tells me that in the perhaps one dozen colloquia on the device that he has presented in recent years, he has been presented with several different attempts at alternative coding schemes after the lectures. "It's great fun," he adds, "convincing their inventors that they can't work. At times I've had large groups of people, all shouting at each other, making tables, diagrams, etc." The only special merit of the examples mentioned in Ref. 28 is that they were thought of by students.

[28]Suppose now, the student argued, that we change the coding/detection scheme by reducing the critical angle to something less than 45°. Would this improve the performance of the model by raising the proportion of times that unequal colors are obtained on different switch settings? Curiously, the answer is no. [See Fig. 2(b)]. The proportion of 33% $GG + RR$, 67% $GR + RG$, is unchanged by changing this angle. In addition, while the switch settings are treated symmetrically, green and red are now on an unequal footing. This coding scheme is inferior even to the triplet model. Another student came up with a quadruplet color coding model, in which each photon carries four colors, some constrained to be unequal. An elaborate set of logic gates was devised to choose three for the final determination. Another suggestion again was the encoding of each photon with three numbers, say, 0, 1, 2 in any order, the order defining the choice of code; this triplet of numbers was flung up against a similar triplet of different numbers in the detector (say, 1, 2, 3), whose order was controlled by the switch setting. The choice of color was then determined from these numbers by an algorithm whose details are arbitrary, except that the manifest symmetries should be satisfied. For example, the products in pairs could be added, and divided by a prime; the remainder then determining the color on some appropriate convention.

[29]The original Bell inequality held only within the triplet coding scheme. Clauser and others (see Refs. 14, 15, and 18) produced a more general inequality, relating only to observables, and clearly violated by quantum theory (and experiment). Its proof is also easy (Ref. 14), but still involves a level of generality which obscures the simplicity of the present case. It shows that a device with *two* position switches would be adequate, although less symmetrical.

[30]It is at this point that the comment of Ref. 3 has its maximal impact.

On the motion of an ice hockey puck

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Some easily observed, but surprising properties, of a homogeneous, circular ring or disk sliding on a smooth horizontal surface under the action of friction are pointed out and discussed.

I. INTRODUCTION

Textbook problems dealing with friction are usually not very exciting and seldom lead to unexpected results. We therefore think that it may be of interest to point out some properties of the motion of a circular ring or disk sliding on a smooth, horizontal surface under the action of friction, which are often found to be surprising and even paradoxical and therefore motivating for a theoretical investigation.

The following properties may be observed in the motion of an ice hockey puck sliding on a horizontal ice surface and may easily be demonstrated on a table top with any circular, homogeneous ring or disk at hand:

(a) When started with a pure translation, the disk will continue in a pure rectilinear translation without being set into rotation, until it comes to rest.

(b) Likewise, when started as a pure rotation around its center, the disk will continue to rotate while the center remains at rest, until it stops.

(c) The time it takes for the disk to come to rest will in both cases increase with increasing value of the initial velocity.

(d) When started with a combination of translation and rotation, the center will continue in a rectilinear motion, until:

(e) The translation and the rotation come to a simultaneous stop, regardless of the initial velocities.

The first four properties, (a)–(d), will usually not seem very surprising at first glance. The last property, (e), will, however, often be found surprising because one, on the basis of the first four properties, will be inclined to expect the stopping times for translation and rotation generally to be different. This false expectation clearly stems from, and would be justified under, the assumption that the translational and rotational motions are independent. One is easily induced, more or less consciously, to this assumption because it would account for the observed properties (a), (b), and (d). In Sec. II, however, it is seen from the equations of motion that the assumption of independence does not hold, but that the translational and rotational motions are strongly coupled. It is also shown, by a simple argument, that the effect of the coupling is to increase the stopping times for both translation and rotation. But why these increases in stopping times should be such that they always become equal remains an open question until it is treated in Sec. VI.

Having realized the coupling between the translational and rotational motions, the properties (a), (b), and (d) will no longer seem obvious, but will call for an explanation. We shall, therefore, first, in Secs. III–V, establish the conditions under which these properties should hold.

In Sec. VI we shall show the equality of stopping times for translation and rotation in a composite motion. In this connection we shall show the existence of a particular, rollinglike motion which also is approached in the final stages of all composite motions, regardless of initial velocities.

II. BASIC EQUATIONS

We shall generally consider a thin slate of arbitrary shape and distribution of mass that slides on a smooth, horizontal surface. We assume that on each infinitesimal element of the slate there acts a force of friction

$$d\mathbf{F} = -\mu g dm \hat{\mathbf{u}}, \tag{1}$$

where μ is the coefficient of friction, g is the acceleration of gravity, dm is the mass of the element, and $\hat{\mathbf{u}}$ is the unit vector in the direction of its velocity $\mathbf{u}$. The velocity $\mathbf{u}$ may be expressed as

$$\mathbf{u} = \mathbf{v} + \boldsymbol{\omega}\times\mathbf{r}, \tag{2}$$

where $\mathbf{v}$ is the velocity of the center of mass, $\boldsymbol{\omega}$ is the angular velocity of the slate, and $\mathbf{r}$ is the position vector of the element from the center of mass. Newton's 2nd law then gives

$$\mathbf{F} = \int d\mathbf{F} = -\mu g \int \hat{\mathbf{u}} dm = m\dot{\mathbf{v}}, \tag{3}$$

where m is the mass of the slate. The law of angular momentum likewise gives

$$\boldsymbol{\tau} = \int \mathbf{r}\times d\mathbf{F} = -\mu g \int \mathbf{r}\times\hat{\mathbf{u}} dm = I\dot{\boldsymbol{\omega}}, \tag{4}$$

where $\boldsymbol{\tau}$ is the torque of the forces of friction around the center of mass and I is the moment of inertia around the axis through this point.

When we substitute Eq. (2) into Eqs. (3) and (4) we see easily that the translational and rotational motions are strongly coupled. We realize also that F is maximal in a pure translation and that τ is maximal in a pure rotation. Since the stopping times for translation and rotation must decrease with increasing values of F and τ, respectively, it follows that in a composite motion the stopping times must be larger than in the corresponding cases of pure translation and pure rotation. But, it is not clear why the increases in stopping times should be such that they always become equal to each other. We shall return to this question in Sec. VI.

Another consequence of the coupling is that the properties (a), (b), and in particular (d), no longer seem obvious. Why does, for instance, the center of the disk move in a straight line, even when the disk is rotating? We shall therefore establish the conditions under which these properties should hold before we return to the question of the equality of stopping times.

III. PURE TRANSLATION

If we put $\boldsymbol{\omega} = 0$ in the expression $\mathbf{F}$ and $\boldsymbol{\tau}$ in Eqs. (3) and (4) and use the equation $\int \mathbf{r}\, dm = 0$, we find $\dot{\omega} = 0$ and

$$\dot{\mathbf{v}} = -\mu g \hat{\mathbf{v}}, \tag{5}$$

where $\hat{\mathbf{v}}$ is the unit vector in the direction of $\mathbf{v}$. We see that a slate that starts with a pure translation will continue with a pure, rectilinear translation without being put into rotation [property (a)]. From Eq. (5) we find for the stopping time

$$T = v_0/(\mu g), \tag{6}$$

where v_0 is the initial velocity.

It should be noted that no assumptions on the shape or distribution of mass of the slate have been made.

IV. PURE ROTATION

If we put $\mathbf{v} = 0$ in the expression for $\mathbf{F}$ in Eq. (3), we find

$$\mathbf{F} = -\mu g \hat{\boldsymbol{\omega}} \times \int \hat{\mathbf{r}}\, dm = m\dot{\mathbf{v}}, \tag{7}$$

where $\hat{\boldsymbol{\omega}}$ and $\hat{\mathbf{r}}$ are unit vectors in the directions of $\boldsymbol{\omega}$ and $\mathbf{r}$, respectively. The force is seen to vanish if

$$\int \hat{\mathbf{r}} dm = 0. \tag{8}$$

A motion that is started as a pure rotation around the center of mass will in this case remain a pure rotation until it stops. The property (b) will therefore hold for the restricted class of slates defined by Eq. (8). This class includes homogeneous, circular rings and disks and all slates with rotational symmetry. It also includes thin, straight rods, and quite generally it consists of slates where the distribution of mass is such that a displacement of its elements in the direction of the center of mass to the same distance from it will leave the center of mass unmoved.

One might ask whether a pure rotation around a fixed point other than the center of mass is possible. In this case, the center of mass would have to move in a circle, which would require a centripetal force equal to $m\omega^2\rho$, where ρ is the radius of the circle. We see that as the angular velocity would decrease due to friction, so would the centripetal force have to decrease. But, it is clear from Eq. (1) that the force of friction is independent of ω and this force will therefore not be able to sustain the motion. A pure rotation will therefore be possible only around the center of mass or not at all.

If we put $\mathbf{v} = 0$ in the expression for $\boldsymbol{\tau}$ in Eq. (4), we find

$$\dot{\omega} = -\left(\frac{\mu g}{I}\right) \int r dm. \tag{9}$$

For a homogeneous, circular ring with radius R, when we introduce $w = \omega R$, this gives

$$\dot{w} = \dot{\omega} R = -\mu g. \tag{10}$$

This gives the stopping time for a pure rotation

$$T = w_0/(\mu g), \tag{11}$$

where w_0 is the inital value of w. Comparison with Eq. (6) shows that the stopping times for translation and rotation are equal when the initial values v_0 and w_0 are equal.

For a homogeneous disk we find likewise that

$$\dot{w} = -(\tfrac{4}{3})\mu g, \tag{12}$$

which gives the stopping time

$$T = (\tfrac{3}{4}) w_0/(\mu g). \tag{13}$$

V. RECTILINEARITY OF TRANSLATION IN COMPOSITE MOTION

We consider first a homogeneous, circular ring which at a given time has an angular velocity ω in the clockwise direction and a translational velocity v in the x direction of Fig. 1. From Eq. (3) we find the component of F in the y direction to be zero:

$$F_y = \frac{\mu g}{2\pi} \int_0^{2\pi} w \sin\varphi\, (v^2 + w^2 + 2vw\cos\varphi)^{-1/2}\, d\varphi = 0, \tag{14}$$

i.e., that the center will continue to move in the x direction.

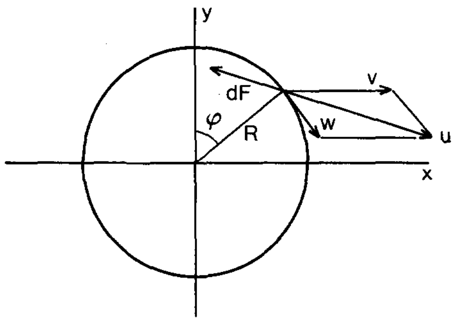

Fig. 1. The force of friction dF on an infinitesimal element of a circular ring.

The same result will clearly hold for all slates with a rotationally symmetric distribution of mass, including homogeneous disks. The rotational symmetry is essential for this result. Consider, for instance, a thin, homogeneous rod with a length $2R$ that at a given instant makes an angle φ with the y axis while the center has a velocity v in the x direction. We find in this case

$$F_y = \frac{m}{2R}\mu g \sin\varphi \int_{-R}^{R} dr\, \omega r \times (v^2 + \omega^2 r^2 + 2v\omega r \cos\varphi)^{-1/2}, \tag{15}$$

which is seen to vanish only when $\varphi = 0, \pi/2, \pi$, or $3\pi/2$. The center of a rotating rod will therefore not move along a straight line. The class of slates for which property (d) holds is therefore even more restricted than the class for which property (b) holds.

VI. EQUALITY OF STOPPING TIMES AND FINAL STAGES OF COMPOSITE MOTIONS

We shall now return to the question of the equality of stopping times in composite motions and will first consider the homogeneous, circular ring. Equations (3) and (4) give for this case

$$\frac{F}{m} = -\frac{\mu g}{2\pi}\int_0^{2\pi} (v + w\cos\varphi)(v^2 + w^2 + 2vw\cos\varphi)^{-1/2}\, d\varphi = \dot v \tag{16}$$

and

$$\frac{\tau}{mR} = -\frac{\mu g}{2\pi}\int_0^{2\pi} (w + v\cos\varphi)(v^2 + w^2 + 2vw\cos\varphi)^{-1/2}\, d\varphi = \dot w. \tag{17}$$

We shall consider the ratio w/v and have generally

$$\frac{d}{dt}\left(\frac{w}{v}\right) = \left(\frac{\dot w}{\dot v} - \frac{w}{v}\right)\frac{\dot v}{v}. \tag{18}$$

By means of Eqs. (16) and (17) we get

$$v\left(\frac{d}{dt}\right)\left(\frac{w}{v}\right) = \left(\frac{\dot w}{\dot v} - \frac{w}{v}\right)\dot v = \frac{\mu g}{2\pi}\left[\left(\frac{w}{v}\right)^2 - 1\right]\int_0^{2\pi} \cos\varphi \times \left[1 + \left(\frac{w}{v}\right)^2 + 2\left(\frac{w}{v}\right)\cos\varphi\right]^{-1/2} d\varphi. \tag{19}$$

We see from Eq. (19) that w/v will be stationary if $w/v = 1$. Kinematically this motion may be described as a pure rolling of the ring along a tangent line. From Eqs. (16) and (17) we find in this case that

$$\dot v = \dot w = -(2/\pi)\mu g, \tag{20}$$

which gives the stopping time

$$T = \frac{\pi}{2}\frac{v_0}{(\mu g)} = \frac{\pi}{2}\frac{w_0}{(\mu g)}, \tag{21}$$

where $v_0 = w_0$ are the initial velocities of v and w. The stopping time is seen to be larger than the corresponding stopping times for the cases of pure translation and pure rotation given by Eqs. (6) and (11), respectively, in accordance with the general result in Sec. II.

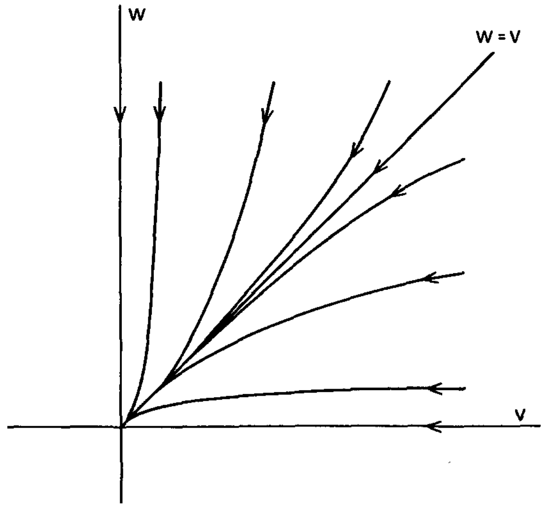

Fig. 2. The rotation velocity $w = \omega r$ as a function of the translation velocity v for various initial conditions.

We find generally the integral in Eq. (19) to be negative when we assume v and w to be positive. Thus we get that $d(w/v)/dt \gtrless 0$ according to whether $w/v \lessgtr 1$. It follows that if $w_0/v_0 < 1$, w/v will increase but not exceed unity. Since w and v both decrease with time, it follows that they must vanish simultaneously, as illustrated in Fig. 2. A corresponding argument holds when $w_0/v_0 > 1$. It follows that for all composite motions the translational and rotational motions will stop simultaneously, regardless of initial velocities.

It is interesting to consider the final stages of composite motions. Since v and w approach zero simultaneously, we have

$$\lim (w/v) = \lim (\dot w/\dot v). \tag{22}$$

Thus by taking the limit of Eq. (19) when $v,w \to 0$, we get the result

$$\lim (w/v) = 1, \tag{23}$$

which shows that in the final stage all composite motions will approach the pure rollinglike motion described earlier.

In Fig. 2 w is shown as a function of v for various initial velocities, based upon numerical integration of the following, which is obtained from Eqs. (16) and (17):

$$\frac{dw}{dv} = \left\{\int_0^{2\pi}\left[\left(\frac{w}{v}\right) + \cos\varphi\right]\left[1 + \left(\frac{w}{v}\right)^2 + 2\left(\frac{w}{v}\right)\cos\varphi\right]^{-1/2} d\varphi\right\}\left\{\int_0^{2\pi}\left[\left(\frac{v}{w}\right) + \cos\varphi\right] \times \left[1 + \left(\frac{v}{w}\right)^2 + 2\left(\frac{v}{w}\right)\cos\varphi\right]^{-1/2} d\varphi\right\}^{-1}. \tag{24}$$

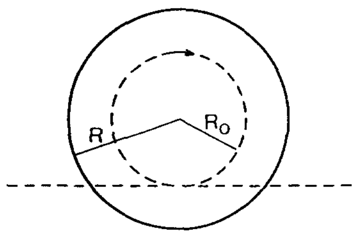

Fig. 3. The rollinglike motion of an ice hockey puck. $R_0 \simeq 0.65\ R$.

The projective character and the (v,w) symmetry of Eq. (24) and the integral curves should be noted. The curves $v = 0$, $w = 0$, and $w = v$ correspond to the cases of pure rotation, pure translation, and pure rolling, respectively.

A similar treatment of the homogeneous disk is given in the Appendix and leads to similar results. In particular, it is shown that the stopping times for the translational and rotational motions also in this case will be equal and that the ratio w/v approaches a stationary value $\alpha \simeq 1.53$. This motion may be described as a pure rolling of a circle with a radius $R_0 = R/\alpha \simeq 0.65\ R$ along a tangent line (Fig. 3).

The equality of stopping times may also be shown by an alternative argument of a more general character. The argument is based upon the familiar uniqueness property of the dynamical equations: that initial conditions that specify the position and velocities of a mechanical system will normally be sufficient for a unique determination of the future, as well as past, motion of the system. Let us consider a composite motion of a slate and assume that its rotation were to stop at a given time t_0, before the translation stops. The slate would then at the time t_0 be in a state of pure translation with velocities $\omega_0 = 0$ and $v_0 \neq 0$. We may consider this state as an initial state, but we know from our previous treatment in Sec. III that a pure translation before and after t_0 will satisfy the dynamical equations as well as the "initial" conditions. According to the uniqueness property the former, assumed motion will be impossible and the rotation can therefore not stop before the translation stops. By a similar argument it may be shown that neither can the translation stop before the rotation stops. The stopping times must therefore be equal.

An objection to this argument is that it clearly does not hold for the "initial" condition $v_0 = \omega_0 = 0$ since this is, in fact, the final state of all motions. Ought we not to expect the argument to fail also for initial states where v_0 *or* ω_0 vanish? The answer is that the force of friction and its torque have a discontinuity at the point $v = \omega = 0$, which violates the conditions for the uniqueness property to hold. At all other points, including the axes $v = 0$ and $\omega = 0$, the force and the torque are continuous and satisfy the uniqueness conditions. If one, on the other hand, again makes the false assumption that the translational and rotational motions are independent, there will be discontinuities also on the axes and our argument would fail. The coupling therefore plays an essential role also in this argument for the equality of stopping times.

VII. CONCLUSION

The seemingly paradoxical character of the properties (a)–(d) is seen to be caused by the easily induced, but false assumption that the translational and rotational motions of the disk are independent. The properties have been found to be in full agreement with the dynamical equations, and their range of validity has been established. In addition, we have shown the existence of a pure rollinglike motion and that this motion is approached in the final stages of all composite motions.

APPENDIX

For the case of the homogeneous disk, Eq. (3), when we introduce polar coordinates (r,φ) gives

$$\dot v = -\frac{\mu g}{\pi R^2}\int_0^{2\pi} d\varphi \int_0^R dr\, r(v+\omega r\cos\varphi)^{-1/2}.\times(v^2+\omega^2 r^2+2v\omega r\cos\varphi)^{-1/2}. \quad \text{(A1)}$$

Likewise, Eq. (4) gives

$$\dot\omega = -\frac{2\mu g}{\pi R^4}\int_0^{2\pi} d\varphi \int_0^R dr\, r^2(\omega r+v\cos\varphi)\times(v^2+\omega^2 r^2+2v\omega r\cos\varphi)^{-1/2}. \quad \text{(A2)}$$

Introducing $w = \omega R$ and $s = \omega r/v$, Eq. (A1) may be written as

$$\dot v = -\left(\frac{\mu g}{\pi}\right)\left(\frac{v}{w}\right)^2 G\left(\frac{w}{v}\right), \quad \text{(A3)}$$

where

$$G\left(\frac{w}{v}\right) = \int_0^{2\pi} d\varphi \int_0^{w/v} ds\, s(1+s\cos\varphi)\times(1+s^2+2s\cos\varphi)^{-1/2} \quad \text{(A4)}$$

Likewise, we may write Eq. (A2) as

$$\dot w = -\left(\frac{2\mu g}{\pi}\right)\left(\frac{v}{w}\right)^3 H\left(\frac{w}{v}\right), \quad \text{(A5)}$$

where

$$H\left(\frac{w}{v}\right) = \int_0^{2\pi} d\varphi \int_0^{w/v} ds\, s^2(s+\cos\varphi)\times(1+s^2+2s\cos\varphi)^{-1/2}. \quad \text{(A6)}$$

From Eqs. (A3) and (A5) we find

$$v\left(\frac{d}{dt}\right)\left(\frac{w}{v}\right) = \left(\frac{\dot w}{\dot v} - \frac{w}{v}\right)\dot v = \left(\frac{\mu g}{\pi}\right)\left(\frac{v}{w}\right)\times\left[G\left(\frac{w}{v}\right) - 2\left(\frac{v}{w}\right)^2 H\left(\frac{w}{v}\right)\right]. \quad \text{(A7)}$$

We find that w/v has a stationary value α defined by

$$G(\alpha) - 2\alpha^{-2}H(\alpha) = 0, \quad \text{(A8)}$$

which gives numerically $\alpha \simeq 1.53$. The motion may be described as a pure rolling of a circle with a radius $R_0 = R/\alpha \simeq 0.65\ R$ and concentric with the disk, along a tangent line. From Eqs. (A3) and (A5) we find in this case

$$\dot v = \alpha^{-1}\dot w = -(\beta/\pi)\mu g, \quad \text{(A9)}$$

where

$$\beta = \alpha^{-2}G(\alpha) = 2\alpha^{-4}H(\alpha) \simeq 1.93. \quad \text{(A10)}$$

This gives the stopping time

$$T = \frac{\pi}{\beta}\frac{v_0}{\mu g} = \frac{\pi}{\alpha\beta}\frac{w_0}{\mu g}, \quad \text{(A11)}$$

where v_0 and w_0 are the initial velocities. The stopping time is seen to be larger than the stopping times for pure translation and pure rotation given by Eqs. (6) and (13), respectively, in accordance with the result of Sec. II.

It may further be shown that

$$G\left(\frac{w}{v}\right) - 2\left(\frac{v}{w}\right)^2 H\left(\frac{w}{v}\right) \gtrless 0, \tag{A13}$$

according to whether $w/v \lesseqgtr \alpha$. If w and v are positive, it then follows from Eq. (A7) that $d(w/v)dt \gtrless 0$ according to whether $(w/v) \lesseqgtr \alpha$. By the same argument as we used in the case of the ring in Sec. VI, it then follows that for all composite motions of the disk, the stopping times for the translational and the rotational motions must be equal, regardless of the initial velocities. It also follows by the same argument that all composite motions in the final stage will approach a motion which is a pure rolling of a circle of radius $\simeq 0.65\ R$ along a tangent line.

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The principle of equivalence and a theory of gravitation

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We examine a well-known thought experiment often used to explain why we should expect a ray of light to be bent by gravity; according to this the light bends downward in the gravitational field because this is just what an observer would see if there were no field and he were accelerating upward instead. We show that this description of the action of Newtonian gravity in a flat space-time corresponds to an old two-index symmetric tensor field theory of gravitation.

I. INTRODUCTION

There is a well-known thought experiment which shows why we should expect a ray of light to be bent by gravity. This elegant argument, which might be called the accelerating-elevator thought experiment, appears in the popular book by Einstein and Infeld, *The Evolution of Physics*;[1] it seems to be one of the most frequently reproduced in physics, both in the technical literature and in elementary expositions. According to the argument, light bends in the gravitational field because this is just what an observer would see if there were no gravitational field and if he were accelerating upward instead. The bending of the light appears as an abberation effect. Since the background of this thought experiment is flat space-time, it is not the complete explanation as given by general relativity nor, presumably, was it meant to be. Because of its ubiquity, however, it is interesting to inquire if the argument is equivalent to some field theory of gravitation in flat space-time, and, if there is such a theory, just what its predictions are, for example, for the three main tests of general relativity.

In this paper we analyze the motion of a relativistic particle in a Newtonian gravitational field by using the principle of equivalence in this way; we obtain the equations of motion of a relativistic particle by examining what happens as the particle moves across a differential-sized accelerating elevator, and we show that the thought experiment is equivalent to the description of gravitation by means of a two-index symmetric tensor field theory. The Lorentz covariant equations of motion we obtain resemble the linearized geodesic equations of motion of general relativity. Many theories of gravitation have been proposed over the years as model theories or as rivals of general relativity. The particular theory which arises here is, in fact, one originally proposed in 1942 by Birkhoff.[2]

In Sec. II we consider the Kepler problem for a relativistic particle of rest mass $m \geqslant 0$, carry out the equivalence-principle argument quantitatively, and obtain the differential equations of motion of the particle. In Sec. III we obtain the corresponding Lorentz covariant field theory of gravitation. In Sec. IV we comment briefly on the theory.

For the so-called three main tests of general relativity, namely the gravitational red shift, the deflection of light by gravity, and the advance of the planetary perihelion, the theory gives, respectively, the same result, one-half the result, and one-third the result of general relativity. (In connection with these predictions, however, see Birkhoff's original paper and the remarks in Sec. IV.)

That such results for the main tests can be obtained using special relativity seems to be part of the folklore of the subject, but we do not think that the appearance of symmetric two-index tensor fields in the description of gravitation is generally known to arise so naturally, short of general relativity.

II. THE EQUATIONS OF MOTION

We consider the motion of a particle of rest mass $m > 0$ in the gravitational field of the sun. By symmetry, the motion is in a plane passing through the center of the sun, and we take polar coordinates r, θ in this plane with this point as origin. We are interested in the bending of the path of the particle as it moves from a point $Q = (r,\theta)$ to the point $Q' = (r + dr, \theta + d\theta)$. Let $d\sigma$ be the distance from Q to Q'.