Contribution to the discussion of the problem 05.96 - Motion of hockey puck

To simplify the problem we use the following dimensionless quantities:

$$V = \frac{v}{\sqrt{g.R}} \;;\; \Omega = \frac{\omega}{\sqrt{g/R}} \;;\; T = \frac{t.\mu}{\pi\sqrt{R/g}} \tag{1}$$

where v is the puck's velocity, ω - its angular velocity, R – the puck's radius, g – the acceleration of gravity and μ the friction coefficient. By considering the puck as a thin disk and assuming that the reaction forces between the puck and the ice are uniformly distributed on the surface of the puck, we can easily derive the equation of motion:

$$\frac{dV}{dT} = -F(x) \;;\; \frac{d\Omega}{dT} = -G(x) \tag{2.1}$$

where:

$$x = \frac{V}{\Omega}$$

$$F(x) = \int_{-\pi/2}^{\pi/2}\int_{0}^{1} \frac{(x - r.\sin(\varphi)).r}{\sqrt{x^2 - 2x.r.\sin(\varphi) + r^2}}.dr.d\varphi \tag{2.2}$$

$$G(x) = 2\int_{-\pi/2}^{\pi/2}\int_{0}^{1} \frac{(r - x.\sin(\varphi)).r^2}{\sqrt{x^2 - 2x.r.\sin(\varphi) + r^2}}.dr.d\varphi$$

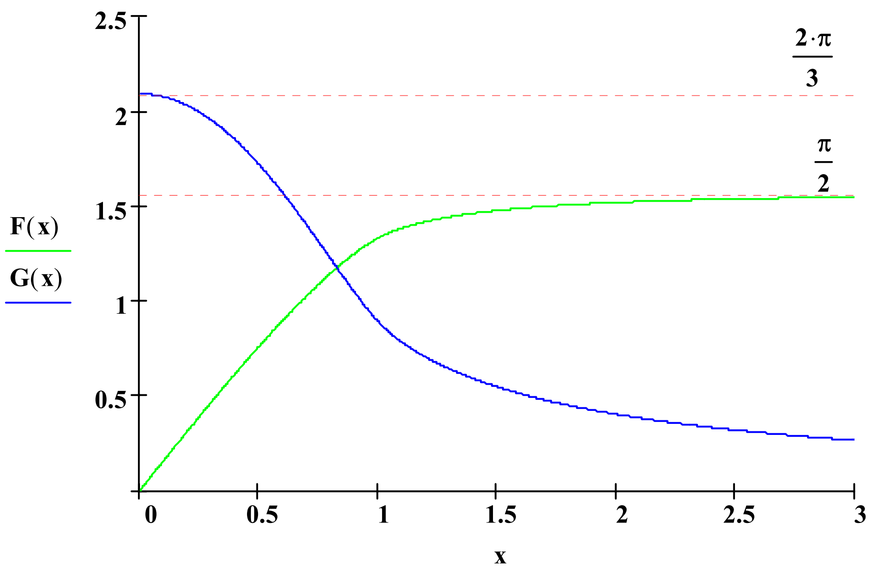

Fig. 1

The graphics of F(x) and G(x) are presented in Fig. 1 and we would like to emphasize that these functions are smooth, monotonic and strictly positive (all these facts follow from the theory of integral depending on parameter). Indeed differentiating under the sign of integrals:

$$\frac{dF(x)}{dx} = \int_{-\pi/2}^{\pi/2}\int_{0}^{1} \frac{r^3.\cos^2(x)}{\sqrt{x^2 - 2x.r.\sin(\varphi) + r^2}^{\,3}}.dr.d\varphi$$

$$\frac{dG(x)}{dx} = -\int_{-\pi/2}^{\pi/2}\int_{0}^{1} \frac{r^3.x.\cos^2(x)}{\sqrt{x^2 - 2x.r.\sin(\varphi) + r^2}^{\,3}}.dr.d\varphi \tag{3}$$

we can see that F(x) is increasing function while G(x) decreases with x. Additionally we have:

$$F(0)=0; \quad \lim_{x\to+\infty} F(x)=\frac{\pi}{2}$$
$$G(0)=\frac{2.\pi}{3}; \quad \lim_{x\to+\infty} G(x)=0 \tag{4}$$

which completes the proof.

Now we would like to show that neither V nor Ω could became zero at a moment when other velocity is positive (of course we suppose that the initial velocities are positive and as it follows from (2) decreasing functions of time). By dividing the 2 equations in (2.1) we have:

$$\frac{dV}{d\Omega}=\frac{F(x)}{G(x)} \tag{5}$$

where naturally V is considered as a function of Ω. If $V=0$ at the moment $\overline{\Omega}>0$, we have $\frac{dV}{d\Omega}(\overline{\Omega})=0$ which is contradictory to the theorem of uniqueness of the solution of ordinary differential equation with regular right-hand side (apart from the supposed solution we have also the trivial solution V = 0). Similar considerations prove the it is not possible to have $\Omega=0$ for V > 0.

At that point the only possibility we have is that the two velocities become zero simultaneously for finite or infinite time interval. For more precise analysis lets change the variable:

$$\frac{dx}{d\Omega}=\frac{d}{d\Omega}\left[\frac{V}{\Omega}\right]=\frac{\left(\frac{dV}{d\Omega}\right)\Omega-V}{\Omega^2}=\frac{\frac{F(x)}{G(x)}-x}{\Omega}=\frac{F(x)-x.G(x)}{\Omega.G(x)} \tag{6.1}$$

$$-\frac{d\Omega}{\Omega}=\frac{G(x).dx}{x.G(x)-F(x)} \tag{6.2}$$

Similarly:

$$\frac{dx}{dT}=\frac{x.G(x)-F(x)}{\Omega(x)} \tag{7.1}$$

$$dT=\frac{\Omega(x)}{x.G(x)-F(x)}.dx \tag{7.2}$$

where $\Omega(x)$ is the solution of (6.2). It is obvious from (6) and (7) that the character of the function $x.G(x)-F(x)$ is of great importance for the problem. Its graphics is presented in Fig. 2.

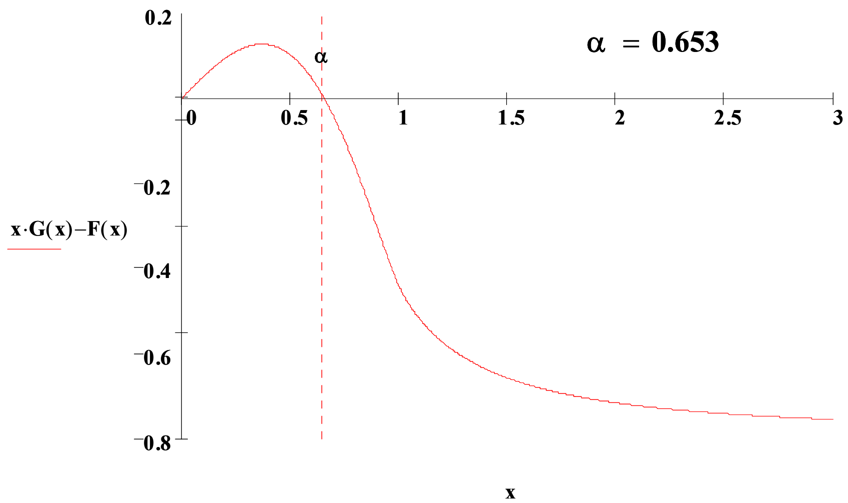

Fig. 2

The only positive root of this function is $\alpha \approx 0.653$. Since $d\Omega < 0$, from (6.2) it follows that if the initial value of $x_0 = \frac{V_0}{\Omega_0} > \alpha$, then $dx < 0$ while if $x_0 = \frac{V_0}{\Omega_0} < \alpha$, then $dx > 0$. In both cases x is approaching α and therefore it is important to clarify the asymptotic of the solution of (6.2) at α:

$$\frac{\Omega}{\Omega_0} = \exp\left\{\int_{x_0}^{x} \frac{-G(p).dp}{p.G(p)-F(p)}\right\} \tag{8}$$

Using the analytical character of F(p) and G(p) the integrant in (8) could be presented around α as:

$$\frac{-G(p)}{p.G(p)-F(p)} = \frac{A}{p-\alpha} + H(p) \tag{9.1}$$

$$A = \frac{-G(\alpha)}{\frac{d}{dp}\left(p.G(p)-F(p)\right)_{p=\alpha}} = -\frac{-G(\alpha)}{G(\alpha)+\alpha.G'(\alpha)-F'(\alpha)} \approx 1.626 \tag{9.2}$$

where H(p) is analytical and continuous function. It is easy to show from (9) that the asymptotic of $\Omega(x)$ at α is:

$$\Omega(x) \sim |x-\alpha|^{A} \tag{10}$$

and therefore $\lim_{x\to\alpha}\Omega(x) = 0$.

Now it follows from (7.2) that the total time of motion is finite. Indeed the integral:

$$T_{total} = \int_{x_0}^{\alpha} \frac{\Omega(x)}{x.G(x)-F(x)}.dx \tag{11}$$

is converging, which is the consequence of the asymptotic presentation (10).

Finally, we would like to mention the special case $x_0 = \alpha$. Now x is a constant and we simply have $V = \alpha.\Omega$.

Additionally in the attached “hockey.avi” file I present the animation of the puck motion for V = 8 and $\Omega = 3$, resulted from the numerical solution of the equation of motion (2).