Infinite Free Products of Profinite Groups as Absolute Galois Groups

In Erinnerung an Jürgen Ritter (1943 – 2021)

Moshe Jarden, Tel Aviv University, Tel Aviv, Israel
jarden@tauex.tau.ac.il

2 September 2024

Abstract

It is known that the free product of finitely many absolute Galois groups of fields is the absolute Galois group of a field.

We give examples of free products of infinitely many nontrivial absolute Galois groups of fields which are not isomorphic to the absolute Galois group of any field.

On the other hand, we give examples of free products of infinitely many nontrivial absolute Galois groups of fields which are the absolute Galois group of a field.

Introduction

This note deals with free products of profinite groups as absolute Galois groups of fields. To this end we denote the separable closure of a field K by $K_{\rm sep}$ and call $\mathrm{Gal}(K) := \mathrm{Gal}(K_{\rm sep}/K)$ the **absolute Galois group** of K. If $\mathrm{char}(K) = 0$, which is here our main concern, $K_{\rm sep}$ is the algebraic closure $\tilde{K}$ of K, so $\mathrm{Gal}(K) = \mathrm{Gal}(\tilde{K}/K)$. In any case, $\mathrm{Gal}(K) = \varprojlim \mathrm{Gal}(L/K)$ is the inverse limit over all finite Galois groups of Galois extensions L of K, so $\mathrm{Gal}(K)$ is a **profinite group**, that is the inverse image of finite groups [FrJ23, Sec. 1.2].

Following [FrJ23, p. 6, Rem. 1.2.1(g)], we tacitly assume that every homomorphism $\varphi\colon G \to H$ of profinite groups is continuous.

The **free product of finitely many profinite groups** $G_1, \ldots, G_n$ is a profinite group $G := \coprod_{i=1}^n G_i$ satisfying the following conditions:

(a) Each G_i is a closed subgroup of G.

(b) If H is a profinite group and $\varphi_i\colon G_i \to H$ is a homomorphism of profinite groups for $i = 1, \ldots, n$, then there exists a unique homomorphism $\varphi\colon G \to H$ such that $\varphi|_{G_i} = \varphi_i$ [FrJ23, p. 530, Prop. 25.5.1].

The **free product** (in the sense of Binz-Neukirch-Wenzel [BNW71]) of a set $\{G_i\}_{i\in I}$ of profinite groups for an arbitrary set I is a profinite group $G := \coprod^{*}_{i\in I} G_i$ satisfying the following conditions: [1]

(a) Every G_i is a closed subgroup of G.

(b) Every open subgroup H of G contains **almost all** (i.e. all but finitely many) of the groups G_i.

(c) Given a profinite group $\bar{G}$ and a set of homomorphisms $(\varphi_i\colon G_i \to \bar{G})_{i\in I}$ such that for each open subgroup $\bar{H}$ of $\bar{G}$ we have $\mathrm{Im}(\varphi_i) \leq \bar{H}$ for almost all $i \in I$, there exists a unique homomorphism $\varphi\colon G \to \bar{G}$ satisfying $\varphi|_{G_i} = \varphi_i$.

Note that this type of free product of profinite groups will be the only one that appears in this short note.

Jochen Koenigsmann proves in [Koe02] that every free product of finitely many absolute Galois groups is an absolute Galois group of a field. One may find an alternative proof of Koenigsmann's result in [HJK00].

Tamar Bar-On asked in a private communication whether every infinite free product of absolute Galois groups of fields is an absolute Galois group of a field.

The aim of this note is to show that this is not always the case. Indeed, by results of Emil Artin, the absolute Galois group of every real closed field, in particular of the field $\mathbb{R}$ of real numbers, is isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Conversely, every field R with absolute Galois group of order 2 is real closed (Example 1.1). However, that example shows that the free product of infinitely many copies of $\mathbb{Z}/2\mathbb{Z}$ is never an absolute Galois group of a field.

Similarly, we show in Examples 1.2 and 1.4 that for every prime number p and every finite extension K of $\mathbb{Q}_p$ and $\mathbb{Q}$ there exists no field with absolute Galois group isomorphic to the free product of infinitely many copies of $\mathrm{Gal}(K)$.

But these counterexamples are not the rule for an infinite free product of absolute Galois groups of fields to be an absolute Galois group of a field. Indeed, Example 2.2 points out that the free profinite product of finitely many or infinitely many projective groups is the absolute Galois group of a field.

So, we are still looking for a criterion for an infinite free product of absolute Galois groups of fields to be the absolute Galois group of a field.

The author thanks Dan Haran for Example 1.1 and for other useful suggestions. Likewise, the author thanks Aharon Razon for a careful reading of the manuscript and the anonymous referee for useful hints.

1 Negative Examples

{EXMP}

We give three examples of arithmetical nature. Each of them shows that an infinite free product of arbitrary absolute Galois groups of fields need not be an absolute Galois group of a field.

The first example treats the absolute Galois group of real closed fields, with absolute Galois group of order 2. The second one handles the absolute Galois

[1] The original definition of [BNW71] demands that the G_i's are just profinite groups together with homomorphisms $\varphi_i\colon G_i \to G$.

group of a p-adic number field, i.e. a finite extension of the field $\mathbb{Q}_p$, where p is a prime number. The main idea of the proof is similar to the proof in the "real case" but uses stronger means. Finally, the third example considers number fields.

Example 1.1. *Real closed fields* (Dan Haran). Assume toward contradiction that I is an infinite set and the absolute Galois group $\mathrm{Gal}(K)$ of a field K is a free product $\mathop{\ast\!\!\prod}_{i\in I}\mathrm{Gal}(R_i)$ of absolute Galois groups $\mathrm{Gal}(R_i)$ of fields with $2 \le \mathrm{card}(\mathrm{Gal}(R_i)) < \infty$ for each $i \in I$. By [Lan97, p. 299, Cor. 9.3], $\mathrm{card}(\mathrm{Gal}(R_i)) = 2$, and $\widetilde{R_i} = R_i(\sqrt{-1})$. Hence, by [Lan97, p. 452, Prop. 2.4], each of the R_i's is a real closed field. {RELL}

Next consider the quadratic extension $L := K(\sqrt{-1})$ of K. Its absolute Galois group $\mathrm{Gal}(L)$ is an open subgroup of $\mathrm{Gal}(K)$. By definition, almost all of the groups $\mathrm{Gal}(R_i)$ are contained in $\mathrm{Gal}(L)$. Since I is infinite, there exists $i \in I$ such that $\mathrm{Gal}(R_i) \subseteq \mathrm{Gal}(L)$, so $L \subseteq R_i$. Thus, $\sqrt{-1} \in R_i$, hence R_i is not real closed [Lan97, p. 451, Sect. 2]. This contradicts the conclusion of the preceding paragraph. ∎

The method applied in Example 1.1 also works for fields whose absolute Galois groups are isomorphic to $\mathrm{Gal}(M)$, where M is a finite extension of $\mathbb{Q}_p$ for some prime number p.

Example 1.2. *p-adic fields.* Let p be a prime number and let M be a finite extension of $\mathbb{Q}_p$. Assume toward contradiction that I is an infinite set and that there exists a field K such that $\mathrm{Gal}(K) \cong \mathop{\ast\!\!\prod}_{i\in I}\mathrm{Gal}(M_i)$ and $\mathrm{Gal}(M_i) \cong \mathrm{Gal}(M)$ for each $i \in I$. {mADC}

We choose a prime number $q \neq 2, p$ such that $q \nmid [M : \mathbb{Q}_p]$. Let v_p be the standard discrete valuation of $\mathbb{Q}_p$ satisying $v_p(p) = 1$, let v_M be the unique normalized valuation of M that extends v_p [CaF67, p. 56, Thm.], let e be the ramification index of v_M/v_p, and let $f := [\bar{M}_{v_M} : \mathbb{F}_p]$ be its residue degree. Then, [FrJ23, p. 27, (2.10)] gives

$$ef = [M : \mathbb{Q}_p]. \quad (1)$$ {efp}

Claim: *$X^q - p$ has no roots in M.* Indeed, assume by contradiction that there exists $x \in M$ such that $x^q = p$. Then, $qv_M(x) = v_M(p) = ev_p(p) = e$. Hence, by (1), q divides $[M : \mathbb{Q}_p]$, in contrast to the choice of q.

Following the Claim, we choose a q-th root $p^{1/q}$ of p and note that $p^{1/q} \notin M$.

By assumption, $\mathrm{Gal}(M_i) \cong \mathrm{Gal}(M)$ for each i. Hence, by [Koe95, p. 179, Cor. 7.2], each of the fields M_i is **elementarily equivalent** to M in the language of fields [FrJ23, p. 145]. Therefore, since $p^{1/q} \notin M$, it does not belong to any of the fields M_i.

On the other hand, $\mathrm{Gal}(K(p^{1/q}))$ is an open subgroup of $\mathrm{Gal}(K)$. Hence, by the definition of the infinite free product of profinite groups, $\mathrm{Gal}(M_i) \subseteq \mathrm{Gal}(K(p^{1/q}))$ for all but finitely many i's. Since I is infinite, there exists $i \in I$ with $p^{1/q} \in M_i$. This contradicts the preceding paragraph. ∎

{pACF}

Remark 1.3. Let K be a field of characteristic 0. Following Ido Efrat [Efr06, p. 167, Exm. 18.3.4(2)] we write $K_{\mathrm{alg}} := \tilde{\mathbb{Q}} \cap K$ for the algebraic part of K. Of course, K_{alg} is defined only up to isomorphism. So, in order to make K_{alg} unique, we assume that all fields in this note are contained in a fixed universal field Ω (as is done in the classical algebraic geometry [Lan64, p. 21, Chap. II1]).

As a consequence of Krasner's Lemma, Ido Efrat notes that $\widetilde{\mathbb{Q}_p} = \tilde{\mathbb{Q}}\mathbb{Q}_p$ [Efr06, p. 172, Exm. 18.5.4(1)]. By definition, $\mathbb{Q}_p \cap \tilde{\mathbb{Q}} = \mathbb{Q}_{p,\mathrm{alg}}$ and the restriction map res: $\mathrm{Gal}(\mathbb{Q}_p) \to \mathrm{Gal}(\mathbb{Q}_{p,\mathrm{alg}})$ is an isomorphism.

{GLFd}

Example 1.4. *Number fields.* Consider a number field K and a free product $\ast_{i\in I} \mathrm{Gal}(K_i)$ where I is an infinite set and $\mathrm{Gal}(K_i) \cong \mathrm{Gal}(K)$ for each $i \in I$. Assume toward contradiction that $\ast_{i\in I} \mathrm{Gal}(K_i) \cong \mathrm{Gal}(F)$ for some field F.

To this end let N be a finite Galois extension of $\mathbb{Q}$ that contains K. We will apply the Chebotarev density theorem to choose a prime number p that totally splits in N.

Indeed, let S be the set of all prime numbers p which are unramified in N and for which the Artin symbol $\left(\frac{N/\mathbb{Q}}{p}\right)$ [FrJ23, p. 123] is the conjugacy class of $\mathrm{Gal}(N/\mathbb{Q})$ consisting of the unit element. Then, the Dirichlet density of S is $\frac{1}{[N:\mathbb{Q}]}$ [FrJ23, p. 124, Thm. 7.3.1], so that density is positive, hence S is an infinite set [FrJ23, p. 124, lines 6–7]. By definition, each $p \in S$ totally splits in N, so $\mathrm{Gal}(\mathbb{Q}_{p,\mathrm{alg}})$ and all of its conjugates in $\mathrm{Gal}(\mathbb{Q})$ are contained in $\mathrm{Gal}(N)$, hence also in $\mathrm{Gal}(K)$, so $\mathrm{Gal}(\mathbb{Q}_{p,\mathrm{alg}}) \leq \mathrm{Gal}(K_i)$ for each $i \in I$, Hence, by Remark 1.3, $\mathrm{Gal}(\mathbb{Q}_p)$ is isomorphic to a closed subgroup of $\mathrm{Gal}(K_i)$ for each $i \in I$.

Thus, $\ast_{i\in I} \mathrm{Gal}(\mathbb{Q}_p)$ is isomorphic to a closed subgroup of $\ast_{i\in I} \mathrm{Gal}(K_i)$ and therefore also of $\mathrm{Gal}(F)$. Hence $\ast_{i\in I} \mathrm{Gal}(\mathbb{Q}_p) \cong \mathrm{Gal}(E)$ for some algebraic extension E of F. This contradicts Example 1.2. ∎

We call the counterexamples given in the present section "arithmetic".

2 Positive Examples

{PSX}

In contrast to the arithmetic examples given in the first section, we give here "positive profinite groups examples" showing that infinite free products of profinite groups may well be absolute Galois groups of fields.

Specifically, those groups will be "projective". To this end we say that a profinite group G is **projective** if for given epimorphisms $\alpha\colon B \to A$ and $\varphi\colon G \to A$ of profinite groups there exists a homomorphism $\gamma\colon G \to B$ such that $\alpha \circ \gamma = \varphi$ [FrJ23, p. 528, 1st paragraph].

Here is a classical example of an absolute Galois group G of a field such that G is projective.

{SHFr}

Example 2.1. The absolute Galois group of the maximal abelian extension $\mathbb{Q}_{\mathrm{ab}}$ of $\mathbb{Q}$ is projective. See [Jar11, p. 90, Exm. 5.10.5]. ∎

{PRJ}

Example 2.2. Let K be a field and let $G_1, G_2, G_3, \dots$ be a sequence of projective groups and set $G = \mathop{\circledast}_{i=1}^{\infty} G_i$ for their free product in the sense of Binz–Neukirch-Wenzel.

We prove that G is isomorphic to the absolute Galois group of a perfect **PAC field** M that contains K. Thus, every geometrically irreducible variety over M has an M-rational point [FrJ23, p. 203].

Claim: *G is a projective group.* Consider a finite embedding problem

$$(\varphi\colon G \to A, \alpha\colon B \to A), \tag{2}$$ {phl}

such that both A and B are finite groups and both α and φ are epimorphisms [FrJ23, p. 525, Def. 25.3.1]. By definition of the free product, there exists a positive integer n such that $\varphi(G_i) = \mathbf{1}_A$ for all $i > n$. By [FrJ23, p. 530, Prop. 25.5.1(c)], the closed subgroup $\mathop{\circledast}_{i=1}^{n} G_i$ of G is a projective group. Let $\varphi_n\colon \mathop{\circledast}_{i=1}^{n} G_i \to A$ be the restriction of φ to $\mathop{\circledast}_{i=1}^{n} G_i$. Then, there exists a homomorphism $\gamma_0\colon \mathop{\circledast}_{i=1}^{n} G_i \to B$ such that $\varphi_n = \alpha \circ \gamma_0$. By definition (see the Introduction), γ_0 extends to a homomorphism $\gamma\colon G \to B$ such that $\gamma(G_i) = \mathbf{1}_B$ for each $i > n$. Then, $\alpha \circ \gamma = \varphi$, so γ is a **weak solution** [FrJ23, p. 525] of the embedding problem (2). By [FrJ23, p. 525, Lemma 25.3.2], G is projective, as claimed.

End of proof: Since G is projective, it follows from a theorem of Alexander Lubotzky and Lou van der Dries [FrJ23, p. 570, Cor. 26.1.2] that G is isomorphic to the absolute Galois of a perfect PAC field that contains K, as desired. ∎

{FP}

Example 2.3. Let $G = \mathop{\circledast}_{i=1}^{\infty} G_i$ be the free product of free profinite groups and let K be a field. Then, by [FrJ23, p. 529, Cor. 25.4.5], each G_i is a projective group. Hence, by Example 2.2, G is the absolute Galois group of a perfect PAC field L that contains K.

Moreover, each G_i is a closed subgroup of G, so the fixed field L_i of G_i in $\tilde{L}$ is an algebraic extension of L. Hence, by a theorem of James Ax and Peter Roquette [FrJ23, p. 207, Cor. 12.2.5], each of the fields L_i is perfect and PAC. ∎

References

[BNW71] E. Binz, J. Neukirch, and G. H. Wenzel, *A subgroup theorem for free products of profinite groups*, Journal of Algebra **19** (1971), 104–109.

[CaF67] J. W. S. Cassels and A. Fröhlich, *Algebraic Number Theory*, Academic Press, London, 1967.

[Efr06] I. Efrat, *Valuations, Orderings, and Milnor K-Theory*, Mathematical surveys and monographs **124**, American Mathematical Society, Providence, 2006.

[FrJ23] M. D. Fried and M. Jarden, *Field Arithmetic, 4th edition, revised by Moshe Jarden,* Ergebnisse der Mathematik (3) **11**, Springer, Heidelberg, 2023.

[HJK00] D. Haran, M. Jarden, J. Koenigsmann, *Free products of absolute Galois groups,* https://arxiv.org/pdf/math/0005311.pdf, 31 May 2000.

[Jar11] M. Jarden, *Algebraic patching,* Springer Monographs in Mathematics, Springer, Heidelberg, 2011.

[Koe95] J. Koenigsmann, *From p-rigid elements to valuations (with a Galois-characterization of p-adic fields),* Journal für die reine und angewandte Mathematik **465** (1995), 165–182.

[Koe02] J. Koenigsmann, *Relatively projective groups as absolute Galois groups,* Israel Journal of Mathematics **127** (2002), 93–129.

[Lan64] S. Lang, *Introduction to Algebraic Geometry,* Interscience Publishers, New York, 1964.

[Lan97] S. Lang, *Algebraic Number Theory (third edition),* Addison-Wesley, Reading, 1997.