 \\ <0,1> \to 1 \\ <0,0> \to 1 \end{pmatrix}$$

$F_M(\wedge)$, $F_M(\vee)$ and $F_M(\to)$ are **two-place truth function.**

2. Recursive semantics for L_1.

We define **for every expression** α of L_1, $[\![\alpha]\!]_M$, **the interpretation of α in model M**:

1. If $\alpha \in$ LEX, then $[\![\alpha]\!]_M = F_M(\alpha)$

2. If $P \in PRED^1$ and $\alpha \in$ NAME then:
   $[\![P(\alpha)]\!]_M = 1$ iff $[\![\alpha]\!]_M \in [\![P]\!]_M$; 0 otherwise.

3. if $R \in PRED^2$ and $\alpha,\beta \in$ NAME then:
   $[\![R(\alpha,\beta)]\!]_M = 1$ iff $<[\![\alpha]\!]_M, [\![\beta]\!]_M> \in [\![R]\!]_M$; 0 otherwise.

4. If $\varphi \in$ FORM then:
   $[\![\neg\varphi]\!]_M = [\![\neg]\!]_M ([\![\varphi]\!]_M)$
5. If $\varphi,\psi \in$ FORM then:
   $[\![(\varphi \wedge \psi)]\!]_M = [\![\wedge]\!]_M (<[\![\varphi]\!]_M, [\![\psi]\!]_M>)$
6. If $\varphi,\psi \in$ FORM then:
   $[\![(\varphi \vee \psi)]\!]_M = [\![\vee]\!]_M (<[\![\varphi]\!]_M, [\![\psi]\!]_M>)$
7. If $\varphi,\psi \in$ FORM then:
   $[\![(\varphi \to \psi)]\!]_M = [\![\to]\!]_M (<[\![\varphi]\!]_M, [\![\psi]\!]_M>)$

COMPOSITIONALITY AND SEMANTIC GLUE.

If you're interested in the lexical meanings of predicates and relations, the semantics of L_1 is disappointing. The semantics for L_1 has nothing interesting to say about that.

Let us assume that you already know how naming works and what the meanings of the predicates and relations in L_1.
So, you're a grown-up person, so you know what kissing is: you know how to distinguish situations where it is kissing from situations where it is not.
And you know that KISS means when it **is**.

What else do you need to know in order to know the semantics of L_1?

Two things:
1. The meaning of the semantic glue.
2. The meanings of the connectives $\neg, \wedge, \vee, \rightarrow$.

The meaning of the semantic glue is the most universal bit.
Remember, compositionality says:

$$
\begin{aligned}
&[\![P(\alpha)]\!]_M &&= \text{OPERATION}_1\ [\ [\![P]\!]_M, [\![\alpha]\!]_M\ ]\\
&[\![R(\alpha,\beta)]\!]_M &&= \text{OPERATION}_2\ [\ [\![R]\!]_M, [\![\alpha]\!]_M, [\![\beta]\!]_M\ ]\\
&[\![\neg\varphi]\!]_M &&= \text{OPERATION}_3\ [\ [\![\neg]\!]_M, [\![\varphi]\!]_M\ ]\\
&[\![(\varphi \wedge \psi)]\!]_M &&= \text{OPERATION}_4\ [\ [\![\wedge]\!]_M, [\![\varphi]\!]_M, [\![\psi]\!]_M\ ]
\end{aligned}
$$

In order to master the semantics of L_1, you need to know what the operations OPERATION_1... OPERATION_4 are.

The **idea** of the semantics given is that there really is only one operation which is the interpretation of the semantic glue:

> $\text{OPERATION}[\ F, A_1,...,A_n\ ] = F(A_1,...,A_n)$
> the result of applying function entity F to argument entities $A_1...A_n$

So:

> In the semantics for L_1, the semantic glue is interpreted as function-argument application.

This idea applies **directly** to $OPERATION_3$ and $OPERATION_4$:

-we interpret $\neg$ as a truth function $[\![\neg]\!]_M$: $\{0,1\} \rightarrow \{0,1\}$
and any φ as a truth value $[\![\varphi]\!]_M \in \{0,1\}$.

$[\![\neg\varphi]\!]_M$ = OPERATION[$[\![\neg]\!]_M$, $[\![\varphi]\!]_M$] =
$[\![\neg]\!]_M$ ($[\![\varphi]\!]_M$)
$[\![\neg]\!]_M$ ($[\![\varphi]\!]_M$) $\in \{0,1\}$

-we interpret $\wedge$ as a truth function $[\![\wedge]\!]_M$: $\{0,1\}\times\{0,1\} \rightarrow \{0,1\}$
and any φ and ψ as a truth values $[\![\varphi]\!]_M$, $[\![\varphi]\!]_M \in \{0,1\}$.

$[\![(\varphi \wedge \psi]\!]_M$ = OPERATION[$[\![\wedge]\!]_M$, $[\![\varphi]\!]_M$, $[\![\psi]\!]_M$] =
$[\![\wedge]\!]_M$ ($[\![\varphi]\!]_M$, $[\![\psi]\!]_M$)
$[\![\wedge]\!]_M$ ($[\![\varphi]\!]_M$, $[\![\psi]\!]_M$) $\in \{0,1\}$.

The idea applies **indirectly** to $OPERATION_1$ and $OPERATION_2$.
The first argument of the operation is not a function, but a set
(a set of individuals for $OPERATION_1$, a set of ordered pairs of individuals for $OPERATION_2$).

But we have learned that we can switch between sets and characteristic functions.
Instead of letting OPERATION operate on X, we can let OPERATION operate on ch_X:

-If $X \subseteq D_M$, then ch_X: $D_M \rightarrow \{0,1\}$
for every $d \in D_M$: $ch_X(d) = 1$ iff $d \in X$

So: $ch_{[\![P]\!]M}$: $D_M \rightarrow \{0,1\}$
for every $d \in D_M$: $ch_{[\![P]\!]}(d) = 1$ iff $d \in [\![P]\!]_M$

-If $Y \subseteq D_M \times D_M$, then ch_Y: $D_M \times D_M \rightarrow \{0,1\}$
for every $<d_1,d_2> \in D_M \times D_M$: $ch_Y(<d_1,d_2<) = 1$ iff $<d_1,d_2> \in Y$

So: $ch_{[\![R]\!]M}$: $D_M \times D_M \rightarrow \{0,1\}$
for every $<d_1,d_2> \in D_M \times D_M$: $ch_{[\![R]\!]}(<d_1,d_2>) = 1$ iff $<d_1,d_2> \in [\![R]\!]_M$

Now we can assume that $OPERATION_1$ and $OPERATION_2$ are **the very same** operation OPERATION of functional application:

$[\![P(\alpha)]\!]_M$ = OPERATION [$ch_{[\![P]\!]M}$, $[\![\alpha]\!]_M$] =
$ch_{[\![P]\!]M}$ ($[\![\alpha]\!]_M$)
$ch_{[\![P]\!]M}$ ($[\![\alpha]\!]_M$) $\in \{0,1\}$

This specifies **exactly** what we specified in the semantics for L_1:

$[\![P(\alpha)]\!]_M = 1$ iff $[\![\alpha]\!]_M \in [\![P]\!]_M$; 0 otherwise.

$$[\![R(\alpha,\beta)]\!]_M = \text{OPERATION}\ [\ ch_{[\![R]\!]M},\ [\![\alpha]\!]_M,\ [\![\beta]\!]_M\ ] =$$
$$ch_{[\![R]\!]M}\ (\ [\![\alpha]\!]_M,\ [\![\beta]\!]_M\ )$$
$$ch_{[\![R]\!]M}\ (\ [\![\alpha]\!]_M,\ [\![\beta]\!]_M\ ) \in \{0,1\}$$

This specifies **exactly** what we specified in the semantics for L_1:

$$[\![R(\alpha,\beta)]\!]_M = 1 \text{ iff } <[\![\alpha]\!]_M,\ [\![\beta]\!]_M> \in [\![R]\!]_M;\ 0 \text{ otherwise.}$$

Thus, the first thing we need to know to master the semantics of L_1 is the interpretation of the semantic glue:

The semantic glue in L_1 is function-argument application.

Function-argument application is one of the basic operations for building meanings.
Later in this class, we will see (one instance of) a second basic operation for building meanings: **functional abstraction. General functional abstraction,** and also other operations, like **function composition** and **type shifting operations** we will not discuss in this class: they are discussed in Advanced Semantics.

So, if you have learned the meanings of the lexical items of L_1 (including those of the connectives), and, say, function-argument application is a universal cognitive capacity, then **the only thing** you need to learn to master the semantics of L_1 is
the **syntax-semantics map**:

How to properly divide a complex expression into an expression denoting a funcion, and expressions denoting its arguments.

Arguably, this is eminently learnable: natural languages provide ample clues for this, in L_1 it is by and large written into the notation of the language.

This means that, we can prove for L_1 that if the meanings of the lexical items are learnable (and why shouldn't they), the semantics of the whole language is learnable.

The second thing we need to know is what the semantics of L_1 is **really** a theory about: the meanings of the connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$.
Really the only interesting **predictions** of the semantics given for L_1 concern the interrelations between those meanings:

3. Entailment for L_1

Let $\varphi, \psi \in$ FORM, $\Delta \subseteq$ FORM

We write $\varphi \Rightarrow \psi$ for φ **entails** ψ:

$\boldsymbol{\varphi \Rightarrow \psi}$ iff for every M: if $[\![\varphi]\!]_M = 1$ then $[\![\psi]\!]_M = 1$
on every model where φ is true, ψ is true as well.

$\boldsymbol{\Delta \Rightarrow \psi}$ iff for every M: if for every $\varphi \in \Delta$: $[\![\varphi]\!]_M = 1$ then $[\![\psi]\!]_M = 1$
on every model where all the premises in Δ are true, ψ is true as well.

φ and ψ are **equivalent**, $\boldsymbol{\varphi \Leftrightarrow \psi}$ iff $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$.

So:

$\varphi \Leftrightarrow \psi$ iff for every M: $[\![\varphi]\!]_M = 1$ iff $[\![\psi]\!]_M = 1$
φ and ψ are true on exactly the same models.

FACT:
For any $\varphi \in$ FORM:

$$\neg\neg\varphi \Leftrightarrow \varphi$$

Namely:
For every M:
(1) $[\![\neg\neg\varphi]\!]_M = 1$ iff

(2) $[\![\neg]\!]_M$ ($[\![\neg\varphi]\!]_M$) = 1 iff

(3) $F_M(\neg)$($[\![\neg\varphi]\!]_M$) = 1 iff

(4) $\begin{pmatrix} 1 \to 0 \\ 0 \to 1 \end{pmatrix}$ ($[\![\neg\varphi]\!]_M$) = 1 iff

(5) $[\![\neg\varphi]\!]_M = 0$ iff

(6) $[\![\neg]\!]_M$ ($[\![\varphi]\!]_M$) = 0 iff

(7) $F_M(\neg)$($[\![\varphi]\!]_M$) = 0 iff

(8) $\begin{pmatrix} 1 \to 0 \\ 0 \to 1 \end{pmatrix}$ ($[\![\varphi]\!]_M$) = 0 iff

(9) $[\![\varphi]\!]_M = 1$

FACT:
Let $\varphi, \psi \in$ FORM:

$$\{ (\varphi \vee \psi), \neg\varphi \} \Rightarrow \psi$$

Namely:(1) Assume $[\![(\varphi \vee \psi)]\!]_M = 1$ and $[\![\neg\varphi]\!]_M = 1$.

(2) Then $[\![\vee]\!]_M (< [\![\varphi]\!]_M, [\![\psi]\!]_M >) = 1$ and $[\![\neg]\!] ([\![\varphi]\!]_M) = 1$.

(3) Then $F_M(\vee) (< [\![\varphi]\!]_M, [\![\psi]\!]_M >) = 1$ and $F_M(\neg) ([\![\varphi]\!]_M) = 1$.

(4) Then $\begin{pmatrix} <1,1> \rightarrow 1 \\ <1,0> \rightarrow 1 \\ <0,1> \rightarrow 1 \\ <0,0> \rightarrow 0 \end{pmatrix} (< [\![\varphi]\!]_M, [\![\psi]\!]_M >) = 1$ and $\begin{pmatrix} 1 \rightarrow 0 \\ 0 \rightarrow 1 \end{pmatrix} ([\![\varphi]\!]_M) = 1$.

(5) Then $[\![\varphi]\!]_M = 0$ and one of the following three holds:

a. $[\![\varphi]\!]_M = 1$ and $[\![\psi]\!]_M = 1$

b. $[\![\varphi]\!]_M = 1$ and $[\![\psi]\!]_M = 0$

bc. $[\![\varphi]\!]_M = 0$ and $[\![\psi]\!]_M = 1$

(6) Then, since, the (a) and the (b) cases are impossible, the (c) case holds, so:
$[\![\varphi]\!]_M = 0$ and $[\![\psi]\!]_M = 1$.

(7) Then $[\![\psi]\!]_M = 1$.

Other facts:

$(\varphi \wedge \psi) \Leftrightarrow \neg(\neg\varphi \vee \neg\psi)$ De Morgan Laws
$(\varphi \vee \psi) \Leftrightarrow \neg(\neg\varphi \wedge \neg\psi)$

$\neg(\varphi \rightarrow \psi) \Leftrightarrow (\varphi \wedge \neg\psi)$ Problematic. We discuss this later.

$(\varphi \rightarrow \psi) \Leftrightarrow (\neg\varphi \vee \psi)$

Notice discourse anaphora:

(1) a. If this house has *a study*, ***it*** is in a strange place. $\varphi \rightarrow \psi$

b. Either this house has *no study*, or ***it*** is in a strange place. $\neg\varphi \vee \psi$

cf.

(2) a. ??The house has *a bathroom*, or ***it*** is in a strange place. $\varphi \vee \psi$

b. ?? If the house doesn't have *a bathroom*, ***it*** is in a strange place. $\neg\varphi \rightarrow \psi$

This fits with the equivalence.

Notice a difficult case for theories of discourse anaphora:

(3) If he doesn't have *a car*, he can't give me a ride in ***it.*** $\neg\varphi \rightarrow \neg\psi$

III. QUANTIFIERS AND VARIABLES (Frege)

(1) a. Sasha purrs.
b. PURR(s)

$[\![\text{PURR(s)}]\!]_M = 1$ iff $F_M(s) \in F_M(\text{PURR})$

(2) a. Everybody purrs.
b. PURR(everybody)
(3) a. Somebody purrs.
b. PURR(somebody)
(4) a. Nobody purrs.
b. PURR(nobody).

$[\![\text{PURR}(\alpha)]\!]_M = 1$ iff $F_M(\alpha) \in F_M(\text{PURR})$
So: F_M(everybody), F_M(somebody), F_M(nobody) $\in D_M$

Problem 1: F_M(nobody) $\in D_M$?

Alice: I saw nobody on the road.
The white king: I wish I had your eyes.

Problem 2: No predictions about entailment patterns:

I.	Every cat purrs.	PURR(every cat)
	Sasha is a cat.	CAT(s)
entails	Sasha purrs.	PURR(s)

II.	No cat purrs.	PURR(no cat)
	Sasha is a cat.	CAT(s)
entails	Sasha doesn't purr.	$\neg$ PURR(s)

III.	Some dog chases every cat.	CHASE(some dog,every cat)
	Sasha is a cat.	CAT(s)
entails	Some dog chases Sasha.	CHASE(some dog,s)

Problem 3: Wrong predictions about entailment patterns.

Let us use for clarity a non-vague predicate like *be completely red*, CR ∈ PRED[1].

$F_M(CR) \cup (D_M - F_M(CR)) = D_M$
$F_M(CR) \cap (D_M - F_M(CR) = \emptyset$

Hence:
for every model M and every $\alpha \in$ NAME: $[\![CR(\alpha) \vee \neg CR(\alpha)]\!]_M = 1$
$CR(\alpha) \vee \neg CR(\alpha)$ is a **tautology.**
for every model M and every $\alpha \in$ NAME: $[\![CR(\alpha) \wedge \neg CR(\alpha)]\!]_M = 0$
$CR(\alpha) \wedge \neg CR(\alpha)$ is a **contradiction.**
Ok for names:

(5) a. Sasha is completely red or Sasha isn't completey red. Tautology
b. Sasha is completely red and Sasha isn't completely red. Contradiction

But not for the others:

(6) Every cat is completely red or every girl isn't completely red. No tautology
(7) Some cat is completely red and some cat isn't completely red. No contradiction

Problem: (6) is **predicted** to be a tautology, (7) is **predicted** to be a contradiction.

Aristotle: partial account of the entailment problem:
Stipulation of a set of entailment rules (syllogisms).
Every man is mortal
Socrates is a man
hence, Socrates is mortal

Problems:
-Rules are stipulated, not explained by the meanings of the expressions involved.
-Only for noun phrases in subject position: 2000 years of logic failed to come up with a satisfactory set of rules for entailments like those in (III) or the following:

(8) a. Some boy gave every girl her favorite flower
b. Mary is a girl and her favorite flower is a Lily.
c. Some boy gaves Mary a Lily.

All these problems were solved once and for all in 1879 in Gottlob Frege's *Begriffschrift*.

Frege's solution: quantifiers and variables.
Frege: Do not analyse *Everybody purrs* as PURR(everybody), but analyse *Everybody purrs* in two stages:

STAGE 1: Replace *everybody* in *Everybody purrs* by a **pronoun**: *(s)he*:
(s)he purrs PURR(x)
This is a sentence whose truth value depends on what you are pointing at.

STAGE 2: Let *everybody* express a **constraint** on what you are pointing at:
For *every* pointing with *(s)he*: *(s)he purrs* $\forall x[PURR(x)]$

Note: this is not Frege's notation, and while Frege gave the idea of the semantics intuitively, he didn't give the semantics: he gave a set of inference rules **fitting** this semantics.

Everybody purrs.
For *every* pointing with *(s)he*: *(s)he* purrs
$\forall x[PURR(x)]$

Somebody purrs.
For *some* pointing with *(s)he*: *(s)he* purrs
$\exists x[PURR(x)]$

Nobody purrs.
For *no* pointing with *(s)he*: *(s)he* purrs
$\neg\exists x[PURR(x)]$

Every cat purrs.
For *every* pointing with *(s)he*: if *(s)he* is a cat, then *(s)he* purrs
$\forall x[CAT(x) \rightarrow PURR(x)]$

Some cat purrs.
For *some* pointing with *(s)he*: *(s)he* is a cat and *(s)he* purrs.
$\exists x[CAT(x) \wedge PURR(x)]$

No cat purrs.
For *no* pointing with *(s)he*: *(s)he* is a cat and *(s)he* purrs.
$\neg\exists x[CAT(x) \wedge PURR(x)]$

-Frege's inference rules for these expressions predict the entailments in I and II.

I $\forall x[CAT(x) \rightarrow PURR(x)]$
$CAT(s)$
entails $PURR(s)$

II $\neg\exists x[CAT(x) \wedge PURR(x)]$
$CAT(s)$
entails $\neg PURR(s)$

-Frege's solves the problem of tautologies and contradictions:

(6) Every cat is completelyt red or every cat is not completely red.
(s)he is completely red or *(s)he* is not completely red
CR(x) or ¬CR(x).

for every pointing with *(s)he* to a cat: *(s)he* is completely red or
for every pointing with *(s)he* to a cat: *(s)he* isn't complety red
$\forall x[CAT(x) \rightarrow CR(x) \vee \forall x[CAT(x) \rightarrow \neg CRx)]$.

The trick is to analyse *every cat* in *every cat is not completely red* **after** *isn't* the same for *some catl* in *some cat isn't completely red*:

$\forall x[CAT(x) \rightarrow CR(x)] \vee \forall x[CAT(x) \rightarrow \neg CR(x)]$ No tautology.

(7) Some cat is completely red and some cat isn't completely red.

$\exists x[CAT(x) \wedge CR(x)] \wedge \exists x[CAT(x) \wedge \neg SCR(x)]$ No contradiction.

-Frege solves the problem of entailments for noun phrases not in subject position.
Frege's solution: apply **the same analysis** in stages: (I make the cats feminine and the dogs masculin for readability):

Some dog chases every cat.
Stage 1a. Replace ***every cat*** in this by a pronoun ***she (her)***:
Some dog chases ***her***. Truth value depends on pointings for *she*
Some dog chases y

Stage 1b: *every cat* constrains pointings for *she*:
For *every* pointing with *she*: if *she* is a cat, some dog chases *her*
$\forall y[CAT(y) \rightarrow$ some dog chases y]

Stage 2a. Now replace *some dog* by a pronoun *he*:
For *every* pointing with *she*: if *she* is a cat, ***he*** chases *her*
Truth value depends on pointings for *he*
$\forall y[CAT(y) \rightarrow CHASE(x,y)]$

Stage 2b. *some dog* constrains pointings for *he*:
For *some* pointing with *he*: *he* is a dog and for *every* pointing with *she*:
if *she* is a cat, *he* chases *her*.
$\exists x[DOG(x) \wedge \forall y[CAT(y) \rightarrow CHASE(x,y)]]$

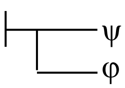
-With this analysis, Frege doesn't have to stipulate anything separate for entailments for sentences with quantifiers not in subject position: the same inference rules predict the entailment pattern in III:

III $\exists x[DOG(x) \wedge \forall y[CAT(y) \rightarrow CHASE(x,y)]]$
$CAT(s)$
entails $\exists x[DOG(x) \wedge CHASE(x,s)]$

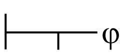
After 2000 years of failure, this is **very** impressive!

Afred Tarski developed the semantics for Frege's analysis in *The Concept of Truth in Formalized Languages*, first publised in Polish in 1932. He did this by precisely specifying the notions of 'truth relative to a pointing for pronoun *(s)he*'
and the notion of quantifiers as 'constraints on pointings for pronoun *(s)he*.'
Frege told us what the meanings of quantifiers and variables **do**.
Tarski told us, given that, what the meanings of quantifiers and variables **are**.

Frege's notation:



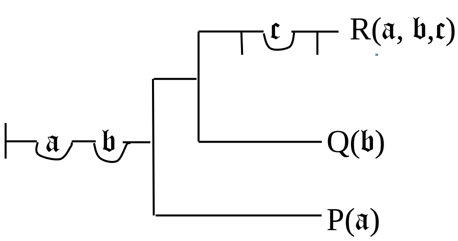
$(\varphi \rightarrow \psi)$



$\neg\varphi$

$\forall xP(x)$

Our linear notation:
$\forall x\forall y((P(x) \rightarrow (Q(y) \rightarrow \neg\forall z\neg R(x,y,z)))$
becomes two-dimensional:



And we would need definitions of connectives corresponding to $\wedge$, $\vee$ and $\exists$ to write the equivalent and more legible:

$\forall x\forall y((P(x) \wedge Q(y)) \rightarrow \exists zR(x,y,z))$

Clearly bookprinters preferred the linear notation (which stems by and large from Peano around the beginning of the 20^{th} century).

IV. L_2, A LANGUAGE WITH VARIABLES

Syntax of L_2

1. Lexicon of L_1

NAME = {SASHA,SHUNRA, FIDO,...}

VAR = $\{x_1,x_2,...,x,y,z\}$ An infinite set of variables ('pronouns')

$PRED^1$ = {PURR, MEOUW, CAT, DOG,...}

$PRED^2$ = {CHASE, HUG,...}

NEG = $\{\neg\}$

CONN = $\{\wedge,\vee,\rightarrow\}$

LEX = NAME $\cup$ $PRED^1$ $\cup$ $PRED^2$ $\cup$ NEG $\cup$ CONN

TERM = NAME $\cup$ VAR Terms are names or variables.

2. Sentences of L_2

FORM, the set of all formulas of L_2 is the smallest set such that:

1. If $P \in PRED^1$ and $\alpha \in$ TERM, then $P(\alpha) \in$ FORM.
2. If $R \in PRED^2$ and $\alpha,\beta \in$ TERM, then $R(\alpha,\beta) \in$ FORM.
3. If $\varphi \in$ FORM, then $\neg\varphi \in$ FORM.
4. If $\varphi,\psi \in$ FORM, then $(\varphi \wedge \psi) \in$ FORM.
5. If $\varphi,\psi \in$ FORM, then $(\varphi \vee \psi) \in$ FORM.
6. If $\varphi,\psi \in$ FORM, then $(\varphi \rightarrow \psi) \in$ FORM.

Semantics for L_2

1. Models for L_2

A **Model for** L_2 is a pair $M = \langle D_M, F_M \rangle$, where:

1. D_M, the domain of M, is a (non-empty) set.
2. F_M, the interpretation function for the lexical items of L_2, is given by:

a. F_M: NAME $\rightarrow D_M$

i.e. for every $\alpha \in$ NAME: $F_M(\alpha) \in D_M$.

b. F_M: $PRED^1 \rightarrow$ **pow**(D_M)

i.e. for every $P \in PRED^1$: $F_M(P) \subseteq D_M$.

c. F_M: $PRED^2 \rightarrow$ **pow**$(D_M \times D_M)$

i.e. for every $R \in PRED^2$: $F_M(R) \subseteq D_M \times D_M$.

pow(A) = $\{B: B \subseteq A\}$

$\{a,b\} = \{ \emptyset, \{a\}, \{b\} \{a,b\}\}$

$\{a,b\} \times \{a,b\} = \{\langle a,a\rangle, \langle b,b\rangle, \langle a,b\rangle, \langle b,a\rangle\}$

2. Variable assignments.

Variables are not yet interpreted. We introduce pointing devices and call them **variable assignments**:

A **variable assignment** for L_2 on M is a function g: VAR $\rightarrow D_M$, a function from variables to individuals.

i.e. for every $x \in$ VAR: $g(x) \in D_M$.

3. Recursive semantics for L_2.
We define for every expression α of L_2, $[\![\alpha]\!]_{M,g}$,
the interpretation of α in model M, relative to variable assignment g:

1_a. If $\alpha \in$ LEX, then $[\![\alpha]\!]_{M,g} = F_M(\alpha)$
1_b. If $\alpha \in$ VAR, then $[\![\alpha]\!]_{M,g} = g(\alpha)$
2. If $P \in PRED^1$ and $\alpha \in$ TERM then:
$[\![P(\alpha)]\!]_{M,g} = 1$ iff $[\![\alpha]\!]_{M,g} \in [\![P]\!]_{M,g}$; 0 otherwise.
3. If $R \in PRED^2$ and $\alpha,\beta \in$ TERM then:
$[\![R(\alpha,\beta)]\!]_{M,g} = 1$ iff $<[\![\alpha]\!]_{M,g}, [\![\beta]\!]_{M,g}> \in [\![R]\!]_{M,g}$; 0 otherwise.
4. If $\varphi \in$ FORM then:
$[\![\neg\varphi]\!]_{M,g} = 1$ iff $[\![\varphi]\!]_{M,g} = 0$; 0 otherwise
5. If $\varphi,\psi \in$ FORM then:
$[\![(\varphi \wedge \psi)]\!]_{M,g} = 1$ iff $[\![\varphi]\!]_{M,g} = [\![\psi]\!]_{M,g} = 1$; 0 otherwise
6. If $\varphi,\psi \in$ FORM then:
$[\![(\varphi \vee \psi)]\!]_{M,g} = 1$ off $[\![\varphi]\!]_{M,g} = 1$ or $[\![\psi]\!]_{M,g} = 1$; 0 otherwise
7. If $\varphi,\psi \in$ FORM then:
$[\![(\varphi \rightarrow \psi)]\!]_{M,g} = 1$ iiff $[\![\varphi]\!]_{M,g} = 0$ or $[\![\psi]\!]_{M,g} = 1$; 0 otherwise

4. Truth for L_2. (Independent of assignments)
We define, for formulas of L_2, in terms of the recursively defined notion of 'interpretation in M relative to g' ($[\![\ ]\!]_{M,g}$), the notions of 'truth in M' ($[\![\ ]\!]_M = 1$) and 'falsity in M' ($[\![\ ]\!]_M = 0$).
So $[\![\ ]\!]_M$ is defined in terms of $[\![\ ]\!]_{M,g}$

Let $\varphi \in$ FORM:
$[\![\varphi]\!]_M = 1$ iff for every assignment g for L_2: $[\![\varphi]\!]_{M,g} = 1$
$[\![\varphi]\!]_M = 0$ iff for every assignment g for L_2: $[\![\varphi]\!]_{M,g} = 0$

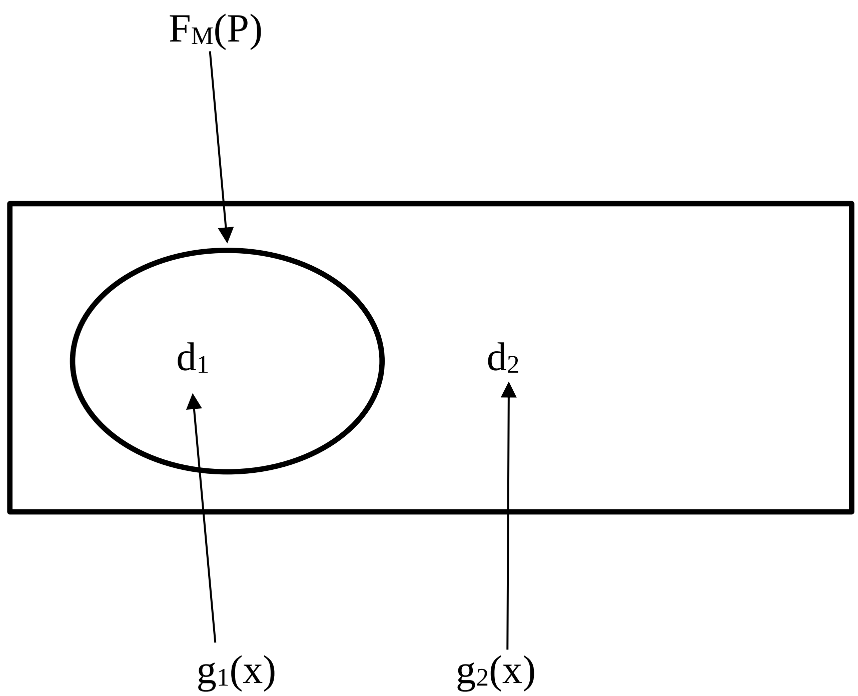
3. Entailment for L_2: Defined in terms of $[\![\ ]\!]_M$.
Let $\varphi, \psi \in$ FORM
φ **entails** ψ, $\varphi \Rightarrow \psi$ iff for every model M for L_2: if $[\![\varphi]\!]_M = 1$ then $[\![\psi]\!]_M = 1$

For formulas without variables we have:

FACT: if φ is a formula without variables, then:
for every model M: either $[\![\varphi]\!]_M=1$ or $[\![\varphi]\!]_M = 0$

Formulas with variables are often neither true, nor false on a model (but undefined), because their truth varies with assignment functions.

Example:
Let $F_M(P) \subseteq D_M$, $d_1, d_2 \in D_M$ and $d_1 \in F_M(P)$, $d_2 \notin F_M(P)$.
Let $g_1(x)=d_1$, $g_2(x)=d_2$.



Then: $[\![P(x)]\!]_M \neq 1$, because $[\![P(x)]\!]_{M,g_2} = 0$
$[\![P(x)]\!]_M \neq 0$, because $[\![P(x)]\!]_{M,g_1} = 1$

Not all formulas with variables come out as undefined, though:

$[\![P(x) \vee \neg P(x)]\!]_M = 1$ iff
for every g: $[\![P(x) \vee \neg P(x)]\!]_{M,g} = 1$ iff

for every g: $g(x) \in F_M(P)$ or $g(x) \notin F_M(P)$ iff

for every g: $g(x) \in F_M(P)$ or $g(x) \in D_M - F_M(P)$ iff

for every g: $g(x) \in D_M$. Which is true.

So: $[\![P(x) \vee \neg P(x)]\!]_M = 1$

Note:

$[\![P(x) \vee \neg P(x)]\!]_{M,g_1} = 1$ because $[\![P(x)]\!]_{M,g_1} = 1$
$[\![P(x) \vee \neg P(x)]\!]_{M,g_2} = 1$ because $[\![\neg P(x)]\!]_{M,g_2} = 1$

Similarly:
$[\![P(x) \wedge \neg P(x)[\![_M = 0$ iff
for no g: $g(x) \in F_M(P)$ and $g(x) \in D_M - F_M(P)$
So: $[\![P(x) \wedge \neg P(x)]\!]_M = 0$

Hence, $P(x) \vee \neg P(x)$ is a tautology, and $P(x) \wedge \neg P(x)$ is a contradiction.
Later we will follow the logical tradition in defining entailment only for formulas whose truth doesn't vary with assignments (formulas without free occurrences of variables). But it is important to note that the **technique** applies correctly to formulas with free variables as well.

The **technique** of defining truth in M as truth relative to all variation parameters, and falsity as falsity relative to all variation parameters plays an important role in semantics (for instance in the analysis of vagueness). It is called the technique of **super valuations**. (van Fraasen)

Excursus: Vagueness as a problem for many-valued logic (Kamp)

We assume that our domain consists only of humans (for simplicity)

(1) Bob is male
(2) Bob is a typical adolescent, borderline between grown-up and not-grown up
(3) A grown-up male is a man
A non-grown-up male is a boy

$[\![\mathit{male}(\text{bob})]\!]_{M,g} = 1$
$[\![\mathit{grown\text{-}up}(\text{bob})]\!]_{M,g} = \perp$ and $[\![\neg\mathit{grown\text{-}up}(\text{bob})]\!]_{M,g} = \perp$
(undefined, or any intermediate value between 1 and 0, this is allowed in many-valued logic)

$[\![\mathit{man}(\text{bob})]\!]_{M,g} = 1$ iff $[\![\mathit{male}(\text{bob})]\!]_{M,g} = 1$ and $[\![\mathit{grown\text{-}up}(\text{bob})]\!]_{M,g} = 1$

$[\![\mathit{boy}(\text{bob})]\!]_{M,g} = 1$ iff $[\![\mathit{male}(\text{bob})]\!]_{M,g} = 1$ and $[\![\mathit{grown\text{-}up}(\text{bob})]\!]_{M,g} = 0$

So:

$[\![\mathit{man}(\text{bob})]\!]_{M,g} = \perp$
$[\![\mathit{boy}(\text{bob})\ ]\!]_{M,g} = \perp$

Kamp's problem of conditionals:
Intuitively (1a) and (1b) are true:

(1) a. ***If*** Bob is grown-up, he is a man.
b. ***If*** Bob is not grown-up, he is boy.

We show the problem with material conditional, though the problem can be reconstructed for other analyses of the conditional as well.

[1] *grown-up*(bob) → *man*(bob)

working out the definition of *man*, this is equivalent to:

[2] *grown-up*(bob) → (*male*(bob) ∧ *grown-up*(bob))

using the truth-table for the material implication ($\varphi \rightarrow \psi$ is equivalent to $\neg\varphi \vee \psi$), this is equivalent to:

[3] ¬*grown-up*(bob) ∨ (*male*(bob) ∧ *grown-up*(bob))

using the distributive law ($(\varphi \vee (\psi \wedge \chi))$ is equivalent to $((\varphi \vee \psi) \wedge (\varphi \vee \chi))$), this is equivalent to:

[4] (¬*grown-up*(bob) ∨ *male*(bob)) ∧ (¬*grown-up*(bob) ∨ *grown-up*(bob))

The first conjunct: ¬*grown-up*(bob) ∨ *male*(bob)
$[\![\neg \textit{grown-up}(\text{bob})]\!]_{M,g} = \bot$ and $[\![\textit{male}(\text{bob})]\!]_{M,g} = 1$
In the strong-systems of many-valued logic, this means that the disjunction is true (one true disjunct is enough). So, let's assume that: the first conjunct is true.

We want [4] itself to come out as true. By the same reasoning as for disjunction, we will need **both conjuncts** in [4] to come out as true: in many-valued logic, if one of the conjuncts has a value less than 1, the conjunction will itself have a value less than 1, which is not good enough for us, because, with Kamp, we want (1a) to come out as true. So we need the second conjunct to come out as true.

The second conjunct: ¬*grown-up*(bob) ∨ *grown-up*(bob)
We are now back to the problem of tautologies. The value of *grown-up*(bob) is intermediate between 0 and 1 (in three-valued logic ⊥, but the problem is the same for theories with more values, like fuzzy logic). This means by necessity in many-valued logic that the value of ¬*grown-up*(bob) is less than 1 (namely, depending on your definition of ¬, 0 or intermediate).

But in many-valued logic, the disjunction of two values that are less than one is never 1. This means that the tautology does not come out as 1, and with that (1a) does not come out as true. So (1a) and (1b) do not come out as true.

Sketch of the solution (Kamp, Fine):

-Truth of sentences with vague predicates depends on a contextual *standard of precision* and the ways in which this standard of precision can be *refined.*

-Three-valued logic: Bob is a borderline case of a man (boy), because our standard of precision doesn't count him among the grown-ups, nor among the non-grown-ups.

-Supervaluations: This means that *some* refinements of our standard of precision make him a grown-up, and *some* refinements of our standard of precision make him a non-grown-up. A ***completion*** of s is a refinement of s that makes all the predicates involved completely precise.

-Truth is 'super-truth': if s is our standard of precision, we define:

φ is supertrue in s iff every completion of s makes φ true

φ is superfalse in s iff every completion of s makes φ false

Results:

male(bob) is supertrue	it is already true in s
grown-up(bob) is not supertrue	some completion makes *grown-up*(bob) false
¬*grown-up*(bob) is not supertrue	some completion makes *grown-up*(bob) true

but: *grown-up*(bob) ∨ ¬*grown-up*(bob) is supertrue,
because tautologies are true in all completions

and: *grown-up*(bob) → *man*(bob) is supertrue
because in every completion where bob is counted among the *grown-ups* he is a grown-up male, hence a man.

Notice the parallel with the definition of truth/falsity as truth/falsity relative to all assignments.

WH(x) *that one wears a hat*

⟦WH(x)⟧$_{M,g}$ = 1 iff g(x) ∈ F_M(WH)

V. L_3, A LANGUAGE WITH QUANTIFIERS AND VARIABLES

Syntax of L_3:

L_3 has the same syntax as L_2, except that we **add** two more clauses to the definition of FORM:

7. If x ∈ VAR and φ ∈ FORM, then ∀xφ ∈ FORM
8. If x ∈ VAR and φ ∈ FORM, then ∃xφ ∈ FORM

Semantic for L_3:

The notion of model for L_3 and variable assignment for L_3 on a model are the same as for L_2.

Note on compositionality:
I introduce the symbols ∀ and ∃ in the formula definition and not in the lexicon (such symbols are called **syncategorematic,** meaning, not part of a lexical category).
Similarly, I will specify the truth conditions of sentences with these symbols, but not give an explicit interpretation for them, i.e. their interpretation will be specified **implicitly.**
This is **solely** for your convenience. Just as in L_2 I defined explicitly F_M(¬) as a function, I **can** explicitly define F_M(∀) and F_M(∃) as functions.
But doing this is technically more involved.
The reason is that, whereas the operations introduced so far (like ¬, ∧, ∨) are **extensional with respect to assignment functions** (meaning that the interpretation of a complex in M relative to g, depends on the interpretations of the parts in M relative to **that same** g), the quantifiers are **intensional with respect to assignment functions** (meaning that the interpretation of a **quantificational** complex in M relative to g, depends on the interpretations of the parts in M relative to **other assignments** g').
And this means that if we want to introduce the interpretations of quantifiers explicitly, we need to introduce for their interpretations complex functions from sets of assignment functions to sets of assignment functions.
Since this is too technical at this point of the exposition, we explain for your convenience what a quantifier **does** in the theory, rather than what a quantifier **is** in the theory.
Importantly: this doesn't mean that the semantics for L_3 given is not compositional; it only means that for your convenience I won't work out all compositional details.

Resetting values of variables.
Let g be a variable assignment for L_3 on M, g: VAR → D_M
We define: g_x^d, the result of **resetting** the value of variable x in assignment g to object d.

$\mathbf{g_x^d}$ = the assignment function such that:
1. for every y ∈ VAR−{x}: $g_x^d(y) = g(y)$
2. $g_x^d(x) = d$

i.e. g_x^d assigns to all variables except for x **the same value** as g assigns, but it assigns to variable x object d, it **varies** the value for variable x.

Example:

$$g = \begin{pmatrix} x_1 \to d_1 \\ x_2 \to d_2 \\ x_3 \to d_1 \\ x_4 \to d_2 \\ \ldots \end{pmatrix} \quad g_{x_2}^{d_1} = \begin{pmatrix} x_1 \to d_1 \\ \mathbf{x_2 \to d_1} \\ x_3 \to d_1 \\ x_4 \to d_2 \\ \ldots \end{pmatrix} \quad g_{x_2\,x_1}^{d_1\,d_2} = \begin{pmatrix} \mathbf{x_1 \to d_2} \\ x_2 \to d_1 \\ x_3 \to d_1 \\ x_4 \to d_2 \\ \ldots \end{pmatrix}$$

$$g_{x_2\,x_1\,x_2}^{d_1\,d_2\,d_2} = \begin{pmatrix} x_1 \to d_2 \\ \mathbf{x_2 \to d_2} \\ x_3 \to d_1 \\ x_4 \to d_2 \\ \ldots \end{pmatrix} \quad g_{x_2\,x_1\,x_2\,x_1}^{d_1\,d_2\,d_2\,d_1} = \begin{pmatrix} \mathbf{x_1 \to d_1} \\ x_2 \to d_2 \\ x_3 \to d_1 \\ x_4 \to d_2 \\ \ldots \end{pmatrix}$$

Compositional semantics for $\forall x[P(x)]$.
The truth value $[\![P(x)]\!]_{M,g}$ is not enough to define compositionally the truth value $[\![\forall xP(x)]\!]_{M,g}$.
What you need is not the **extension** of P(x) in M relative to g, the truth value in M relative to g, but **the pattern of variation** of the extension, the truth value, of P(x) in M, when you vary the value of x.

Given D_M and $g(x)=d_1$.
The pattern of variation of the value of x over domain D_M is the list:

$$\begin{pmatrix} g_x^{d_1}: & x \to d_1 \\ g_x^{d_2}: & x \to d_2 \\ g_x^{d_3}: & x \to d_3 \\ \ldots & \text{for all } d \in D_M. \end{pmatrix}$$

The pattern of variation of the truth value of P(x) over domain D_M is the list:

$$\begin{pmatrix} < g_x^{d_1}:\ [\![P(x)]\!]_{M,g_x^{d_1}} > \\ < g_x^{d_2}:\ [\![P(x)]\!]_{M,g_x^{d_2}} > \\ < g_x^{d_3}:\ [\![P(x)]\!]_{M,g_x^{d_3}} > \\ \ldots \text{ for all } d \in D_M. \end{pmatrix}$$

$[\![\forall xP(x)]\!]_{M,g} =$ 1 iff you get truth value 1 everywhere in the list.

Equivalently: iff for every $d \in D_M$: $[\![P(x)]\!]_{M,g_x^d} = 1$; 0 otherwise

$[\![\exists xP(x)]\!]_{M,g} =$ 1 iff you get truth value 1 somewhere in the list.

Equivalently: iff for some $d \in D_M$: $[\![P(x)]\!]_{M,g_x^d} = 1$; 0 otherwise

Moral: The meanings of expressions in predicate logic are not extensions, but these lists of assignment-extension pairs.

Explanation:

$[\![\forall x\varphi]\!]_{M,g} = 1/0?$ $[\![\exists x\varphi]\!]_{M,g} = 1/0?$

1. Form the list which varies in g the value of x through the domain:

$$\begin{array}{ccccc} [\![\varphi]\!]_{M,g_x^{d_1}} & [\![\varphi]\!]_{M,g_x^{d_2}} & [\![\varphi]\!]_{M,g_x^{d_3}} & [\![\varphi]\!]_{M,g_x^{d_4}} & \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \\ g_x^{d_1} & g_x^{d_2} & g_x^{d_3} & g_x^{d_4} & \ldots \end{array}$$

2. Add the truth value of $[\![\varphi]\!]_{M,h}$ relative to all these assignmens h:, say:

$$\begin{array}{ccccc} 1 & 1 & 1 & 0 & \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \\ [\![\varphi]\!]_{M,g_x^{d_1}} & [\![\varphi]\!]_{M,g_x^{d_2}} & [\![\varphi]\!]_{M,g_x^{d_3}} & [\![\varphi]\!]_{M,g_x^{d_4}} & \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \\ g_x^{d_1} & g_x^{d_2} & g_x^{d_3} & g_x^{d_4} & \ldots \end{array}$$

so:

$$\begin{array}{ccccc} 1 & 1 & 1 & 0 & \ldots \\ \uparrow & \uparrow & \uparrow & \uparrow & \\ g_x^{d_1} & g_x^{d_2} & g_x^{d_3} & g_x^{d_4} & \ldots \end{array} \quad PV(\varphi)$$

3. This is the relevant pattern of variation for φ. $PV(\varphi)$

The truth conditions say the following:

$[\![\forall x\varphi]\!]_{M,g} = 1$ if you only get 1's in $PV(\varphi)$
$[\![\forall x\varphi]\!]_{M,g} = 0$ if you get one or more 0 in $PV(\varphi)$

$[\![\exists x\varphi]\!]_{M,g} = 1$ if you get one or more 1 in PV(φ)
$[\![\exists x\varphi]\!]_{M,g} = 0$ if you only get 0's in PV(φ)

This means:

$[\![\forall x\varphi]\!]_{M,g} = 1$ iff for every $d \in D_M$: $[\![\varphi]\!]_{M,g_x^d} = 1$
0 iff for some $d \in D_M$: $[\![\varphi]\!]_{M,g_x^d} = 0$

$[\![\exists x\varphi]\!]_{M,g} = 1$ iff for some $d \in D_M$: $[\![\varphi]\!]_{M,g_x^d} = 1$
0 iff for everyd $\in D_M$: $[\![\varphi]\!]_{M,g_x^d} = 0$

Tarski's formalization of Frege's intuition:
A Frege-Tarski-quantifier like ∀x is a function that does **two things simultaneously:**

1. The quantifier binds all occurrences of variable x that are free in the input.
What corresponds to this semantically is: the quantifier sets up **a pattern of variation for the input**. The occurrences of the variable x are bound in this pattern of variation.
This bit is the same for all quantifiers.

2. The quantifier expresses a quantificational constraint, its particular lexical meaning.
What corresponds to this semantically is: the quantifier expresses **a constraint** on the pattern of variation for the input. (i.e. the meaning of ∀x tells you that you need to get value 1 at **every** place in the list, the meaning of ∃x that you need to get value 1 at **some** place in the list.

I will argue later that natural language semantics took off in the 1960s, when this analysis of quantification and binding was rejected for a similar, but nevertheless different analysis. But to understand that, we need to understand the Frege-Tarski analysis first.

Recursive semantics for L_3:
We define $[\![\alpha]\!]_{M,g}$ in exactly the same way as for L_2, except that we **add** two interpretation clauses:

8. If $x \in VAR$ and $\varphi \in FORM$ then:
   $[\![\forall x\varphi]\!]_{M,g} = 1$ iff for every $d \in D_M$: $[\![\varphi]\!]_{M,g_x^d} = 1$; 0 otherwise
9. If $x \in VAR$ and $\varphi \in FORM$ then:
   $[\![\exists x\varphi]\!]_{M,g} = 1$ iff for some $d \in D_M$: $[\![\varphi]\!]_{M,g_x^d} = 1$; 0 otherwise

Truth and entailment: see below

VI. L_4, FULL PREDICATE LOGIC WITH IDENTITY

Syntax of L_4

CON = $\{c_1,c_2,...\}$ The set of individual constants (= names)
For every n>0: $PRED^n = \{P^n_1,P^n_2,...\}$ The set of n-place predicates.
(For CON and each $PRED^n$ you choose which and how many elements these sets have in L_4.)
VAR = $\{x_1,x_2,...\}$ The set of variables.
(VAR contains infinitely many variables.)
NEG = $\{\neg\}$, CONN = $\{\wedge,\vee,\rightarrow\}$
LEX = CON $\cup$ $PRED^n$ $\cup$ NEG $\cup$ CONN (for each n>0)
TERM = CON $\cup$ VAR

FORM is the smallest set such that:

1. If $P \in PRED^n$ and $\alpha_1,...\alpha_n \in$ TERM, then $P(\alpha_1,...,\alpha_n) \in$ FORM
2. If $\alpha_1,\alpha_2 \in$ TERM, then $(\alpha_1=\alpha_2) \in$ FORM
3. If $\varphi,\psi \in$ FORM, then $\neg\varphi$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi) \in$ FORM
4. If $x \in$ VAR and $\varphi \in$ FORM, then $\forall x\varphi$, $\exists x\varphi \in$ FORM

Semantics for L_4.

A **model for** L_4 is a pair $M = \langle D_M,F_M \rangle$, where:

1. D_M, the domain of M, is a non-empty set.
2. F_M, the interpretation function for M, is given by:
   a. for every $c \in$ CON: $F_M(c) \in D_M$
   b. for every $P \in PRED^n$: $F_M(P) \subseteq (D_M)^n$

Here $(D_M)^1 = D_M$
$(D_M)^2 = D_M \times D_M$
$(D_M)^3 = D_M \times D_M \times D_M$
etc.

d. $F_M(\neg)$: $\{0,1\} \rightarrow \{0,1\}$

$$F_M(\neg) = \begin{bmatrix} 0 \rightarrow 1 \\ 1 \rightarrow 0 \end{bmatrix}$$

e. $F_M(\wedge)$: $\{0,1\} \times \{0,1\} \rightarrow \{0,1\}$

$$F_M(\wedge) = \begin{pmatrix} \langle 1,1\rangle \rightarrow 1 \\ \langle 1,0\rangle \rightarrow 0 \\ \langle 0,1\rangle \rightarrow 0 \\ \langle 0,0\rangle \rightarrow 0 \end{pmatrix}$$

f. $F_M(\vee)$: $\{0,1\} \times \{0,1\} \rightarrow \{0,1\}$

$$F_M(\vee) = \begin{pmatrix} \langle 1,1\rangle \rightarrow 1 \\ \langle 1,0\rangle \rightarrow 1 \\ \langle 0,1\rangle \rightarrow 1 \\ \langle 0,0\rangle \rightarrow 0 \end{pmatrix}$$

g. $F_M(\rightarrow)$: $\{0,1\} \times \{0,1\} \rightarrow \{0,1\}$

$$F_M(\rightarrow) = \begin{pmatrix} <1,1> \rightarrow 1 \\ <1,0> \rightarrow 0 \\ <0,1> \rightarrow 1 \\ <0,0> \rightarrow 1 \end{pmatrix}$$

A **variable assignment** for L_4 on M is a function g: VAR $\rightarrow D_M$

Let g be a variable assignment for L_4.

$\mathbf{g_x^d}$ = the assignment function such that:

1. for every $y \in \text{VAR}-\{x\}$: $g_x^d(y) = g(y)$
2. $g_x^d(x) = d$

Recursive specification of $\llbracket\alpha\rrbracket_{M,g}$, the interpretation of α in model M, relative to assignment g, for every expression of L_4:

0. If $\alpha \in \text{LEX}$, then $\llbracket\alpha\rrbracket_{M,g} = F_M(\alpha)$
   If $\alpha \in \text{VAR}$, then $\llbracket\alpha\rrbracket_{M,g} = g(\alpha)$

1. If $P \in \text{PRED}^n$ and $\alpha_1,\dots,\alpha_n \in \text{TERM}$ then:
   $\llbracket P(\alpha_1,\dots,\alpha_n)\rrbracket_{M,g} = 1$ iff $< \llbracket\alpha_1\rrbracket_{M,g},\dots,\llbracket\alpha_n\rrbracket_{M,g} > \in \llbracket P\rrbracket_{M,g}$; 0 otherwise.

2. If $\alpha_1,\alpha_2 \in \text{TERM}$, then:
   $\llbracket(\alpha_1=\alpha_2)\rrbracket_{M,g} = 1$ iff $\llbracket\alpha_1\rrbracket_{M,g} = \llbracket\alpha_2\rrbracket_{M,g}$; 0 otherwise.

3. If $\varphi,\psi \in \text{FORM}$ then:
   $\llbracket\neg\varphi\rrbracket_{M,g} = \llbracket\neg\rrbracket_{M,g}(\llbracket\varphi\rrbracket_{M,g})$
   $\llbracket(\varphi \wedge \psi)\rrbracket_{M,g} = \llbracket\wedge\rrbracket_{M,g}(<\llbracket\varphi\rrbracket_{M,g}, \llbracket\psi\rrbracket_{M,g}>)$
   $\llbracket(\varphi \vee \psi)\rrbracket_{M,g} = \llbracket\vee\rrbracket_{M,g}(<\llbracket\varphi\rrbracket_{M,g}, \llbracket\psi\rrbracket_{M,g}>)$
   $\llbracket(\varphi \rightarrow \psi)\rrbracket_{M,g} = \llbracket\rightarrow\rrbracket_{M,g}(<\llbracket\varphi\rrbracket_{M,g}, \llbracket\psi\rrbracket_{M,g}>)$

4. If $x \in \text{VAR}$ and $\varphi \in \text{FORM}$ then:
   $\llbracket\forall x\varphi\rrbracket_{M,g} = 1$ iff for every $d \in D_M$: $\llbracket\varphi\rrbracket_{M,g_x^d} = 1$; 0 otherwise
   $\llbracket\exists x\varphi\rrbracket_{M,g} = 1$ iff for some $d \in D_M$: $\llbracket\varphi\rrbracket_{M,g_x^d} = 1$; 0 otherwise

Note, we have introduced = syncategorematically. We could also assume that
$= \in \text{PRED}^2$,
specify its semantics as: $F_M(=) = \{<d,d>: d \in D_M\}$,
and introduce a notation convention: $(\alpha = \beta) := \ =(\alpha,\beta)$
(:= means 'is by definition')

Truth and entailment: See below.

VII: QUANTIFIER SCOPE: BOUND AND FREE VARIABLES

The **construction tree** of a formula of L_4 is the tree showing how the formula is built from L_4-expressions.
Rather than defining this notion precisely, I indicate in the following example what the construction trees look like.

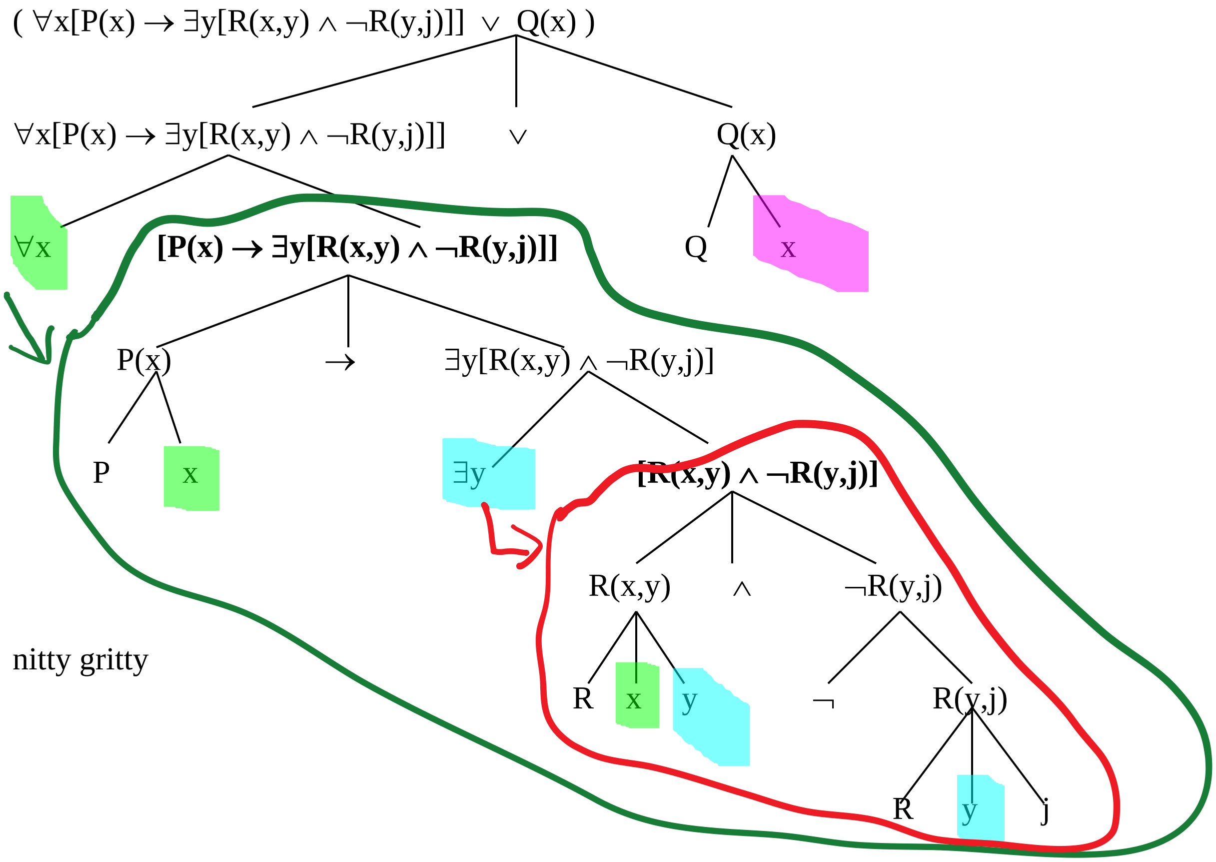
Let $x,y \in VAR$, $j \in CON$, $P,Q \in PRED^1$, $R \in PRED^2$

$(\forall x(P(x) \rightarrow \exists y(R(x,y) \wedge \neg R(y,j))) \vee Q(x)) \quad \in FORM$

We usually change the notation a bit to make the formula more legible. This can involve not write some brackets where this doesn't lead to confusion, adding some brackets to bring out the structure more clearly, or change the form of the brackets, so that you see more clearly which brackets belong together.
So I write the above formula as:

$(\forall x[P(x) \rightarrow \exists y[R(x,y) \wedge \neg R(y,j)]] \ \vee \ Q(x))$

Its construction tree is:



Note that in this tree all nodes are labeled by expressions of L_4, except for the nodes with labels ∀x and ∃y, which are not L_4-expressions. As remarked earlier, we set up L_4 in this way to make the semantics simpler to read and understand for you.
For the purpose of the construction tree, we **will** assume that ∀x and ∃y are L_4 expressions, we call them **universal and existential quantifiers.**

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For the purpose of the construction tree, we **will** assume that $\forall x$ and $\exists y$ are L_4 expressions, we call them **universal and existential quantifiers.**

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Important: for the purpose of the notionsg defined below, we will **not** decompose $\forall x$ into $\forall$ and x, the same for $\exists y$.

This means that, while we normally call $\forall$ the universal quantifier and $\exists$ the existential quantifier, we will **here** call $\forall x$ a universal quantifier and $\exists y$ an existential quantifier.

Thus, on this mode of speech, L_4 contains infinitely many different universal quantifiers, and infinitely many existential quantifiers:

$\forall x_1, \forall x_2, \forall x_3,\ldots$

$\exists x_1, \exists x_2, \exists x_3,\ldots$

FACT about L_4: each formula of L_4 has a unique construction tree.

We say: L_4 is **syntactically unambiguous**.

> Let φ be an L_4 formula and α an L_4 expression.
> α **occurs in** φ iff there is a node in the construction tree of φ labeled by α.

If φ and ψ are formulas and ψ occurs in φ, we call ψ a **subformula** of φ.

b

> Let α be an L_4 expression and φ an L_4 formula.
> an **occurrence of** α **in** φ is a node in the construction tree of φ labeled by α.

So an expression α may occur more than once, say, twice, in a formula φ. In that case there are two occurrences of α in φ, and these two occurrences are nodes in the construction tree of φ.

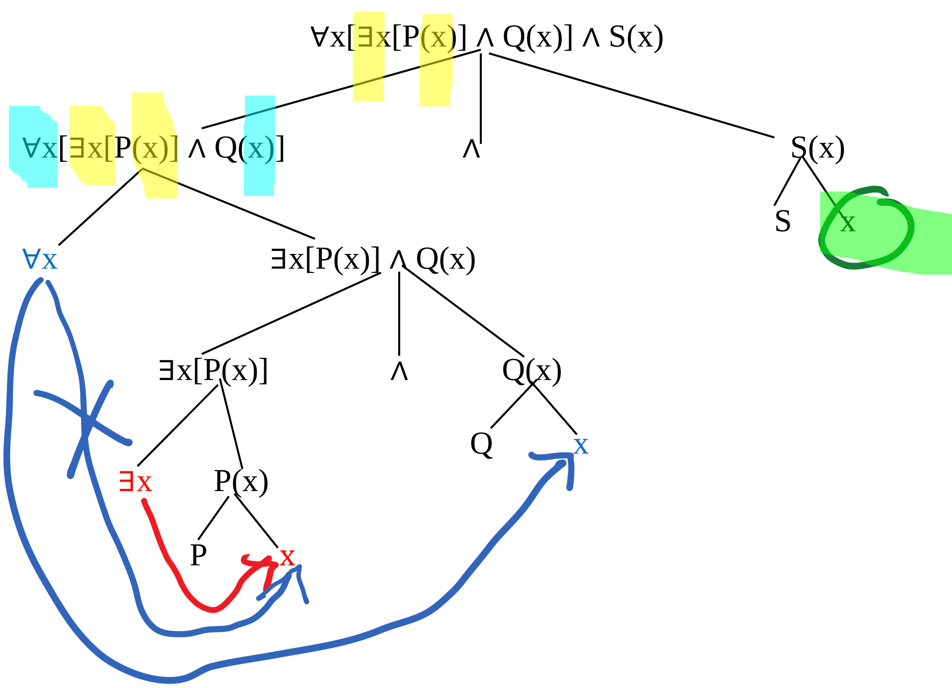
> Let φ be an L_4 formula, $x \in$ VAR.
> Let α be an occurrence of a quantifier $\forall x$ or $\exists x$ in φ (that is, α is a node in the construction tree of φ labeled by $\forall x$ or by $\exists x$).
>
> The **scope of** α **in** φ is the sisternode of α in the construction tree of φ.
>
> Let β be a node in the construction tree of φ.
> β **is in the scope of** α iff β is a daughternode of the scope of α.

Example: In the above formula, there is an occurrence of quantifier $\forall x$. Its scope is the sister node which is boldfaced. In the formula, there are three occurrences of variable x, two of these occurrences of x are in the scope of the occurrence of $\forall x$, one occurrence of x is not in the scope of the occurrence of $\forall x$.

There is an occurrence of quantifier ∃y in the formula. Its scope is its boldfaced sister node. There are two occurrences of variable y in the formula. Both these occurrences are in the scope of the occurrence of ∃y.

> Let φ be an L_4 formula,
> let α be an occurrence of quantifier ∀x or ∃x in φ,
> let β be an occurrence of variable x in φ.
>
> β is **bound by** α in φ iff
> 1. β is in the scope of α.
> 2. There is no occurrence γ of either ∀x or of ∃x in φ such that **both** (a.) and (b.) hold:
>   a. γ is in the scope of α.
>   b. β is in the scope of γ.

This means that an occurence β of a variable x is bound by an occurrence α of a quantifier ∀x or ∃y in φ if β is in the scope of α, and there is no occurrence of a quantifier with the **same variable** x (i.e. ∀x or ∃x) **in between** α and β in φ.
Thus an occurrence of x is bound by **the closest occurrence** of ∀x or ∃x in φ that it is in the scope of.



Note that this means that an occurrence of a variable x is **never** bound by an occurrence of a quantifier which is **not** in variable x (i.e. never by ∀y or ∃y).

Let φ be an L_4 formula.

Occurrence β of variable x in φ is **free for** occurrence α of quantifier $\forall$x or $\exists$x in φ iff β is not bound by α in φ

Occurrence β of variable x in φ is **bound in** φ iff β is bound by some occurrence of quantifier $\forall$x or $\exists$x in φ.

Occurrence β of variable x in φ is **free in** φ iff β is not bound in φ.

Variable x **occurs bound in** φ iff some occurrence of x in φ is bound in φ.
Variable x **occurs free in** φ iff some occurrence of x in φ is free in φ.

Variable x **is bound in** φ iff every occurrence of x in φ is bound in φ.
Variable x **is free in** φ iff every occurrence of x in φ is free in φ.

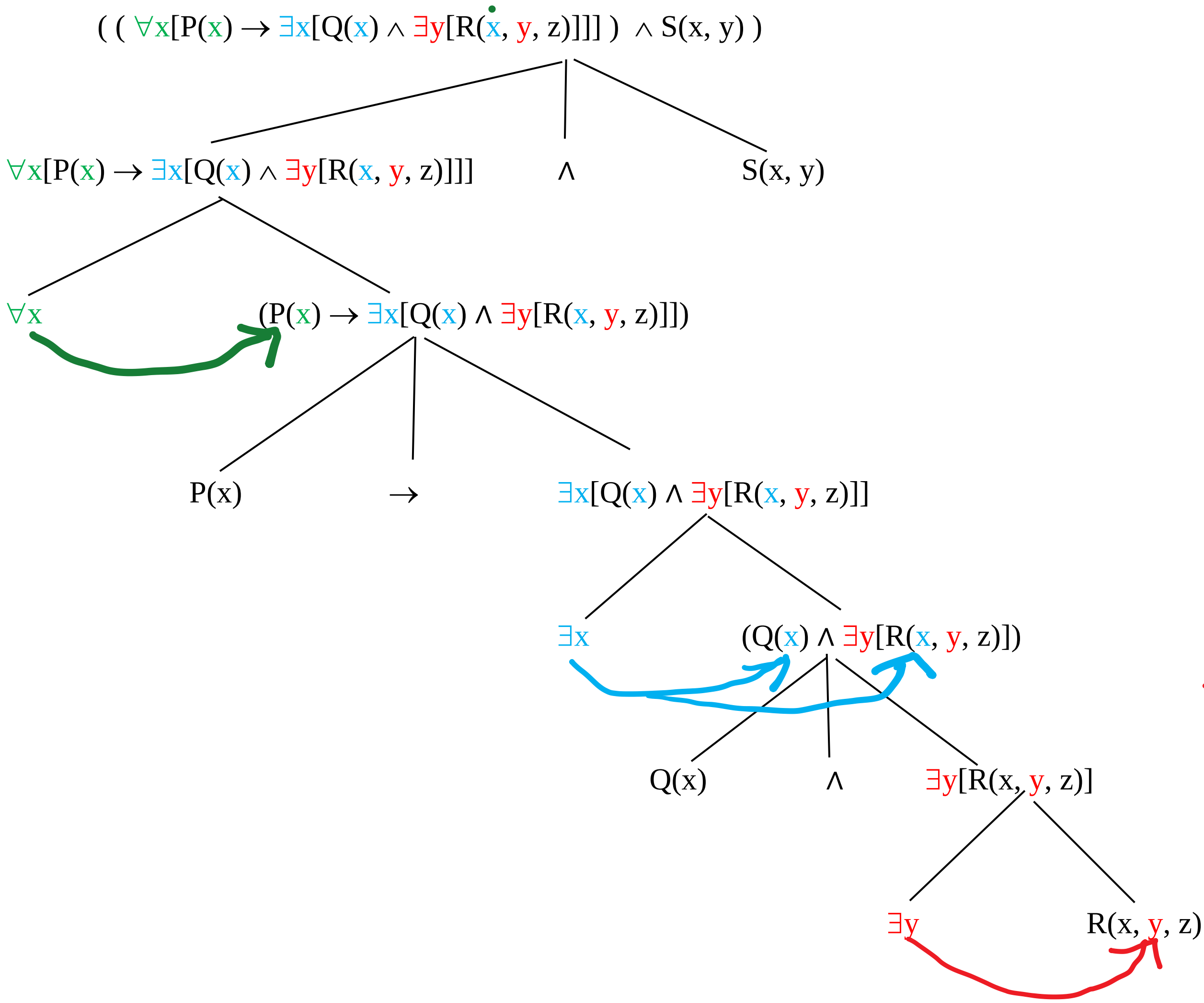
Example:
Let φ be the following L_4 formula:

$$(\forall x[P(x) \rightarrow \exists x[Q(x) \wedge \exists y[R(x,y,z)]]]) \wedge S(x,y)$$

We write **∀x** for the occurrence of ∀x in φ, similarly for the other quantifiers.
Let's indicate the occurrences of the variables in φ by superscripts:

(∀x[P(x) → ∃x[Q(x) ∧ ∃y[R(x, y, z)]]]) ∧ S(x, y)
↓ ↓ ↓ ↓ ↓ ↓ ↓
x^1 x^2 x^3 y^1 z^1 x^4 y^2

occurrence x^1 is bound by occurrence **∀x** in φ
occurrence x^2 is bound by occurrence **∃x** in φ
occurrence x^3 is bound by occurrence **∃x** in φ
occurrence y^1 is bound by occurrence **∃y** in φ
occurrence z^1 is free in φ
occurrence x^4 is free in φ
occurrence y^2 is free in φ
variable z is free in φ
variables x,y are neither free, nor bound in φ, they **occur** both free and bound in φ.


((∀x[P(x) → ∃x[Q(x) ∧ ∃y[R(x, y, z)]]]) ∧ S(x, y))
∀x[P(x) → ∃x[Q(x) ∧ ∃y[R(x, y, z)]]]
∧
S(x, y)
∀x
(P(x) → ∃x[Q(x) ∧ ∃y[R(x, y, z)]])
P(x)
→
∃x[Q(x) ∧ ∃y[R(x, y, z)]]
∃x
(Q(x) ∧ ∃y[R(x, y, z)])
Q(x)
∧
∃y[R(x, y, z)]
∃y
R(x, y, z)

A formula φ of L_4 is a **sentence** of L_4 iff every variable occuring in φ is bound in φ.

SENT = $\{\varphi \in \text{FORM}: \varphi \text{ is a sentence of } L_4\}$

Truth for L_4

Let $\varphi \in$ FORM:

$[\![\varphi]\!]_M = 1$ iff for every assignment g for L_2: $[\![\varphi]\!]_{M,g} = 1$

$[\![\varphi]\!]_M = 0$ iff for every assignment g for L_2: $[\![\varphi]\!]_{M,g} = 0$

FACT: If $\varphi \in$ **SENT** then for every model M for L_4: $[\![\varphi]\!]_M = 1$ or $[\![\varphi]\!]_M = 0$

i.e. formulas in which every variable occurring is bound are true or false **independent of assignment functions**.

Thus, even though the truth conditions of the formula $(P(x) \rightarrow Q(x))$ depend on assignment functions, the truth conditions of the sentence $\forall x[P(x) \rightarrow Q(x)]$, built from it, **do not** depend on assignment functions.

Entailment for L_4

Let $\varphi, \psi \in$ SENT

φ **entails** ψ, $\varphi \Rightarrow \psi$ iff for every model M for L_2: if $[\![\varphi]\!]_M = 1$ then $[\![\psi]\!]_M = 1$

φ and ψ are **equivalent,** $\varphi \Leftrightarrow \psi$ iff $\varphi \Rightarrow \psi$ and $\psi \Rightarrow \varphi$

Let $\Delta \subseteq$ SENT, $\psi \in$ SENT

We write $\Delta \backslash \psi$ for an **argument** with as **premises** the sentences in Δ, and as **conclusion** the sentence ψ.

Argument $\Delta \backslash \psi$ is **valid**, $\Delta \Rightarrow \psi$ iff for for every model M for L_3:

if for every $\varphi \in \Delta$: $[\![\varphi]\!]_M = 1$, then $[\![\psi]\!]_M = 1$

i.e. $\Delta \backslash \psi$ is valid iff in every model where all the premises in Δ are true, the conclusion ψ is true.

ψ is **valid**, $\Rightarrow \psi$, iff $\emptyset \Rightarrow \psi$

i.e. ψ is valid iff ψ is true in every model.

VIII. THE SEMANTICS OF BOUND AND FREE VARIABLES

$\exists x[P(x) \wedge R(x,y)] \wedge Q(x)$

$\downarrow$ bound (first x), $\downarrow$ bound (second x), $\downarrow$ free (y), $\downarrow$ free (last x)

1. $[\![\exists x[P(x) \wedge R(x,y)] \wedge Q(x)]\!]_{M,g} = 1$ iff

2. $[\![\wedge]\!]_{M,g}\,(<[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g}, [\![Q(x)]\!]_{M,g}>) = 1$

2. $[\![\wedge]\!]_{M,g}\,(<[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g}, [\![Q(x)]\!]_{M,g}>) = 1$ iff

3. $F_M(\wedge)(<[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g}, [\![Q(x)]\!]_{M,g}>) = 1$ iff

4. $\begin{pmatrix} <1,1> \rightarrow 1 \\ <1,0> \rightarrow 0 \\ <0,1> \rightarrow 0 \\ <0,0> \rightarrow 0 \end{pmatrix}(<[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g}, [\![Q(x)]\!]_{M,g}>) = 1$ iff

5. $[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g} = 1$ and $[\![Q(x)]\!]_{M,g} = 1$

5. $[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g} = 1$ and $[\![Q(x)]\!]_{M,g} = 1$ iff

6. $[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g} = 1$ and $[\![x]\!]_{M,g} \in [\![Q]\!]_{M,g}$

6. $[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g} = 1$ and $[\![x]\!]_{M,g} \in [\![Q]\!]_{M,g}$ iff

7. $[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g} = 1$ and $g(x) \in [\![Q]\!]_{M,g}$

7. $[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g} = 1$ and $g(x) \in [\![Q]\!]_{M,g}$ iff

8. $[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g} = 1$ and $g(x) \in F_M(Q)$

8. $[\![\exists x[P(x) \wedge R(x,y)]]\!]_{M,g} = 1$ and $g(x) \in F_M(Q)$ iff

9. for some $d \in D_M$: $[\![P(x) \wedge R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$

9. for some $d \in D_M$: $[\![P(x) \wedge R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$ iff

10. for some $d \in D_M$: $[\![\wedge]\!]_{M,g}$ $(< [\![P(x)]\!]_{M,g_x^d}, [\![R(x,y)]\!]_{M,g_x^d} >) =1$ and $g(x) \in F_M(Q)$

10. for some $d \in D_M$: $[\![\wedge]\!]_{M,g}$ $(< [\![P(x)]\!]_{M,g_x^d}, [\![R(x,y)]\!]_{M,g_x^d} >) =1$ and $g(x) \in F_M(Q)$ iff

11. for some $d \in D_M$: $F_M(\wedge)$ $(< [\![P(x)]\!]_{M,g_x^d}, [\![R(x,y)]\!]_{M,g_x^d} >) =1$ and $g(x) \in F_M(Q)$ iff

12. for some $d \in D_M$: $\begin{pmatrix} <1,1> \to 1 \\ <1,0> \to 0 \\ <0,1> \to 0 \\ <0,0> \to 0 \end{pmatrix}$ $(< [\![P(x)]\!]_{M,g_x^d}, [\![R(x,y)]\!]_{M,g_x^d} >) =1$ and $g(x) \in F_M(Q)$ iff

13. for some $d \in D_M$: $[\![P(x)]\!]_{M,g_x^d} = 1$ and $[\![R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$

13. for some $d \in D_M$: $[\![P(x)]\!]_{M,g_x^d} = 1$ and $[\![R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$ iff

14. for some $d \in D_M$: $[\![x]\!]_{M,g_x^d} \in [\![P]\!]_{M,g_x^d}$ and $[\![R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$

14. for some $d \in D_M$: $[\![x]\!]_{M,g_x^d} \in [\![P]\!]_{M,g_x^d}$ and $[\![R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$ iff

15. for some $d \in D_M$: $[\![x]\!]_{M,g_x^d} \in F_M(P)$ and $[\![R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$

15. for some $d \in D_M$: $[\![x]\!]_{M,g_x^d} \in F_M(P)$ and $[\![R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$ iff

16. for some $d \in D_M$: $g_x^d(x) \in F_M(P)$ and $[\![R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$ iff

17. for some d $\in D_M$: d $\in F_M(P)$ and $[\![R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$

17. for some d $\in D_M$: d $\in F_M(P)$ and $[\![R(x,y)]\!]_{M,g_x^d} = 1$ and $g(x) \in F_M(Q)$ iff

18. for some d $\in D_M$: d $\in F_M(P)$ and $< [\![x]\!]_{M,g_x^d}, [\![y]\!]_{M,g_x^d} > \in [\![R]\!]_{M,g_x^d}$ and $g(x) \in F_M(Q)$

18. for some d $\in D_M$: d $\in F_M(P)$ and $< [\![x]\!]_{M,g_x^d}, [\![y]\!]_{M,g_x^d} > \in [\![R]\!]_{M,g_x^d}$ and $g(x) \in F_M(Q)$ iff

19. for some d $\in D_M$: d $\in F_M(P)$ and $< [\![x]\!]_{M,g_x^d}, [\![y]\!]_{M,g_x^d} > \in F_M(R)$ and $g(x) \in F_M(Q)$

19. for some d $\in D_M$: d $\in F_M(P)$ and $< [\![x]\!]_{M,g_x^d}, [\![y]\!]_{M,g_x^d} > \in F_M(R)$ and $g(x) \in F_M(Q)$ iff

20. for some d $\in D_M$: d $\in F_M(P)$ and $<g_x^d(x), [\![y]\!]_{M,g_x^d} > \in F_M(R)$ and $g(x) \in F_M(Q)$ iff

21. for some d $\in D_M$: d $\in F_M(P)$ and $<d, [\![y]\!]_{M,g_x^d} > \in F_M(R)$ and $g(x) \in F_M(Q)$

21. for some d $\in D_M$: d $\in F_M(P)$ and $<d, [\![y]\!]_{M,g_x^d} > \in F_M(R)$ and $g(x) \in F_M(Q)$ iff

22. for some d $\in D_M$: d $\in F_M(P)$ and $<d, g_x^d(y)> \in F_M(R)$ and $g(x) \in F_M(Q)$ iff

23. for some d $\in D_M$: d $\in F_M(P)$ and $<d, g(y)> \in F_M(R)$ and $g(x) \in F_M(Q)$

.

Assume that for every M: $F_M(P)$ is the set of boys in M,

$F_M(Q)$ is the set of girls in M,

$F_M(R)$ is the love relation in M,

g(y) = YOU THERE and

g(x) = YOU OVER THERE

Then $\exists x[P(x) \wedge R(x,y)] \wedge Q(x)$ is true in any situation M, relative to g, where some boy loves you there and you over there are a girl.

IX. ENTAILMENT FOR SENTENCES

(1) $(P(m) \rightarrow Q(m))$
(2) $P(m)$

---

(3) $Q(m)$

1. Assume $[\![P(m)]\!]_M = 1$

Then: For every g: $[\![P(m)]\!]_{M,g} = 1$

Then: For every g: $F_M(m) \in F_M(P)$

Then: $F_M(m) \in F_M(P)$

2. Assume $[\![(P(m) \rightarrow Q(m))]\!]_M = 1$

Then: For every g: $[\![(P(m) \rightarrow Q(m))]\!]_{M,g} = 1$

Then: For every g: $[\![P(m)]\!]_{M,g} = 0$ or $[\![Q(m)]\!]_{M,g} = 1$

Then: For every g: $F_M(m) \notin F_M(P)$ or $F_M(m) \in F_M(Q)$

Then: $F_M(m) \notin F_M(P)$ or $F_M(m) \in F_M(Q)$

3. Combining (1) and (2), it follows that:

$F_M(m) \in F_M(Q)$

Hence: For every g: $F_M(m) \in F_M(Q)$

Hence: For every g: $[\![Q(m)]\!]_{M,g} = 1$

Hence: $[\![Q(m)]\!]_M = 1$

This means, by definition of entailment that (1) and (2) entail (3).

{(1),(2)}\3
(1) $\exists x[CAT(x) \wedge \forall y[DOG(y) \rightarrow CHASE(x,y)]]$
(2) DOG(fido)
(3) $\exists x[CAT(x) \wedge CHASE(x,fido)]$

{(1),(2)} $\Rightarrow$ 3 iff for every M: if $[\![(1)]\!]_M = 1$ and $[\![(2)]\!]_M = 1$, then $[\![(3)]\!]_M = 1$

1. $[\![(1)]\!]_M = 1$ iff

2. $[\![\exists x[CAT(x) \wedge \forall y[DOG(y) \rightarrow CHASE(x,y)]]]\!]_M = 1$ iff

3. for every g: $[\![\exists x[CAT(x) \wedge \forall y[DOG(y) \rightarrow CHASE(x,y)]]]\!]_{M,g} = 1$

3. for every g: $[\![\exists x[CAT(x) \wedge \forall y[DOG(y) \rightarrow CHASE(x,y)]]]\!]_{M,g} = 1$ iff

4. for every g: for some $d \in D_M$: $[\![CAT(x) \wedge \forall y[DOG(y) \rightarrow CHASE(x,y)]]\!]_{M,g_x^d} = 1$

4. for every g: for some $d \in D_M$: $[\![CAT(x) \wedge \forall y[DOG(y) \rightarrow CHASE(x,y)]]\!]_{M,g_x^d} = 1$ iff

5. for every g: for some $d \in D_M$: $[\![CAT(x)]\!]_{M,g_x^d} = 1$ and $[\![\forall y[DOG(y) \rightarrow CHASE(x,y)]]\!]_{M,g_x^d} = 1$

5. for every g: for some $d \in D_M$: $[\![CAT(x)]\!]_{M,g_x^d} = 1$ and $[\![\forall y[DOG(y) \rightarrow CHASE(x,y)]]\!]_{M,g_x^d} = 1$ iff

6. for every g: for some $d \in D_M$: $[\![x]\!]_{M,g_x^d} \in [\![CAT]\!]_{M,g_x^d}$ and $[\![\forall y[DOG(y) \rightarrow CHASE(x,y)]]\!]_{M,g_x^d} = 1$ iff

7. for every g: for some $d \in D_M$: $g_x^d(x) \in F_M(CHASE)$ and $[\![\forall y[DOG(y) \rightarrow CHASE(x,y)]]\!]_{M,g_x^d} = 1$ iff

8. for every g: for some $d \in D_M$: $d \in F_M(CAT)$ and $[\![\forall y[DOG(y) \rightarrow CHASE(x,y)]]\!]_{M,g_x^d} = 1$

8. for every g: for some $d \in D_M$: $d \in F_M(CAT)$ and $[\![\forall y[DOG(y) \rightarrow CHASE(x,y)]]\!]_{M,g_x^d} = 1$ iff

9. for every g: for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in D_M$:
$[\![DOG(y) \rightarrow CHASE(x,y)]\!]_{M,g^{d\,b}_{x\,y}} = 1$

9. for every g: for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in D_M$:
$[\![DOG(y) \rightarrow CHASE(x,y)]\!]_{M,g^{d\,b}_{x\,y}} = 1$ iff

10. for every g: for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in D_M$:
$[\![DOG(y)]\!]_{M,g^{d\,b}_{x\,y}} = 0$ or $[\![CHASE(x,y)]\!]_{M,g^{d\,b}_{x\,y}} = 1$

10. for every g: for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in D_M$:
$[\![DOG(y)]\!]_{M,g^{d\,b}_{x\,y}} = 0$ or $[\![CHASE(x,y)]\!]_{M,g^{d\,b}_{x\,y}} = 1$ iff

11. for every g: for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in D_M$:
$[\![y]\!]_{M,g^{d\,b}_{x\,y}} \notin [\![DOG]\!]_{M,g^{d\,b}_{x\,y}}$ or $< [\![x]\!]_{M,g^{d\,b}_{x\,y}}, [\![y]\!]_{M,g^{d\,b}_{x\,y}} > \in [\![CHASE]\!]_{M,g^{d\,b}_{x\,y}}$ iff

12. for every g: for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in D_M$:
$g^{d\,b}_{x\,y}(y) \notin F_M(DOG)$ or $< g^{d\,b}_{x\,y}(x), g^{d\,b}_{x\,y}(y) > \in F_M(CHASE)$ iff

13. for every g: for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in D_M$:
b $\notin F_M(DOG)$ or $<d, b> \in F_M(CHASE)$

13. for every g: for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in D_M$:
b $\notin F_M(DOG)$ or $<d, b> \in F_M(CHASE)$ iff

14. for every g: for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in F_M(DOG)$:
$<d, b> \in F_M(CHASE)$

14. for every g: for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in F_M(DOG)$:
$<d, b> \in F_M(CHASE)$ iff

15. for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in F_M(DOG)$: $<d, b> \in F_M(CHASE)$

15. for some d $\in D_M$: d $\in F_M(CAT)$ and for every b $\in F_M(DOG)$: $<d, b> \in F_M(CHASE)$ iff

16. for some d $\in F_M(CAT)$ for every b $\in F_M(DOG)$: $<d, b> \in F_M(CHASE)$.

1. $\llbracket(2)\rrbracket_M = 1$ **iff**

2. $\llbracket DOG(fido)\rrbracket_M = 1$ iff

3. for every g: $\llbracket DOG(fido)\rrbracket_{M,g} = 1$ iff

4. for every g: $\llbracket fido\rrbracket_{M,g} \in \llbracket DOG\rrbracket_{M,g}$ iff

5. for every g: $F_M(fido) \in F_M(DOG)$ iff

6. $\mathbf{F_M(fido) \in F_M(DOG)}$

1. $\llbracket(3)\rrbracket_M = 1$ **iff**

2. $\llbracket\exists x[CAT(x) \wedge CHASE(x,fido)]\rrbracket_M = 1$ iff

3. for every g: $\llbracket\exists x[CAT(x) \wedge CHASE(x,fido)]\rrbracket_{M,g} = 1$ iff

4. for every g: for some $d \in D_M$: $\llbracket CAT(x) \wedge CHASE(x,fido)\rrbracket_{M,g_x^d} = 1$ iff

5. for every g: for some $d \in D_M$: $\llbracket CAT(x)\rrbracket_{M,g_x^d} = 1$ and $\llbracket CHASE(x,fido)\rrbracket_{M,g_x^d} = 1$ iff

6. for every g: for some $d \in D_M$: $g_x^d(x) \in F_M(CAT)$ and $\langle g_x^d(x),F_M(fido)\rangle \in F_M(CHASE)$ iff

7. for every g: for some $d \in D_M$: $d \in F_M(CAT)$ and $\langle d,F_M(fido)\rangle \in F_M(CHASE)$ iff

8. for some $d \in D_M$: $d \in F_M(CAT)$ and $\langle d,F_M(fido)\rangle \in F_M(CHASE)$ iff

9. **for some** $\mathbf{d \in F_M(CAT)}$**:** $\mathbf{\langle d,F_M(fido)\rangle \in F_M(CHASE)}$**.0**

In sum:

$[\![(1)]\!]_M = 1$ iff for some $d \in F_M(CAT)$ for every $b \in F_M(DOG)$: $<d,b> \in F_M(CHASE)$.

$[\![(2)]\!]_M = 1$ iff $F_M(fido) \in F_M(DOG)$

$[\![(3)]\!]_M = 1$ iff for some $d \in F_M(CAT)$: $<d,F_M(fido)> \in F_M(CHASE)$

Now let M be any model such that $[\![(1)]\!]_M = 1$ and $[\![(2)]\!]_M = 1$.

This means that:

for some $d \in F_M(CAT)$ for every $b \in F_M(DOG)$: $<d, b> \in F_M(CHASE)$ and $F_M(fido) \in F_M(DOG)$.

Then for some $d \in F_M(CAT)$: $<d, F_M(fido)> \in F_M(CHASE)$, hence $[\![(3)]\!]_M = 1$.

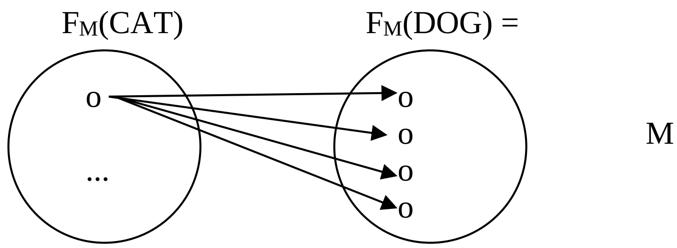
We have shown that the semantics predicts that $\{(1),(2)\} \Rightarrow 3$.

Arguing in a picture.

Some cat chases every dog

(1) $\exists x[CAT(x) \wedge \forall y[DOG(y) \rightarrow CHASE(x,y)]$

$[\![(1)]\!]_M = 1$ iff whatever else holds in M, you find the following:



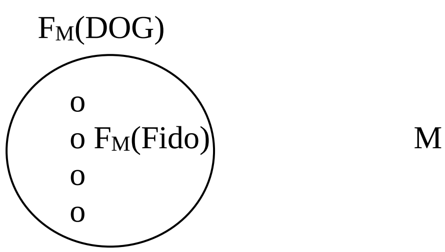
The picture indicates that there could be more cats and we don't know about their chasing, but the righthand side stands for the set of all dogs.

The arrows indicate part of F_M(CHASE)

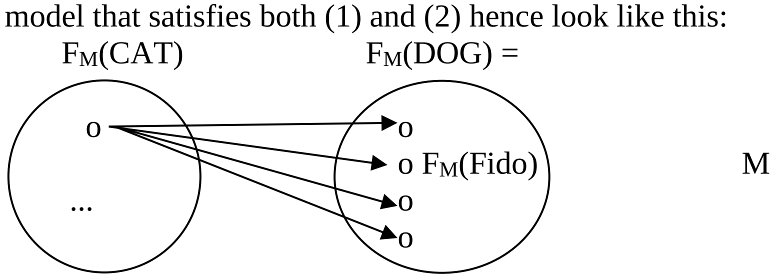
Fido is a dog

(2) DOG(fido)

$[\![(2)]\!]_M = 1$ iff whatever else holds in M, you find the following:



A model that satisfies both (1) and (2) hence look like this:



You can read off this picture that in any such model M (3) is true:

(3) *Some cat chases Fido.*

$\exists x[CAT(x) \wedge CHASE(x,fido)]$

X. ALPHABETIC VARIANTS

Let φ be an L_4 formula, x and y variables.
Let q_x be an occurrence of $\forall x$ or $\exists x$ in φ.
Let $\{ v_{1,x}, \dots, v_{n,x}\}$ be the set of **all** occurrences of variable x bound by q_x in φ.
(So q and $v_1,\dots,v_n$ stand for nodes in the construction tree.)

We call $< q_x, v_{1,x}, \dots, v_{n,x} >$ a **binding relation** in φ.

Crucially, this means that, if $< q_x, v_{1,x}, v_{2,x} >$ is a binding relation in φ, then $< q_x, v_{1,x} >$ is not a binding relation in φ, it got to be **all and only** the bound occurrences to be called a binding relation.

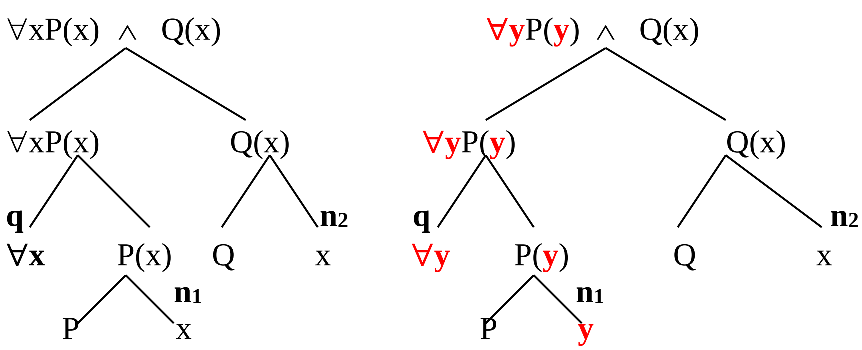
Now take the construction tree for φ, and binding relation $< q_x, v_{1,x}, \dots, v_{n,x} >$ in φ and:

1. replace $< q_x, v_{1,x}, \dots, v_{n,x} >$ by $< q_y, v_{1,y}, \dots, v_{n,y} >$.
2. adjust the nodes above in the tree accordingly.

This gives a formula which we can call: $\varphi_{<q_y, v_{1,y},\dots, v_{n,y}>}$

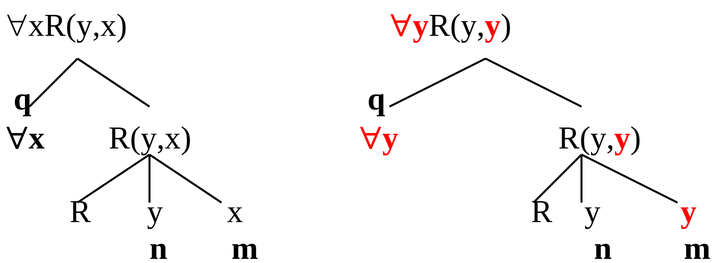
Example 1:

φ $\qquad\qquad$ $\varphi_{<q_y, n_{1,y}>}$



Example 2:

φ $\qquad\qquad$ $\varphi_{<q_y, m_y>}$



Now we define:

Formulas φ and ψ are **basic alphabetic variants** iff

there are variables x and y and there is a binding relation $< q_x, v_{1,x}, \ldots, v_{n,x} >$ in φ such that 1. $\psi = \varphi_{<q_y, v_{1,y}, \ldots v_{n,y}>}$ and

2. $< q_y, v_{1,y}, \ldots, v_{n,y} >$ is a binding relation in ψ.

Again, the requirement that $< q_y, v_{1,y}, \ldots, v_{n,y} >$ is a binding relation in ψ means that $\{v_{1,y}, \ldots, v_{n,y}\}$ is **exactly** the set of occurrences of variable y bound by occurrence q_y in ψ, not less, and not more.

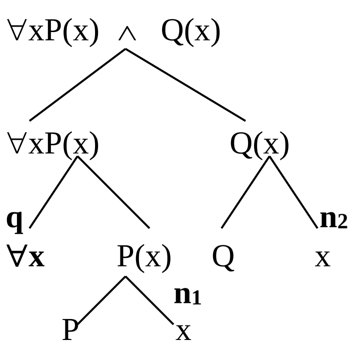
Formulas φ and ψ are **alphabetic variants** iff

there is a sequence of formulas $<\varphi_1, \ldots, \varphi_n>$ such that $\varphi_1 = \varphi$ and $\varphi_n = \psi$ and for every $i<n$: φ_i and φ_{i+1} are basic alphabetic variants.

THEOREM: if φ and ψ are alphabetic variants then φ ⇔ ψ.

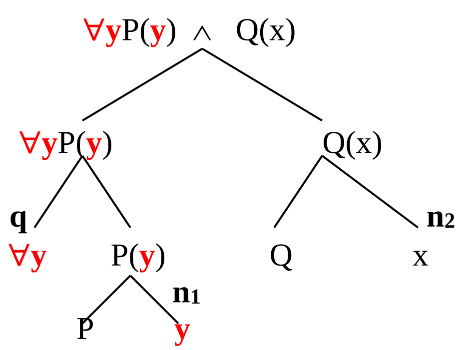
Example 1:

φ



q binds **n1**

$\varphi_{<q_y, n_{1,y}>}$

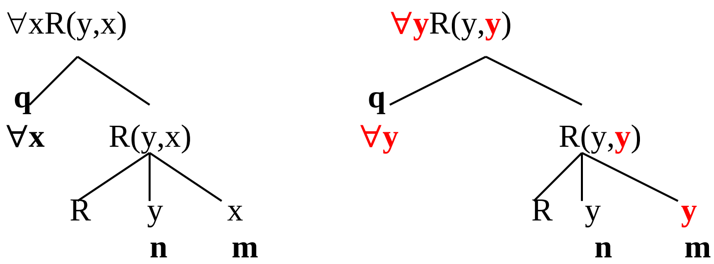


q binds **n1**

$<q_y, n_{1,y}>$ is a binding relation in $\varphi_{<q_y, n_{1,y}>}$, hence φ and $\varphi_{<q_y, n_{1,y}>}$ are basic alphabetic variants, hence alphabetic variants, and so, by the theorem, φ and $\varphi_{<q_y, n_{1,y}>}$ are logically equivalent.

Example 2:

φ $\qquad\qquad\qquad\qquad$ $\varphi_{<q_y,m_y>}$



q binds **m** $\qquad\qquad\qquad\qquad$ **q** binds **n** and **m**

$\varphi_{<q_y,m_y>}$ is not an alphabetic variant of φ, because $\varphi_{<q_y,m_y>}$ is not a basic alphabetic variant of φ.
$\varphi_{<q_y,m_y>}$ is not a basic alphabetic variant of φ, because $<q_y, m_y>$ is not a binding relation in $\varphi_{<q_y,m_y>}$.
$<q_y, m_y>$ is not a binding relation in $\varphi_{<q_y,m_y>}$, because the binding relation is
$<q_y, n_y,m_y>$, not $<q_y, m_y>$.

Hence the theorem does **not** say that $\forall xR(y,x)$ and $\forall \mathbf{y}R(y,\mathbf{y})$ are equivalent, it says nothing about these formulas.
(They are, provably not equivalent, actually, we see that below.)

Alphabetic variants:

$\varphi_1 = \exists x[\ \forall x[P(x)] \wedge Q(x)] \wedge\ S(x)$ $\quad$ φ_1 and φ_2 and φ_3 are alphabetic variants
$\varphi_2 = \exists x[\ \forall y[P(y)] \wedge Q(x)] \wedge\ S(x)$
$\varphi_3 = \exists z[\ \forall y[P(y)] \wedge Q(z)] \ \wedge\ S(x)$

So even though you cannot go in one step from φ_1 to φ_3 by replacing a binding relation, you can in two steps. φ_1 and φ_3 are not basic alphabetic variants, but they are alphabetic variants.

More examples:

$\forall x \exists y[R(x,y)]$ and $\forall u \exists z[R(u,z)]$ are alphabetic variants.

$\forall x \exists y[R(x,y)] \wedge P(x)$ and $\forall u \exists z[R(u,z)] \wedge P(x)$ are alphabetic variants.
(only the first occurrence of x is bound by $\forall x$, so only the first occurrence of x gets changed.)

$\forall x \exists y[R(x,y)] \wedge P(x)$ and $\forall u \exists z[R(u,z)] \wedge P(u)$ are **not** alphabetic variants.
(because you have changed also an occurrence of x which wasn't bound by $\forall x$).
So, P(x) and P(u) are **not** alphabetic variants.

$\forall x \exists y[R(x,y)]$ and $\forall x \exists x[R(x,x)]$ are **not** alphabetic variants.
You change $\exists y$ to $\exists x$ and y to x. But after the change, $\exists x$ binds **not only** the occurrence of x where we changed the label, but also the occurrence of x which was an occurrence of x to start with. This means that we do not satisfy the constraint of basic alphabetic variants.

Hence: $\forall x \exists y[R(x,y)] \Leftrightarrow \forall u \exists z[R(u,z)]$.

Note: $\forall x \exists y[R(x,y)]$ and $\forall x \exists x[R(x,x)]$ are **not** equivalent.
(and if we extend the notion of equivalence to formulas in general, P(x) and P(y) are **not** equivalent.)

Showing the equivalence of alphabetic variants semantically:

(1) **$[\![\forall x\exists y[R(x,y)]]\!]_M = 1$ iff**

(2) for every g: $[\![\forall x\exists y[R(x,y)]]\!]_{M,g} = 1$ iff

(3) for every g: for every $d \in D_M$: $[\![\exists y[R(x,y)]]\!]_{M,g_x^d} = 1$ iff

(4) for every g: for every $d \in D_M$ there is a $b \in D_M$: $[\![R(x,y)]]\!]_{M,g_{x\,y}^{d\,b}} = 1$ iff

(5) for every g: for every $d \in D_M$ there is a $b \in D_M$: $<g_{\mathbf{x}\,\mathbf{y}}^{d\,b}(\mathbf{x}),g_{\mathbf{x}\,\mathbf{y}}^{d\,b}(\mathbf{y})> \in F_M(R)$ iff

(6) for every g: for every $d \in D_M$ there is a $b \in D_M$: $<d,b> \in F_M(R)$ iff

(7) for every g: for every $d \in D_M$ there is a $b \in D_M$: $<g_{\mathbf{u}\,\mathbf{z}}^{d\,b}(\mathbf{u}),g_{\mathbf{u}\,\mathbf{z}}^{d\,b}(\mathbf{z})> \in F_M(R)$ iff

(8) for every g: for every $d \in D_M$ there is a $b \in D_M$: $[\![R(u,z)]]\!]_{M,g_{u\,z}^{d\,b}} = 1$ iff

(9) for every g: for every $d \in D_M$: $[\![\exists z[R(u,z)]]\!]_{M,g_u^d} = 1$ iff

(10) for every g: $[\![\forall u\exists z[R(u,z)]]\!]_{M,g} = 1$ iff

(11) $[\![\forall u\exists z[R(u,z)]]\!]_M = 1$

Hence for every M: $[\![\forall x\exists y[R(x,y)]]\!]_M = 1$ iff $[\![\forall u\exists z[R(u,z)]]\!]_M = 1$,

which means, indeed, that: $\forall x\exists y[R(x,y)] \Leftrightarrow \forall u\exists z[R(u,z)]$.

(1) $[\![\forall x \exists y[R(x,y)]]\!]_M = 1$ iff

(2) for every $d \in D_M$ there is a $b \in D_M$: $\langle d,b\rangle \in F_M(R)$

(1) $[\![\forall x \exists x[R(x,x)]]\!]_M = 1$ iff

(2) for every g: $[\![\forall x \exists x[R(x,x)]]\!]_{M,g} = 1$ iff

(3) for every g: for every $d \in D_M$: $[\![\exists x[R(x,x)]]\!]_{M,g^{d}_{x}} = 1$ iff

(4) for every g: for every $d \in D_M$: there is a $b \in D_M$: $[\![R(x,x)]\!]_{M,g^{d\,b}_{x\,x}} = 1$ iff

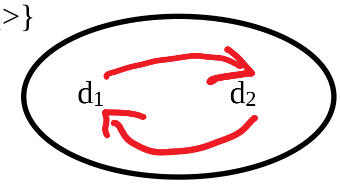
(5) for every g: for every $d \in D_M$: there is a $b \in D_M$: $\langle g^{d\,b}_{x\,x}(x), g^{d\,b}_{x\,x}(x)\rangle \in F_M(R)$ iff

$g^{d}_{x}(x)$ is the result of resetting g(x) to d.
$g^{d\,b}_{x\,x}(x)$ is the result of resetting $g^{d}_{x}(x)$ to b.
So $g^{d\,b}_{x\,x}(x) = g^{b}_{x}(x)$

(6) for every g: for every $d \in D_M$: there is a $b \in D_M$: $\langle b,b\rangle \in F_M(R)$ iff

(7) there is a $b \in D_M$: $\langle b,b\rangle \in D_M(R)$.

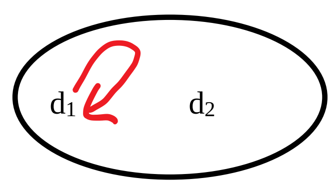
Let M be a model with $D_M = \{d_1,d_2\}$ and $F_M(R) = \{\langle d_1,d_2\rangle,\langle d_2,d_1\rangle\}$



Then $[\![\forall x \exists y[R(x,y)]]\!]_M = 1$ but $[\![\forall x \exists x[R(x,x)]]\!]_M = 0$.
Hence the two are not equivalent.

This model shows that $\forall x \exists y[R(x,y)]$ does not entail $\forall x \exists x[R(x,x)]$.

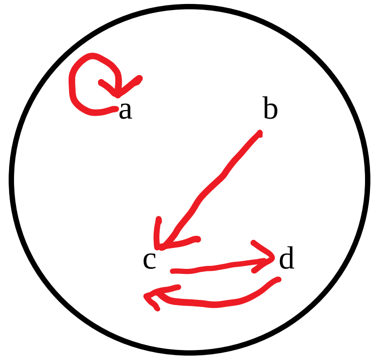
A model M' with $D_M = \{d_1,d_2\}$ and $F_M(R) = \{\langle d_1,d_1\rangle\}$ shows that also $\forall x \exists x[R(x,x)]$ does not entail $\forall x \exists y[R(x,y)]$.



In fact, it is easy to show that: $\forall x \exists x[R(x,x)] \Leftrightarrow \exists x[R(x,x)]$.

We call the quantifier $\forall x$ in $\forall x \exists x[R(x,x)]$ **vacuous**, since it binds no variable.
And we see that semantically the vacuous quantifier doesn't contribute to the meaning of the whole.

THE GAME OF LOVE



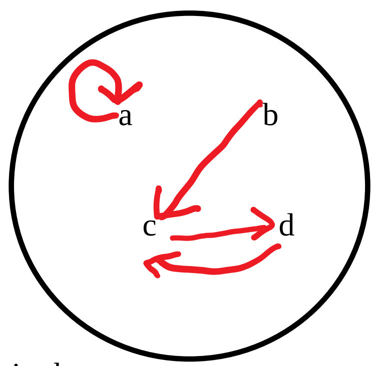
$D_M = \{a,b,c,d\}$
$F_M(\text{LOVE}) = \{\langle a,a\rangle,\langle b,c\rangle,\langle c,d\rangle,\langle d,c\rangle\}$
Game: You win if $[\![\forall x\exists y\text{LOVE}(x,y)]\!]_{M,g} = 1$

iff	for every $d \in D_M$:	$[\![\exists y\text{LOVE}(x,y)]\!]_{M,g^{d}_{x}} = 1$
iff	a:	$[\![\exists y\text{LOVE}(x,y)]\!]_{M,g^{a}_{x}} = 1$
and	b:	$[\![\exists y\text{LOVE}(x,y)]\!]_{M,g^{b}_{x}} = 1$
and	c:	$[\![\exists y\text{LOVE}(x,y)]\!]_{M,g^{c}_{x}} = 1$
and	d:	$[\![\exists y\text{LOVE}(x,y)]\!]_{M,g^{d}_{x}} = 1$

CASE a: To stay in the game you must show that
for some $f \in D_M$: $[\![\text{LOVE}(x,y)]\!]_{M,g^{a\,f}_{x\,y}} = 1$

This means you must get 1 for **one** of:

a_1:	$[\![\text{LOVE}(x,y)]\!]_{M,g^{a\,a}_{x\,y}}$	iff $\langle a,a\rangle \in F_M(\text{LOVE})$
a_2:	$[\![\text{LOVE}(x,y)]\!]_{M,g^{a\,b}_{x\,y}}$	iff $\langle a,b\rangle \in F_M(\text{LOVE})$
a_3:	$[\![\text{LOVE}(x,y)]\!]_{M,g^{a\,c}_{x\,y}}$	iff $\langle a,c\rangle \in F_M(\text{LOVE})$
a_4:	$[\![\text{LOVE}(x,y)]\!]_{M,g^{a\,d}_{x\,y}}$	iff $\langle a,d\rangle \in F_M(\text{LOVE})$

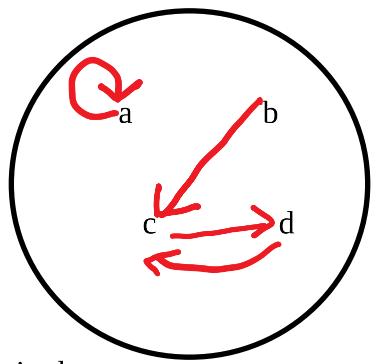


You get 1 at a_1: $\langle a,a\rangle \in F_M(\text{LOVE})$, hence at a, so you stay in the game.

CASE b: To stay in the game you must show that
for some $f \in D_M$: $[\![\text{LOVE}(x,y)]\!]_{M,g_{x\,y}^{b\,f}} = 1$

This means you must get 1 for **one** of:

b_1:	$[\![\text{LOVE}(x,y)]\!]_{M,g_{x\,y}^{b\,a}}$	iff $\langle b,a\rangle \in F_M(\text{LOVE})$
b_2:	$[\![\text{LOVE}(x,y)]\!]_{M,g_{x\,y}^{b\,b}}$	iff $\langle b,b\rangle \in F_M(\text{LOVE})$
b_3:	$[\![\text{LOVE}(x,y)]\!]_{M,g_{x\,y}^{b\,c}}$	iff $\langle b,c\rangle \in F_M(\text{LOVE})$
b_4:	$[\![\text{LOVE}(x,y)]\!]_{M,g_{x\,y}^{b\,d}}$	iff $\langle b,d\rangle \in F_M(\text{LOVE})$

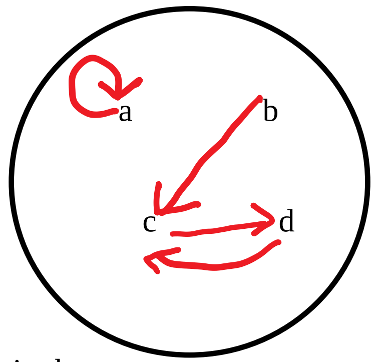


You get 1 at b_3, $\langle b,c\rangle \in F_M(\text{LOVE})$, hence at b, so you stay in the game.

CASE c: To stay in the game you must show that
for some $f \in D_M$: $[\![LOVE(x,y)]\!]_{M,g^{c\,f}_{x\,y}} = 1$

This means you must get 1 for **one** of:

c_1:	$[\![LOVE(x,y)]\!]_{M,g^{c\,a}_{x\,y}}$	iff $<c,a> \in F_M(LOVE)$
c_2:	$[\![LOVE(x,y)]\!]_{M,g^{c\,b}_{x\,y}}$	iff $<c,b> \in F_M(LOVE)$
c_3:	$[\![LOVE(x,y)]\!]_{M,g^{c\,c}_{x\,y}}$	iff $<c,c> \in F_M(LOVE)$
c_4:	$[\![LOVE(x,y)]\!]_{M,g^{c\,d}_{x\,y}}$	iff $<c,d> \in F_M(LOVE)$

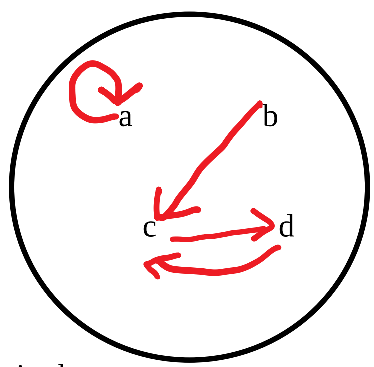


You get 1 at c_4, $<c,d> \in F_M(LOVE)$, hence at c, so you stay in the game.

CASE d: To stay in the game you must show that
for some $f \in D_M$: $[\![LOVE(x,y)]\!]_{M,g^{d\,f}_{x\,y}} = 1$

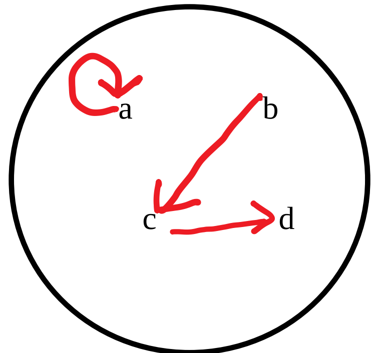
This means you must get 1 for **one** of:

d_1:	$[\![LOVE(x,y)]\!]_{M,g^{d\,a}_{x\,y}}$	iff $<d,a> \in F_M(LOVE)$
d_2:	$[\![LOVE(x,y)]\!]_{M,g^{d\,b}_{x\,y}}$	iff $<d,b> \in F_M(LOVE)$
d_3:	$[\![LOVE(x,y)]\!]_{M,g^{d\,c}_{x\,y}}$	iff $<d,c> \in F_M(LOVE)$
d_4:	$[\![LOVE(x,y)]\!]_{M,g^{d\,d}_{x\,y}}$	iff $<d,d> \in F_M(LOVE)$



You get 1 at d_3, $<d,c> \in F_M(LOVE)$, hence at d, so you stay in the game.
You have gotten 1 at a,b,c,d: YOU WIN! 🏆 🎖 💰

Change the model to:
$D_M = \{a,b,c,d\}$
$F_M(LOVE) = \{<a,a>,<b,c>,<c,d>\}$



The cases a,b,c stay the same, but now on case d you get 0 everywhere in the list, cases d_1,d_2,d_3,d_4. This means you get 0 on d, $<d,c> \notin F_M(LOVE)$, and you lose!.

When you get more experienced, you may do without writing out all the cases and work out the semantics directly:

$[\![\forall x \exists y LOVE(x,y)]\!]_{M,g} = 1$ iff

for every $d \in D_M$ there is an $f \in D_M$: $[\![LOVE(x,y)]\!]_{M,g^{d\,f}_{x\,y}} = 1$ iff

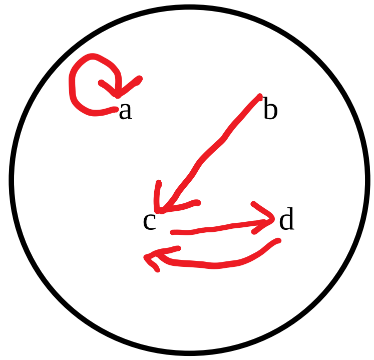
for every $d \in D_M$ there is an $f \in D_M$: $<g^{d\,f}_{x\,y}(x), g^{d\,f}_{x\,y}(y)> \in F_M(LOVE)$ iff

for every $d \in D_M$ there is an $f \in D_M$: $<d,f> \in F_M(LOVE)$ iff

$\mathbf{dom}(F_M(LOVE)) = D_M$

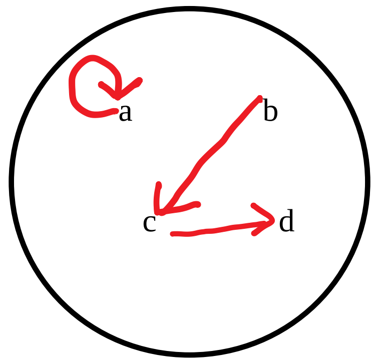
where $\mathbf{dom}(R) = \{d_1 \in D$: for some $d_2 \in D: <d_1, d_2> \in R\}$

You check:
In the first example:



F_M(LOVE) = {<a,a>,<b,c>,<c,d>,<d,c>}
So **dom**(F_M(LOVE)) = {a,b,c,d} = D_M TRUE

In the second example:



F_M(LOVE) = {<a,a>,<b,c>,<c,d>}
So **dom**(F_M(LOVE)) = {a,b,c} $\neq$ D_M FALSE

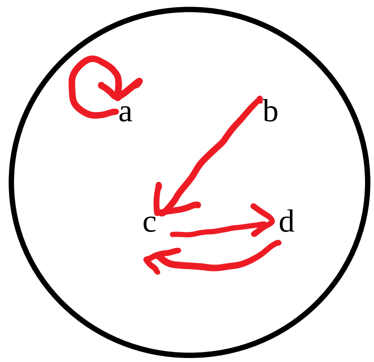
Similarly

$[\![\forall y \exists x LOVE(x,y)]\!]_{Mg} = 1$ iff
for every $f \in D_M$ there is a $d \in D_M$: $<d,f> \in F_M(LOVE)$ iff

$\mathbf{ran}(F_M(LOVE)) = D_M$

where $\mathbf{ran}(R) = \{d_2 \in D: \text{for some } d_1 \in D: <d_1, d_2> \in R\}$

In our example:



$F_M(LOVE) = \{<a,a>,<b,c>,<c,d>,<d,c>\}$

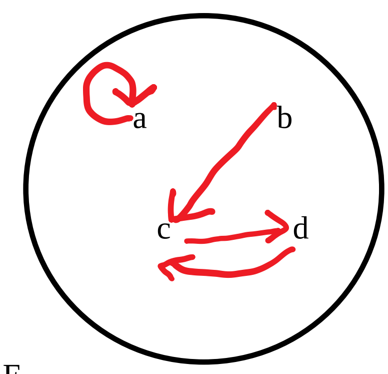
$\mathbf{ran}(F_M(LOVE)) = \{a,c,d\} \neq D_M$ FALSE

$[\![\exists x \forall y LOVE(x,y)]\!]_{M,g} = 1$ iff
for some $d \in D_M$ for every $f \in D_m$: $<d,f> \in F_M(LOVE)$

Let $\mathbf{L}d = \{f \in D_M: <d,f> \in F_M(LOVE)\}$

Hence:
$[\![\exists x \forall y LOVE(x,y)]\!]_{M,g} = 1$ iff
for some $d \in D_M$: $\mathbf{L}_d = D_M$

In our example:

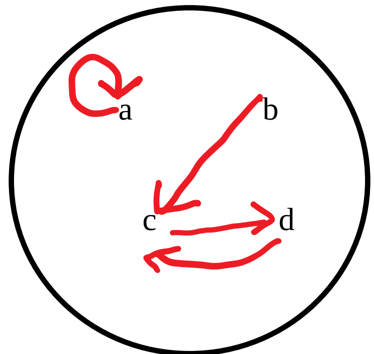


$\mathbf{L}_a = \{a\}$, $\mathbf{L}_b = \{c\}$ $\mathbf{L}_c = \{d\}$, $\mathbf{L}_d = \{c\}$ FALSE

$[\![\exists y \forall x LOVE(x,y)]\!]_{M,g} = 1$ iff
for some $f \in D_M$ for every $d \in D_M$: $\langle d,f \rangle \in F_M(LOVE)$

Let $\mathbf{BL}_d = \{f \in D_M: \langle d,f \rangle \in F_M(LOVE)\}$

Hence:
$[\![\exists y \forall x LOVE(x,y)]\!]_{M,g} = 1$ iff
for some $f \in D_M$: $\mathbf{BL}_f = D_M$



$\mathbf{BL}_a = \{a\}$, $\mathbf{BL}_b = \emptyset$, $\mathbf{BL}_c = \{b,d\}$, $\mathbf{BL}_d = \{c\}$ FALSE

We see already here that $\forall x \exists y LOVE(x,y)$ does not entail $\exists y \forall x LOVE(x,y)$

However, assume a model M' where $\exists y \forall x LOVE(x,y)$ is true.
then for some $f \in D_{M'}$: $BL_f = D_{M'}$,
i.e. for some $f \in D_{M'}$: $\{d \in D_{M'} : \langle d,f \rangle \in F_{M'}(LOVE)\} = D_{M'}$

But, obviously, then $DOM(F_{M'}(LOVE)) = D_{M'}$
and this means that $\forall x \exists y LOVE(x,y)$ is true in M'

This means, that $\exists y \forall x LOVE(x,y)$ entails $\forall x \exists y LOVE(x,y)$.

In words: if somebody loves everybody, then everybody is loved.

XI. EXTENSIONALITY

We define:

$$(\varphi \leftrightarrow \psi) \ := \ (\varphi \rightarrow \psi) \land (\psi \rightarrow \varphi)$$

Let φ,ψ,χ be sentences of L_4 and let α^ψ be an occurrence of ψ in φ (so ψ is a subformula of φ). Let $T(\varphi)$ be the construction tree of φ, Let α^χ be the result of changing the label ψ on α^ψ to χ in $T(\varphi)$, let $T(\varphi)[\alpha^\chi/\alpha^\psi]$ be the resulting construction tree and let $\varphi[\alpha^\chi/\alpha^\psi]$ be the L_4 formula of which $T(\varphi)[\alpha^\chi/\alpha^\psi]$ is the construction tree.

EXTENSIONALITY OF L_4 (for subsentences of sentences):

$$(\psi \leftrightarrow \chi) \Rightarrow (\varphi \leftrightarrow \varphi[\alpha^\chi/\alpha^\psi])$$

Thus, in every model where ψ and χ have the same truth value, φ and the result of substituting χ for ψ in φ have the same truth value.

It follows from this that if $\psi \Leftrightarrow \chi$, then $\varphi \Leftrightarrow \varphi[\alpha^\chi/\alpha^\psi]$).

Let φ be a sentence of L_4 and $t,s \in CON$ and let α^t be an occurrence of t in φ.
Let α^s be the result of changing the label on α^t in $T(\varphi)$ from t to s, and let
$T(\varphi)[\alpha^s/\alpha^t]$ and $\varphi[\alpha^s/\alpha^t]$ be construction tree and formula resulting from this change.

EXTENSIONALITY OF L_4 (for constants in sentences):

$$(t = s) \Rightarrow (\varphi \leftrightarrow \varphi[\alpha^s/\alpha^t])$$

In every model where t and s have the same interpretation, φ and the result of substituting s for t in φ have the same truth value.

It follows from this that if t and s have the same interpretation in **every model** (i.e. $\Rightarrow (t=s)$), then
$\varphi \Leftrightarrow \varphi[\alpha^s/\alpha^t]$).

There are also more general versions of these principles for formulas and terms in general:
Let $x_1,\ldots,x_n$ be exactly the variables occurring free in ψ or χ.

EXTENSIONALITY OF L_4 (for subformulas of formulas):

$$\forall x_1\ldots\forall x_n(\psi \leftrightarrow \chi) \Rightarrow (\varphi \leftrightarrow \varphi[\alpha^\chi/\alpha^\psi])$$

In every model where for every assignment g every resetting of the values of $x_1,\ldots,x_n$ in g gives the same truth value to ψ and χ, in every such model, every assignment g gives φ and $\varphi[\alpha^\chi/\alpha^\psi]$) the same truth value.

Let t,s be terms, x_1,x_2 the variables occurring in t,s (since in our version of predicate logic we don't have complex terms, x_1, x_2 can only possibly occur in t,s if t or s is x_1 or x_2.

EXTENSIONALITY OF L_4 (for terms in formulas):

$$\forall x_1\forall x_2(t = s) \Rightarrow (\varphi \leftrightarrow \varphi[\alpha^s/\alpha^t])$$

In every model where for every assignment g every resetting of the values of x_1,x_2 in g assigns t and s the same interpretation, in every such model, every assignment g gives φ and $\varphi[\alpha^s/\alpha^t]$) the same truth value.

XII. CONNECTIONS BETWEEN CONNECTIVES AND QUANTIFIERS

$\forall xP(x) \Leftrightarrow \neg\exists x\neg P(x)$.

(1) $[\![\forall xP(x)]\!]_M = 1$ iff

(2) for every g: $[\![\forall xP(x)]\!]_{M,g} = 1$

(2) for every g: $[\![\forall xP(x)]\!]_{M,g} = 1$ iff

(3) for every g: for every $d \in D_M$: $[\![P(x)]\!]_{M,g_x^d} = 1$

Now: $[\![P(x)]\!]_{M,g_x^d} = 1$

iff

$\begin{bmatrix} 1 \rightarrow 0 \\ 0 \rightarrow 1 \end{bmatrix} (\backslash[\![P(x)]\!]_{M,g_x^d}) = 0$

So:

(3) for every g: for every $d \in D_M$: $[\![P(x)]\!]_{M,g_x^d} = 1$ iff

(4) for every g: for every $d \in D_M$: $\begin{bmatrix} 1 \rightarrow 0 \\ 0 \rightarrow 1 \end{bmatrix} ([\![P(x)]\!]_{M,g_x^d}) = 0$

(4) for every g: for every $d \in D_M$: $\begin{bmatrix} 1 \rightarrow 0 \\ 0 \rightarrow 1 \end{bmatrix} ([\![P(x)]\!]_{M,g_x^d}) = 0$ iff

(5) for every g: for every $d \in D_M$: $F_M(\neg)$ $([\![P(x)]\!]_{M,g_x^d}) = 0$ iff

(6) for every g: for every $d \in D_M$: $[\![\neg]\!]_{M,g_x^d}$ $([\![P(x)]\!]_{M,g_x^d}) = 0$

(6) for every g: for every $d \in D_M$: $[\![\neg]\!]_{M,g_x^d}$ $([\![P(x)]\!]_{M,g_x^d}) = 0$ iff

(7) for every g: for every $d \in D_M$: $[\![\neg P(x)]\!]_{M,g_x^d} = 0$

(7) for every g: for every $d \in D_M$: $[\![\neg P(x)]\!]_{M,g_x^d} = 0$ iff

(8) for every g: for no $d \in D_M$: $[\![\neg P(x)]\!]_{M,g_x^d} = 1$

(8) for every g: for no $d \in D_M$: $[\![\neg P(x)]\!]_{M,g_x^d} = 1$ iff [**semantics of ∃**]

(9) for every g: $[\![\exists x \neg P(x)]\!]_{M,g} = 0$

(9) for every g: $[\![\exists x \neg P(x)]\!]_{M,g} = 0$ iff

(10) for every g: $\begin{pmatrix} 1 \to 0 \\ 0 \to 1 \end{pmatrix}([\![\exists x \neg P(x)]\!]_{M,g}) = 1$

(10) for every g: $\begin{pmatrix} 1 \to 0 \\ 0 \to 1 \end{pmatrix}([\![\exists x \neg P(x)]\!]_{M,g}) = 1$ iff

(11) for every g: $F_M(\neg)([\![\exists x \neg P(x)]\!]_{M,g}) = 1$ iff

(12) for every g: $[\![\neg]\!]_{M,g}\ ([\![\exists x \neg P(x)]\!]_{M,g}) = 1$

(12) for every g: $[\![\neg]\!]_{M,g}\ ([\![\exists x \neg P(x)]\!]_{M,g}) = 1$ iff

(13) for every g: $[\![\neg \exists x \neg P(x)]\!]_{M,g} = 1$ iff

(14) $[\![\neg \exists x \neg P(x)]\!]_M = 1$

Similarly $\exists x P(x) \Leftrightarrow \neg \forall x \neg P(x)$.

Note that ∀ generalizes ∧ and ∃ generalizes ∨:
Let $D_M = \{d_1,\ldots,d_n)$ and $F_M(c_1)=d_1,\ldots,F_M(c_n)=d_n$
Then:

$[\![\forall x P(x)]\!]_{M,g} = 1$ iff $[\![P(c_1) \wedge \ldots \wedge P(c_n)]\!]_{M,g} = 1$
$[\![\exists x P(x)]\!]_{M,g} = 1$ iff $[\![P(c_1) \vee \ldots \vee P(c_n)]\!]_{M,g} = 1$

This explains the similarity between $\forall x P(x) \Leftrightarrow \neg \exists x \neg P(x)$ and the de Morgan law which says that: $(\varphi \wedge \psi) \Leftrightarrow \neg(\neg\varphi \vee \neg\psi)$.

(1) a. Every cat is smart.
    b. $\forall x[CAT(x) \rightarrow SMART(x)]$
(2) a. Some cat is smart.
    b. $\exists x[CAT(x) \wedge SMART(x)]$

Question: Does (1) entail (2)?
Answer: (1b) does not entail (2b).

Namely, assume: $F_M(CAT) = \emptyset$.

Then $[\![\exists x[CAT(x) \wedge SMART(x)]]\!]_M = 0$

But, $[\![\forall x[CAT(x) \rightarrow SMART(x)]]\!]_M = 1$ iff for every $d \in F_M(CAT)$: $d \in F_M(SMART)$,

and this is **trivially** the case:

$$[\![\forall x[CAT(x) \rightarrow SMART(x)]]\!]_M = 1$$

Hence (1b) does not entail (2b).

FACT: $\{ \forall x[CAT(x) \rightarrow SMART(x)], \exists x[CAT(x)]\} \Rightarrow \exists x[CAT(x) \wedge SMART(x)]$

So: on every model **where there are cats** and every cat is smart, there is indeed a smart cat.

Question: Why don't we make this part of the meaning?
Why don't we change the semantics of *every* to:

(1) c. $\exists x[CAT(x)] \wedge \forall x[CAT(x) \rightarrow SMART(x)]$

Answer: Because we think that *It is not the case that every cat is smart* **should be equivalent to** *some cat isn't smart.*

FACT: $\neg\forall x[CAT(x) \rightarrow SMART(x)] \Leftrightarrow \exists x[CAT(x) \wedge \neg SMART(x)]$

Namely:

$\neg\forall x[CAT(x) \rightarrow SMART(x)] \Leftrightarrow$ (as we saw above)
$\exists x\neg[CAT(x) \rightarrow SMART(x)] \Leftrightarrow$ $(\neg(\varphi \rightarrow \psi) \Leftrightarrow (\varphi \wedge \neg\psi))$
$\exists x[CAT(x) \wedge \neg SMART(x)]$

FACT: $\neg(\exists x[CAT(x) \wedge \forall x[CAT(x) \rightarrow SMART(x)]) \Leftrightarrow$
$(\neg\exists x[CAT(x)] \vee \exists x[CAT(x) \wedge \neg SMART(x)])$

So: *It is not the case that every cat is smart* would mean: either there are no cats, or some cat is not smart. And this seems too weak: if anything, you would want it to mean:

$\exists x[CAT(x)] \wedge \neg\forall x[CAT(x) \rightarrow SMART(x)]$

But this is just equivalent to: $\neg\forall x[CAT(x) \rightarrow SMART(x)]$.

(1) a. Not every picture ascribed to Rembrandt is by Rembrandt.
b. Some picture ascribed to Rembrandt is not by Rembrandt.

(the inference from 1b to 1a is trivial, the one from 1a to 1b is the relevant inference)

Not a question of knowledge, but of *fact*:

(2) We have no techniques available to tell of any picture ascribed to Rembrandt that it is not by Rembrandt, but still I claim (1a)/but still I claim (1b).

(1a) and (1b) make the same claim in this context.

Question: why don't we make it a pressupposition?
Every cat is smart **presupposes** that there are cats.

Answer: Some people do. But the more standard view is that that is too strong.

We all agree that there **is** an effect: **normally**, when we assert (1a), we commit ourselves to (2a) as well.

But the effect can be canceled:

(3) [I run a crackpot lottery, and solemnly swear in court:]
a. Every person who has come to me over the last year, has gotten a prize.
[aside:]
Fortunately, I was away on a polar expedition all year.

My statement of (3a) may be insincere, but it is not infelicitous or false.
It would be false, if *every* **entails** *some*, it would be infelicitous, if *every* **presupposes** *some*.
But it is neither, it is only insincere, because I am well aware that my statement of (3a) is **trivially true**.

With the semantics given, we can explain the effect pragmatically as an implicature:

1. My semantics is the standard semantics for *every* which does not entail *some*.
2. I obey Grice's Maxim of Quality: "Do not say what you know to be false."
So I do claim (3a) to be true.
3. But I violate part of Grice's Maxim of Quantity: "Do not give less information than is required."

I violate this, because, in fact, I knowingly give **no information at all**, because I well know that the content of my statement is **trivial**. Since I violate the maxim of quantity to **mislead** the judge and jury, I am insincere.
4. But this explains directly, why, **in normal contexts**, *every* **conversationally implicates** *some*: The maxim of Quantity entails a maxim of:

> **Avoid Triviality**: make your statement non-trivial.

We go back to (1) and (2).

> (1) Every cat is smart.
> (2) some cat is smart.

Since (1b) is trivial if there are no cats, the assumption that (1b) is asserted in accordance with Grice's maxims entails that there are cats, and this means that:

> **Even though (1b) does not entail (2b), (1b) conversationally implicates (2b)**.

And this is enough to explain the effect.

MORE EXTENDED: ENTAILMENT, PRESUPPOSITION, IMPLICATURE

ENTAILMENT?
Let p be a contingent sentence.
If φ 'implies' p and ¬φ 'implies' p then p **cannot** be an entailment of φ:

$$\begin{array}{ll} \varphi\ )\ p & \text{every model where } \varphi \text{ is true p is true} \\ \neg\varphi\ )\ p & \text{every model where } \neg\varphi \text{ is true p is true} \\ \hline & \text{p is true in every model (hence not contingent)} \end{array}$$

Every cat is smart 'implies' *there are cats*
Not every cat is smart 'implies' *there are cats*
So: *there are cats* is **not** an entailment.

PRESUPPOSITION OR IMPLICATURE?
Let ψ entail $\neg$p.
If p is a presupposition of φ, then φ is only felicitous in a context that already contains p.
This means that I **cannot** felicitously assert: $\varphi \wedge \psi$, because ψ entails $\neg$p, and φ requires p, this gives, $p \wedge \neg p$.
The conjunction test is a **test for presuppositions**:

Example:
I knew that John was rich 'implies' *John was rich*
I didn't know that John was rich 'implies' *John was rich*

John was poor entails *John was not rich*.

Check:
I knew that John was rich, even though he was poor.

If this feels **inconsistent (a contradiction),** the implication relation is **presupposition** (given that it is not entailment).
If it is **consistent**, the implication relation is **implicature** (and can, apparently, be **canceled**).

We check:

φ_1 The *one person* *who presented me with a winning lottery ticket last year* got a prize.

φ_2 The *three persons* *who presented me with a winning lottery ticket last year* got a prize.

φ_3 The *persons* *who presented me with a winning lottery ticket last year* got a prize.

φ_4 Every *person* *who presented me with a winning lottery ticket last year* got a prize.

ψ: Fortunately, I was away all year on a polar expedition.

(We assume that in the relevant context ψ entails that nobody could have presented me with a winning lottery ticket last year.)

Now we check the intuitions:

$\varphi_1 \wedge \psi$	inconsistent	*the one N*	**presupposes** $N \neq \varnothing$
$\varphi_2 \wedge \psi$	inconsistent	*the three N*	**presupposes** $N \neq \varnothing$
$\varphi_3 \wedge \psi$ `	consistent	*the Ns*	**implicates** $N \neq \varnothing$
$\varphi_4 \wedge \psi$	consistent	*every N*	**implicates** $N \neq \varnothing$

The standard theory of *every* and the Boolean theory of plurality and definites (in the version of Landman 2004) predicts these facts.

Confirmation of the facts:

φ_1 in every family, *the boy* goes into the army.
φ_2 in every family, *the three boys* go into the army.

φ_3 in every family, *the boys* go into the army.
φ_4 in every family, *every boy* goes into the army.

Data: φ_1 presupposes: *In every family, there is a boy*
φ_2 presupposes: *In every family, there are three boys*

φ_3, φ_4 do not presuppose *In every family there are boys,*
they only quantify over families in which there are boys, i.e. they mean:
In every family where there are boys, the boys go into the army.

Explanation:
Existence *Presupposition* failure leads to undefinedness, *infelicity*
Existence *Implicature* failure leads to *triviality*.

The universal quantification over families can be seen as a long conjunction:

φ_1
The boy in family 1 goes into the army $\wedge$ … $\wedge$ the boy in family n goes into the army

If in family i there are no boys, the statement *The boy in family i goes into the army* is, as we have seen above, infelicitous.
But then the whole conjunction is infelicitous, and hence φ_1 is infelicitous. hence φ_1 presupposes *In every family there is a boy*.

φ_3
Every boy in family 1 goes into the army $\wedge$ … $\wedge$ Every boy in family n goed into the army

If in family i there are no boys, the statement *Every boy in family i goes into the army* is, as we have seen above, **trivially true**.
But if φ_i is trivially true, $\varphi \wedge \varphi_I$ is equivalent to φ. Thus, the cases of families where there are no boys are truth conditionally irrelevant and drop out of the conjunction. hence φ_3 indeed only quantifies over families where there are boys.

This means that the standard theory of *every* and the boolean theory of plurality and definiteness needs to add nothing to make the right predictions here.

AVOID TRIVIALITY

1. Under quantification the triviality of $\forall x\varphi$ over boys on an empty domain guarantees, as it should, that the quantification over families is restricted in the right way.
2. In some cases we use triviality to stay within the law (tell the truth): violating quantity is not as bad as violating quality.
3. What do we get in **normal** cases?

I say *Every cat is smart*.
-You and I assume that I adhere to quality, so I am assumed to make a true statement.
-You and I assume that I adhere to quantity. Trivial statements give no information, hence violate quantity. This brings in an existence implicature; *There are cats.*

Connections between $\forall, \wedge, \vee$ and $\exists, \wedge, \vee$

(1) $\forall x[SING(x) \vee DANCE(x)]$	Everybody sings or dances
(2) $\forall x SING(x) \vee \forall x DANCE(x)$	Everybody sings or everybody dances
(3) $\forall x SING(x) \wedge \forall x DANCE(x)$	Everybody sings and everybody dances
(4) $\forall x[SING(x) \wedge DANCE(x)]$	Everybody sings and dances

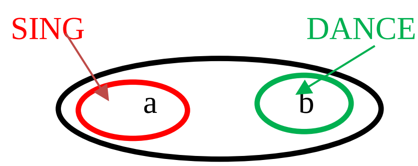
Entailment Pattern for Every(body):

(3) $\Leftrightarrow$ (4):
If everybody sings and dances, then everybody sings.
If everybody sings and dances, then everybody dances.
If everybody sings and everybody dances, then everybody sings and dances.

(3) $\Rightarrow$ (2)
This is just: $\varphi \wedge \psi \Rightarrow \varphi \vee \psi$
(2) does not entail (3): again, $\varphi \vee \psi$ does not entail $\varphi \wedge \psi$

(2) $\Rightarrow$ (1)
If everybody sings then everybody sings or dances
If everybody dances then everybody sings or dances
If $\varphi \Rightarrow \chi$ and $\psi \Rightarrow \chi$ then $(\varphi \vee \psi) \Rightarrow \chi$
Hence indeed (2) $\Rightarrow$ (1)

(1) does not entail (2).
Let $D_M = \{a,b\}$,
$F_M(SING) = \{a\}$, $F_M(DANCE) = \{b\}$.
$[\![\forall x[SING(x) \vee DANCE(x)]]\!]_M = 1$
$[\![\forall x SING(x) \vee \forall x DANCE(x)]\!]_M = 0$.



$$\begin{array}{ccc}
(1) & & \forall x[P(x) \vee Q(x)] \\
\Uparrow & & \Uparrow \\
(2) & & \forall x P(x) \vee \forall x Q(x) \\
\Uparrow & & \Uparrow \\
(3) \Leftrightarrow (4) & & \forall x P(x) \wedge \forall x Qx \Leftrightarrow \forall x[P(x) \wedge Q(x)]
\end{array}$$

(1) $\exists x[\text{SING}(x) \vee \text{DANCE}(x)]$	Somebody sings or dances
(2) $\exists x\text{SING}(x) \vee \exists x\text{DANCE}(x)$	Somebody sings or somebody dances
(3) $\exists x\text{SING}(x) \wedge \exists x\text{DANCE}(x)$	Somebody sings and somebody dances
(4) $\exists x[\text{SING}(x) \wedge \text{DANCE}(x)]$	Somebody sings and dances

Entailment Pattern for Some(body):

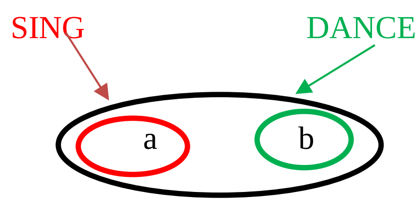
(1) $\Leftrightarrow$ (2)
If somebody sings or dances then somebody sings or somebody dances.

If somebody sings then somebody sings or dances
If somebody dances then somebody sings or dances.
If $\varphi \Rightarrow \chi$ and $\psi \Rightarrow \chi$ then $(\varphi \vee \psi) \Rightarrow \chi$
If somebody sings or somebody dances then somebody sings or dances.

(3) $\Rightarrow$ (2)
same as above $\varphi \wedge \psi \Rightarrow \varphi \vee \psi$

(4) $\Rightarrow$ (3)
If somebody sings and dances, somebody sings.
If somebody sings and dances, somebody dances.
If $\varphi \Rightarrow \psi$ and $\varphi \Rightarrow \chi$, then $\varphi \Rightarrow \psi \wedge \chi$
Hence (4) entails (3)

(3) does not entail (4).
The same model as above:



$D_M = \{a,b\}$,
$F_M(\text{SING}) = \{a\}$, $F_M(\text{DANCE}) = \{b\}$.

$[\![\exists x\text{SING}(x) \wedge \exists x\text{DANCE}(x)]\!]_M = 1$
$[\![\exists x[\text{SING}(x) \wedge \text{DANCE}(x)]]\!]_M = 0$

$$\begin{array}{ccc} (1) \Leftrightarrow (2) & \qquad & \exists x[P(x) \vee Q(x)] \Leftrightarrow \exists xP(x) \vee \exists xQ(x) \\ \Uparrow & & \Uparrow \\ (3) & & \exists xP(x) \wedge \exists xQ(x) \\ \Uparrow & & \Uparrow \\ (4) & & \exists x[P(x) \wedge Q(x)] \end{array}$$

(1) $\neg\exists x[\text{SING}(x) \vee \text{DANCE}(x)]$ Nobody sings or dances
(2) $\neg\exists x\text{SING}(x) \vee \neg\exists x\text{DANCE}(x)$ Nobody sings or nobody dances
(3) $\neg\exists x\text{SING}(x) \wedge \neg\exists x\text{DANCE}(x)$ Nobody sings and nobody dances
(4) $\neg\exists x[\text{SING}(x) \wedge \text{DANCE}(x)]$ Nobody sings and dances

Entailment Pattern for No(body):

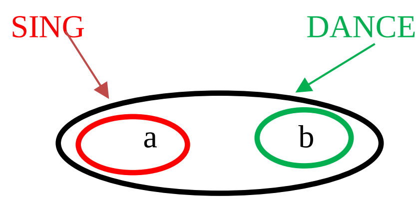
(3) $\Leftrightarrow$ (1)
If nobody sings and nobody dances, nobody sings or dances.
If nobody sings or dances, nobody sings.
If nobody sings or dances, nobody dances.

(3) $\Rightarrow$ (2)
Same as above.

(2) $\Rightarrow$ (4)
If nobody sings or nobody dances, nobody sings and dances.
Assume that nobody sings or nobody dances.
There are three cases:
-nobody sings. In that case obviously nobody sings and dances.
-nobody dances. Also nobody sings and dances.
-nobody sings and nobody dances. The same.

(4) does not entail (2)
The same model:

$D_M = \{a,b\}$,
$F_M(\text{SING}) = \{a\}$, $F_M(\text{DANCE}) = \{b\}$.



(4) is true, since a sings but doesn't dance and b dances but doesn't sing.
(2) is false: it is not the case that nobody sings (since a sings) and it is not the case that nobody dances (since b dances). Hence it is not the case that nobody sings or nobody dances.

(4)	$\neg\exists x[P(x) \wedge Q(x)$
$\Uparrow$	$\Uparrow$
(2)	$\neg\exists xP(x) \vee \neg\exists xQ(x$
$\Uparrow$	$\Uparrow$
(3) $\Leftrightarrow$ (1)	$\neg\exists xP(x) \wedge \neg\exists xQ(x) \Leftrightarrow \neg\exists x[P(x) \vee Q(x)]$

Generalize:

(1) NP sing or dance.
(2) NP sing or NP dance.
(3) NP sing and NP dance.
(4) NP sing and dance.

We saw above that **everybody**, **somebody**, **nobody** have different characteristic patterns. If you try other noun phrases you find that their patterns differ:

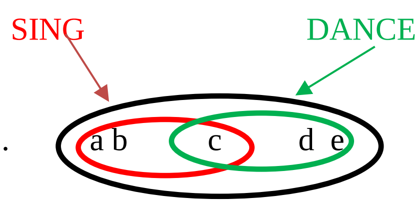
most boys

(1) **Most boys** sing or dance.
(2) **Most boys** sing or **most boys** dance.
(3) **Most boys** sing and **most boys** dance.
(4) **Most boys** sing and dance.

(4) ⇒ (3)
If most boys sing and dance, more than half of the boys sing and dance.
Then more than half of the boys sing and more than half of the boys dance.

(3) does not entail (4)

D_M = {a,b,c,d,e}
F_M(BOY) = {a,b,c,d,e}
F_M(SING) = {a,b,c}, F_M(DANCE) = {c,d,e}.



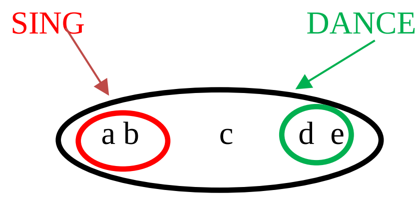
In this model more than half of the boys sing, since {a,b,c} is more than half of {a,b,c,d,e}
Also more than half of the boys dance, since {c,d,e} is more than half of {a,b,c,d,e}.
But less than half of the boys sing and dance, since {c} is less than half of {a,b,c,d,e}.

As usual (3) ⇒ (2).

(2) ⇒ (1)
Assume (2) is true.
There are again three cases:
-More than half of the boys sing.
Since everybody who sings sings or dances, it follows that more than half of the boys sing or dance.
- More than half of the boys dance. A similar argument.
-More than half of the boys sing and more than half of the boys dance. The same argument.

(1) does not entail (2)

$D_M = \{a,b,c,d,e\}$
$F_M(BOY) = \{a,b,c,d,e\}$
$F_M(SING) = \{a,b\}$, $F_M(DANCE) = \{d,e\}$.



(1) is true, since the set of singers together with the set of dancers {a,b,d,e} is more than half of {a,b,c,d,e}.
(2) is false, since the set of singers {a,b} is less than half of the boys, and the set of dancers {d,e} is less than half of the boys.

(1)	Most boys sing or dance
$\Uparrow$	$\Uparrow$
(2)	Most boys sing or most boys dance
$\Uparrow$	$\Uparrow$
(3)	Most boys sing and most boys dance
$\Uparrow$	$\Uparrow$
(4)	Most boys sing and dance

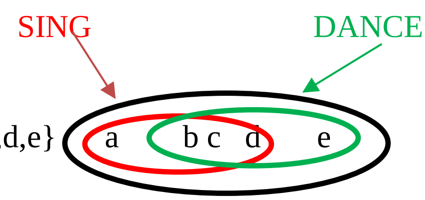
Exactly three boys

(1) **Exactly three boys** sing or dance.
(2) **Exactly three boys** sing or **exactly three boys** dance.
(3) **Exactly three boys** sing and **exactly three boys** dance.
(4) **Exactly three boys** sing and dance.

Here we find only the obvious entailment from (3) to (2), all the others are logically independent.

(4) does not entail (1), (4) doesn't entail (2), (4) doesn't entail (3):

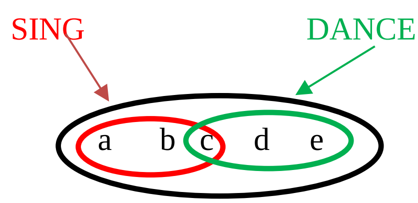
$D_M = \{a,b,c,d,e\}$
$F_M(\text{BOY}) = \{a,b,c,d,e\}$
$F_M(\text{SING}) = \{a,b,c,d\}$, $F_M(\text{DANCE}) = \{b,c,d,e\}$



(4) is true, (1) is false, (2) is false, (3) is false.

(3) does not entail (1), (3) doesn't entail (4)

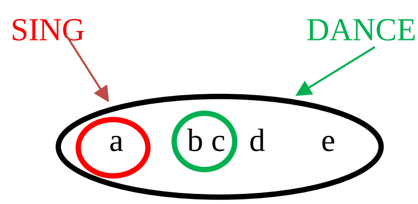
$D_M = \{a,b,c,d,e\}$
$F_M(\text{BOY}) = \{a,b,c,d,e\}$
$F_M(\text{SING}) = \{a,b,c\}$, $F_M(\text{DANCE}) = \{c,d,e\}$



(3) is true, (1) is false, (4) is false.

(1) doesn't entail (2), (1) doesn't entail (3), (1) doesn't entail (4):

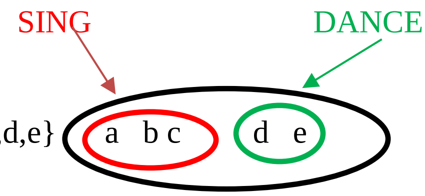
$D_M = \{a,b,c,d,e\}$
$F_M(\text{BOY}) = \{a,b,c,d,e\}$
$F_M(\text{SING}) = \{a,\}$, $F_M(\text{DANCE}) = \{b,c\}$



(1) is true, (2) is false, (3) is false, (4) is false.

(2) doesn't entail (1), (2) doesn't entail (3), (2) doesn't entail (4)

$D_M = \{a,b,c,d,e\}$
$F_M(BOY) = \{a,b,c,d,e\}$
$F_M(SING) = \{a,b,c,d\}$, $F_M(DANCE) = \{b,c,d,e\}$



$F_M(SING) = \{a,b,c\}$, $F_M(DANCE) = \{d,e\}$.
(2) is true, (1) is false, (3) is false, (4) is false.

(1) Exactly three boys sing or dance

$\not\Uparrow \quad \not\Downarrow$

(4) Exactly three boys sing and dance

$\not\Downarrow \quad \not\Uparrow$

(2) Exactly three boys sing or exactly three boys dance

$\Uparrow$

(3) Exactly three boys sing and exactly three boys dance

Inverse logic: if you're not sure whether an expression in a language means , say, **every** or **most,** check how that expression interacts with $\wedge$ and $\vee$. The characteristic pattern will tell you.

Gavagai boys sing or/and dance.

Also, wrt. expressions in different categogies, like adverbials.

Example:
You are a linguist who is doing fieldwork on a peculiar West-Germanic language, and you have discovered an adverb transscribed in your notes as *telkens*, pronounced *telkəns*. You have a hunch that it might be a quantificational adverb, and you want to find out its meaning. You set up the following test:

(1) Telkens als het regent trek ik mijn regenpak aan of steek ik mijn paraplu op.
Telkens when it rains I put on my rainsuit or put up my umbrella.
(2) Telkens als het regent trek ik mijn regenpak aan of telkens als het regent steek ik mijn paraplu op.
Telkens when it rains I put on my rainsuit or
telkens when it rains I put up my umbrella.
(3) Telkens als het regent trek ik mijn regenpak aan en telkens als het regent steek ik mijn paraplu op.
Telkens when it rains I put on my rainsuit and
telkens when it rains I put up my umbrella
(4) Telkens als het regent trek ik mijn regenpak aan en steek ik mijn paraplu op.
Telkens when it rains I put on my rainsuit and put up my umbrella.

You ask your informants for judgements concerning entailments and you find:

$$\begin{array}{c} (1) \\ \Uparrow \\ (2) \\ \Uparrow \\ (3) \Leftrightarrow (4) \end{array}$$

You conclude that *telkens* patterns like *everyone*, and hence is a universal quantifier.

Note: there may be more than one universal quantifier. Dutch has *altijd* and *telkens*, they are both universal quantifiers. Similarly, *every* and *each* are both universal quantifiers. They pattern alike on the above tests, but other tests distinguish between them (we talk about distributivity and collectivity later).