

Chess Board Problem - 07/01

Part (a)

For conductivity σ , we assume $\mathbf{J} = \sigma\mathbf{E}$. From the continuity equation, $\nabla \cdot \mathbf{J} = \frac{d\rho}{dt} = \mathbf{0}$ because we are dealing with a static problem. Thus:

$$\nabla\sigma \cdot \mathbf{E} + \sigma\nabla \cdot \mathbf{E} = \nabla \cdot (\sigma\mathbf{E}) = \nabla \cdot \mathbf{J} = \mathbf{0}$$

Now if we let $\mathbf{E} = -\nabla\phi$, then:

$$\nabla^2\phi = -\frac{\nabla\sigma \cdot \nabla\phi}{\sigma} \quad \text{or} \quad \nabla \cdot \mathbf{E} = -\frac{\nabla\sigma \cdot \mathbf{E}}{\sigma} \tag{1}$$

Inside each square of the chessboard, σ is constant, so $\nabla\sigma$ vanishes. Thus we may assume that inside a black square the potential is given by a harmonic function $b_1(x, y)$ and inside a white square it is given by the harmonic function $w_1(x, y)$. The exact form of these two functions depends on the boundary conditions. However, for a very large chessboard, the boundary conditions will be the same for each black and each white square because they won't vary much from square to square. Since harmonic functions are determined uniquely by their boundary conditions, we can use b_1 for each black square and w_1 for each white square.

Part (b)

Since b_1 and w_1 are harmonic, they can each be treated as real parts of analytic functions B and W , where $B(x, y) = b_1(x, y) + ib_2(x, y)$ and similarly for W . The important thing here is that the real and imaginary parts of any analytic function must obey the Cauchy-Riemann equations:

$$\frac{\partial b_1}{\partial x} = \frac{\partial b_2}{\partial y} \quad \text{and} \quad \frac{\partial b_1}{\partial y} = -\frac{\partial b_2}{\partial x}$$

and similarly for W . These relationships between the derivatives of the real and imaginary parts of B and W imply a relationship between electric fields. Specifically, the fields we would get from b_2 and w_2 by a naive direct calculation are equal to the fields of b_1 and w_1 *rotated* by ninety degrees at each point. Let the original field be \mathbf{E} and current density be \mathbf{J} , while the field and current from b_2 and w_2 are \mathbf{E}' and \mathbf{J}' . Also let $R = \frac{1}{\sigma}$, so:

$$\nabla \cdot \mathbf{J} = 0 \quad \text{and} \quad \nabla \times (R\mathbf{J}) = 0 \tag{2}$$

or

$$\nabla \cdot (\sigma\mathbf{E}) = 0 \quad \text{and} \quad \nabla \times \mathbf{E} = 0 \tag{3}$$

Using the Cauchy-Riemann equations, we find that:

$$\nabla \cdot \mathbf{E}' = 0 \quad \text{and} \quad \nabla \times (\sigma\mathbf{E}') = 0 \tag{4}$$

or

$$\nabla \cdot (R\mathbf{J}') = 0 \quad \text{and} \quad \nabla \times \mathbf{J}' = 0 \tag{5}$$

Thus equations 2 & 4 and 3 & 5 are the same, with $\mathbf{E} \rightarrow \mathbf{J}'$, $\mathbf{J} \rightarrow \mathbf{E}'$, and $\sigma \rightarrow R$, so b_2 and w_2 do solve a different chessboard problem when we switch fields with currents. This is possible because the conditions on \mathbf{E} that guarantee continuity of the potential force the new current \mathbf{E}' to have a continuous normal component across the boundaries, while the continuity of the normal component of \mathbf{J} across the boundaries forces continuity of the new potential (which has negative gradient \mathbf{J}').

Part (c)

We are looking for $\langle \sigma \rangle$, the ‘average’ conductivity, which we can define by:

$$\langle \sigma \rangle = \frac{\langle \sigma \mathbf{E} \rangle}{\langle \mathbf{E} \rangle}$$

However, from comparison of solutions in part (b), we know that:

$$\frac{1}{\langle \sigma \rangle} = \frac{\langle \frac{\mathbf{E}}{\sigma} \rangle}{\langle \mathbf{E} \rangle}$$

Thus if $\langle \sigma \rangle = F(\sigma_1, \sigma_2)$ then:

$$F(\sigma_1, \sigma_2) = \frac{1}{F(1/\sigma_1, 1/\sigma_2)} = \frac{1}{F(1/\sigma_2, 1/\sigma_1)} = \frac{\sigma_1 \sigma_2}{F(\sigma_1 \sigma_2 / \sigma_2, \sigma_1 \sigma_2 / \sigma_1)} = \frac{\sigma_1 \sigma_2}{F(\sigma_1, \sigma_2)}$$

This completes the solution:

$$\langle \sigma \rangle = (\sigma_1 \sigma_2)^{1/2}$$

We have made several assumptions: (i) Existence of $\langle \sigma \rangle$, which is very reasonable for a large chessboard, though not necessarily for a small one (ii) Symmetry when switching black and white squares, $F(a, b) = F(b, a)$, which obviously works for the large chessboard geometry (iii) Scale invariance, so $F(ka, kb) = kF(a, b)$ (iv) Isotropy, which is perhaps more subtle. Technically our final result is (with the first three assumptions):

$$F_x(\sigma_1, \sigma_2) F_y(\sigma_1, \sigma_2) = \sigma_1 \sigma_2$$

because the F 's are evaluated in perpendicular directions. Of course, $F_x = F_y$ is reasonable for the chessboard, but not in general. For instance it is instructive to test our result with a ‘row geometry’, with rows of all black squares and rows of all white squares in a 50-50 mix. With the obvious choice for x and y , $F_x = \frac{\sigma_1 + \sigma_2}{2}$ while $F_y = \frac{2\sigma_1 \sigma_2}{\sigma_1 + \sigma_2}$, so:

$$F_x F_y = \frac{2\sigma_1 \sigma_2}{\sigma_1 + \sigma_2} \frac{\sigma_1 + \sigma_2}{2} = \sigma_1 \sigma_2$$

as predicted. Thus given the formula for the effective resistance of resistors in parallel or resistors in series, our solution gives a very sophisticated proof of the other formula.

Note however that these are the only assumptions we have made, so our result is applicable to many other geometries other than the chessboard - consider, for example, a random 50-50 mixture of two materials.