Sliding Box

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Abstract

A rectangular box leans on a frictionless wall with one corner and rests on a frictionless floor with another corner. It starts sliding down. When will the box become detached from the wall? (Assume that all the dimensions are given.)

1 General analysis

We assume the length and width of the box to be L and d, respectively and its mass to be m. In order to deal with the constraints, we introduce θ to be the angle between side L and the horizontal. We can then express the position x, y of the center of mass (CM) of the box as:

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L & d \\ d & L \end{pmatrix} \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} = \frac{\sqrt{L^2+d^2}}{2} \begin{pmatrix} \cos(\theta_0-\theta) \\ \sin(\theta_0+\theta) \end{pmatrix}, \tag{1}$$

where θ_0 is the angle between side L of the box and the diagonal. Note that θ_0 is the only parameter of the problem and depends on the geometry only.

The conservation of energy in this context is:

$$\frac{m}{2}\,\dot\theta^2\,\frac{L^2+d^2}{4}\,\left\{\sin^2(\theta_0-\theta)+\cos^2(\theta_0+\theta)\right\}+\frac{1}{2}\,\frac{m}{12}\,(L^2+d^2)\,\dot\theta^2 = mg\frac{\sqrt{L^2+d^2}}{2}\,\{1-\sin(\theta_0+\theta)\}\,.$$

Solving for $\dot\theta$, we obtain the angular velocity as a function of the angle:

$$\dot\theta^2 = \frac{4g}{L^2+d^2}\,\frac{1-\sin(\theta_0+\theta)}{4/3+\sin^2(\theta_0-\theta)-\sin^2(\theta_0+\theta)}. \tag{2}$$

The horizontal component of the velocity is, thus, proportional to:

$$\dot x \sim \sin(\theta_0-\theta)\,\sqrt{\frac{1-\sin(\theta_0+\theta)}{4/3+\sin^2(\theta_0-\theta)-\sin^2(\theta_0+\theta)}}. \tag{3}$$

When $\theta_0 = 0$ the expression above reduces to:

$$\dot{x} \sim \sin\theta \sqrt{1 - \sin\theta}, \tag{4}$$

which becomes stationary at $\theta = \arcsin(2/3)$. This is the usual result one obtains in the case of a sliding ladder. Because $\ddot{x}$ is proportional to the normal force exerted by the wall, the box loses contact with the wall when $\dot{x}$ is stationary.

To find the angle θ at which $\dot{x}$ becomes stationary, we have to differentiate (3) with respect to θ and solve a very difficult (nonlinear) trigonometric equation. Because this equation contains parameter θ_0, it is not possible to solve it numerically (if we are not interested for a particular θ_0). Instead, we can find an approximate solution, which can later become as accurate as we like. We consider small perturbations of the "ladder" problem. That is, we expand (3) in series (with respect to θ_0) up to second order. We then differentiate with respect to θ and obtain a relation of the form:

$$\{\text{terms up to second order in } \theta_0\} = 0. \tag{5}$$

We know the exact solution to be $\theta = \arcsin(2/3)$ when $\theta_0 = 0$, so we expect the solution to (5) to be a small deviation from $\arcsin(2/3)$. It is, therefore, reasonable to write $\theta = \arcsin(2/3) + \epsilon$, where ϵ is assumed to be small. We substitute the latter into (5) and obtain a relation of the form:

$$f(\epsilon, \theta_0) = 0. \tag{6}$$

By keeping up to second order terms (with respect to θ) in (6), we can solve for ϵ. The final result can be expressed in the form of a series:

$$\theta_c = c_0 + c_1\ \theta_0 + c_2\ \theta_0^2 + c_3\ \theta_0^3 + \ldots, \tag{7}$$

the first few terms of which are:

$$\theta_c = 0.7297276563 + 0.2444444445\ \theta_0 - 0.4377172082\ \theta_0^2 + O(\theta_0^3), \tag{8}$$

where $\theta_0 = \arcsin\left(\frac{d/L}{\sqrt{1+(d/L)^2}}\right)$. We could continue in the same way and calculate higher order corrections (or improve the accuracy of c_1, c_2). If we are interested in a box with $d/L \approx 1$, we can simply consider the general problem as a perturbation of a sliding "square", rather than box. In any case, the solution can be calculated in the same manner.