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Sliding box - Problem 06/05
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I assume that the box is a parallelepiped with homogenous density, whose dimensions are l and b. The 3rd dimension is not necessary. The geometry of the problem is displayed in Fig.1.

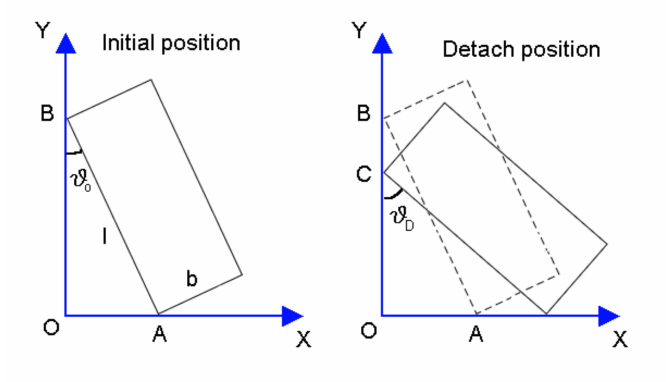

Figure 1: Geometry of the Sliding-box problem

The body can be represented by a rectangle in a cartesian plane Oxy; the initial position is represented in Fig.1 left. Due to gravity, the body will start sliding (no frictions); the vertex A will move along the x axe during all the duration of the movement, while the vertex B will move along the y axe only in the initial phase, just until a certain point, indicated by C in Fig.1 right; in the second phase, i.e. after the vertex B have passed the point C, the body will continue its falling but the vertex B will no more be on y axe.

The question is to determine the detach angle θ_D which will be, in general, function of θ_0, l, and b. It is important to note that there is a minimum starting angle $\theta_{c,m} = \arctan b/l$ under which the body will rotate in the "wrong" direction, i.e. clockwise in Fig.1; there is also a maximum starting angle $\theta_{c,M} = \arctan l/b$ over which the body will not slide on the wall but will detach just at the release; I assume that $\theta_{c,m} < \theta_0 < \theta_{c,M}$.

The following quantities can be defined:

- The Center of Mass (hereinafter CM) of the body has the following coordinates:

$$x_{CM} = \frac{1}{2}\left(l\sin\theta + b\cos\theta\right) \quad (1)$$
$$y_{CM} = \frac{1}{2}\left(l\cos\theta + b\sin\theta\right) \quad (2)$$

- The momentum of inertia of the body with respect to the axe passing through the CM is given by:

$$I_{CM} = \frac{M}{12} \cdot \left(l^2 + b^2\right) \tag{3}$$

- The kinetic energy is:

$$T = \frac{1}{2} \cdot I_{CM} \cdot \dot{\theta}^2 + \frac{1}{2} \cdot M \cdot v^2 = \frac{1}{2} \cdot \frac{M}{12} \cdot \left(l^2 + b^2\right) \cdot \dot{\theta}^2 + \frac{1}{2} \cdot \frac{M}{4} \dot{\theta}^2 \left(l^2 + b^2 - 4bl \sin\theta \cos\theta\right) \tag{4}$$

where v is the velocity of the CM and has been computed as follows:

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{\dot{x}_{CM}^2 + \dot{y}_{CM}^2} = \frac{1}{2} \dot{\theta} \sqrt{l^2 + b^2 - 4bl \sin\theta \cos\theta} \tag{5}$$

- The potential energy is:

$$V = \frac{1}{2} \cdot Mg\left(l \cos\theta + b \sin\theta\right) \tag{6}$$

- The Hamiltonian of the system:

$$H = T + V = \frac{1}{2} \cdot \frac{M}{3} \cdot \left(l^2 + b^2 - 3bl \sin\theta \cos\theta\right) \cdot \dot{\theta}^2 + \frac{1}{2} \cdot Mg\left(l \cos\theta + b \sin\theta\right) \tag{7}$$

H represents a conserved quantity, i.e. the total energy, which can be set equal to the potential energy at start:

$$H = H_0 = \frac{1}{2} \cdot Mg\left(l \cos\theta_0 + b \sin\theta_0\right) \tag{8}$$

Now we can use (7) and (8) to find the general equation for $\dot{\theta}$:

$$\dot{\theta}^2 = \frac{\frac{1}{2} \cdot Mg\left(l \cos\theta_0 + b \sin\theta_0\right) - \frac{1}{2} \cdot Mg\left(l \cos\theta + b \sin\theta\right)}{\frac{1}{2} \cdot \frac{M}{3} \cdot \left(l^2 + b^2 - 3bl \sin\theta \cos\theta\right)} \tag{9}$$

i.e.

$$\dot{\theta} = \sqrt{3g \cdot \frac{\left(l \cos\theta_0 + b \sin\theta_0\right) - \left(l \cos\theta + b \sin\theta\right)}{\left(l^2 + b^2 - 3bl \sin\theta \cos\theta\right)}} \tag{10}$$

[Comment: the equation of motion $\theta(t)$ can be found by integrating (10):

$$\int_{\theta_0}^{\theta} \frac{d\theta'}{\dot{\theta}(\theta')} = t - t_0 \tag{11}$$

where I have imposed the condition $\theta(t = 0) = \theta_0$.]

The problem can be solved once noticed that, starting from the detach point, no force will act along $\overrightarrow{x}$, i.e. $a_{CM,x} = 0$. Now, using the definition of v_x:

$$v_x = \dot{x}_{CM} = \frac{1}{2}\dot{\theta}\big(l\cos\theta - b\sin\theta\big) \tag{12}$$

and (10) we can write the following expression for v_x:

$$v_x = \frac{1}{2}\sqrt{\frac{3g\big(l\cos\theta_0 + b\sin\theta_0 - l\cos\theta - b\sin\theta\big)}{l^2 + b^2 - 3bl\sin\theta\cos\theta}}\big(l\cos\theta - b\sin\theta\big) \tag{13}$$

To compute the value of θ_D we can use the previous condition:

$$a_{CM,x} = \frac{dv_x}{dt} = \frac{dv_x}{d\theta}\cdot\frac{d\theta}{dt} = 0 \tag{14}$$

Being $d\theta/dt > 0$ for $\theta > \theta_0$, this is equivalent to:

$$\frac{dv_x}{d\theta} = 0 \tag{15}$$

which is equivalent to find the maximum of the graph of $v_x(\theta)$. I didn't found an analytic expression for $\theta_D(\theta_0)$, but I computed a numerical solution. In Fig.2, I report, as an example, the graph of θ_D in function of the start angle for l and b respectively equal to 10 and 5 (arbitrary units).

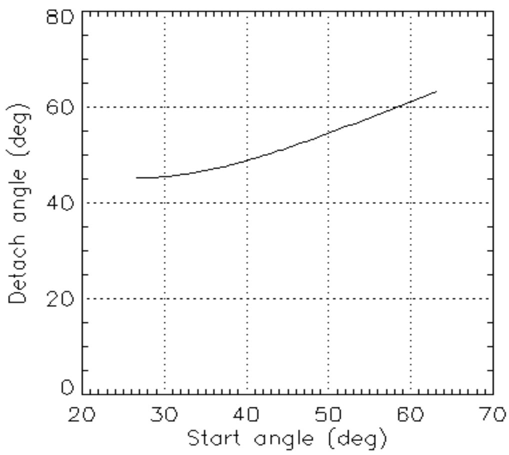

Figure 2: Detach angle for l=10 and b=5