Pseudo Finite Fields with the Laurent Property

In Erinnerung an Wulf-Dieter Geyer (1939–2019)

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Abstract

We construct an algebraic extension F of $\mathbb{Q}$ which is pseudo finite and has the "Laurent property". In addition, F has an extension F^* which is a non-principal ultraproduct of distinct finite fields (so F^* is pseudo finite), F^* has the Laurent property, and F is the algebraic part of F^*.

Introduction

It is well known that the field $F := \tilde{\mathbb{Q}}((t))$ of Laurent series in the variable t over the algebraic closure $\tilde{\mathbb{Q}}$ of $\mathbb{Q}$ has the property that its algebraic closure $\tilde{F}$ is the union $\bigcup_{n=1}^{\infty} F(t^{1/n})$ [Eis95, p. 299, Cor. 13.15]. See also [CaF67, p. 32, special case of Cor. 1].

Note that F is the quotient field of the complete discrete valuation ring $\tilde{\mathbb{Q}}[[t]]$ with t being a prime element of that ring. Hence, by Eisenstein's criterion, $X^n - t$ is irreducible over F for every positive integer n [Lan93, p. 183, Thm. 3.1].

Thus, F has the **Laurent Property**, meaning in general, that $\mathrm{char}(F) = 0$ and F has an element a, such that for all $n \in \mathbb{N}$ the polynomial $X^n - a$ is irreducible, the field $F(a^{1/n})$ is Galois over F of degree n, and $\tilde{F} = \bigcup_{n=1}^{\infty} F(a^{1/n})$. (Note that $F(a^{1/n})$ does not depend on the choice of the nth root of a.) We then say that a is a **Laurent element** of F.

The combination of Theorems 2.4 and 2.6 of Jakub Gismatullin and Katarzyna Tarasek's work [GiT23] provides a non-principal ultraproduct F^* of distinct finite fields that has the Laurent property. Note that F^* is **pseudo finite**, which means that F^* is perfect, F^* is "PAC" (= **pseudo algebraically closed**), and $\mathrm{Gal}(F^*) \cong \hat{\mathbb{Z}}$, where $\hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z}$ [FrJ08, p. 449, Lemma 20.10.1].

To this end we recall that a field K is **PAC** if every geometrically integral variety over K has a K-rational point [FrJ08, p. 192].

That non-principal ultraproducts of finite fields are pseudo finite was used by James Ax in [Ax68] in order to prove that the elementary theory of finite fields is decidable. Then, non-principal ultraproducts of finite fields play a central role in the proof of the "transfer theorem" of the first author saying that the Dirichlet density of the set of primes p for which a given elementary statement θ holds in the fields $\mathbb{F}_p$ is equal to the Haar measure of $\sigma \in \mathrm{Gal}(\mathbb{Q})$ such that θ holds in the fixed field $\tilde{\mathbb{Q}}(\sigma)$ of σ in $\tilde{\mathbb{Q}}$ [Jar72] (see also [FrJ08, p. 447, Thm. 20.9.3]).

Here we prove the existence of an algebraic extension F of $\mathbb{Q}$ which is pseudo finite and has the Laurent property. Then we deduce the existence of a non-principal ultraproduct F^* of distinct finite fields, such that F^* has the Laurent property (as [GiT23] did) and, in addition, $F^* \cap \tilde{\mathbb{Q}} = F$.

On the other hand, we show that the σ's in $\mathrm{Gal}(\mathbb{Q})$ with the Laurent property for $\tilde{\mathbb{Q}}(\sigma)$ are "rare" (Theorem 4.1). In particular, "most" of the fields $\tilde{\mathbb{Q}}(\sigma)$ with $\sigma \in \mathrm{Gal}(\mathbb{Q})$ satisfy $\mathrm{Gal}(\tilde{\mathbb{Q}}(\sigma)) \cong \hat{\mathbb{Z}}$ but do not have the Laurent property (Example 4.3).[1]

1 Preparations

Essential tools in the proof of our main theorem are Lemma 1.1 and Lemma 1.3. In these results and in what follows we set $F^n = \{x^n \mid x \in F\}$ for a field F and a positive integer n.

{Irn}

Lemma 1.1 ([Kar89], p. 425, Thm. 1.6)**.** *Let F be a field, n a positive integer, and $a \in F$. Then, $X^n - a$ is irreducible over F if and only if $a \notin F^p$ for all primes p dividing n and $a \notin -4F^4$ whenever $4|n$.*

{Eight}

Corollary 1.2. *The polynomial $p(X) := 8X^8 - 1$ is irreducible in $\mathbb{Z}[X]$.*

Proof. There exists no $a \in \mathbb{Q}$ with $\frac{1}{8} = a^2$, nor there exists $b \in \mathbb{Q}$ with $\frac{1}{8} = -4b^4$. Hence, by Lemma 1.1, $X^8 - \frac{1}{8}$ is irreducible in $\mathbb{Q}[X]$, so $8X^8 - 1$ is irreducible in $\mathbb{Z}[X]$, as claimed[2].

{Ccl}

Lemma 1.3 ([Lan93], p. 289, Thm. Thm. 6.2)**.** *Let K_0 be a field and n be a positive integer prime to* $\mathrm{char}(K_0)$. *Suppose that K_0 contains a primitive n-th root of unity.*

(a) *Let K be a cyclic extension of a field K_0 of degree n. Then, there exist $a \in K_0$ and $x \in K$ such that $K = K_0(x)$ and $x^n = a$.*

(b) *Conversely, let $a \in K_0$ and let x be a root of $X^n - a$. Then, $K_0(x)$ is cyclic over K_0 of degree that divides n.*

In the following lemma and beyond we consider a positive integer n, a field M that contains all roots of unity of order n, and an element $a \in M$. Then we

[1] The authors are indebted to the anonymous referee for useful comments.

[2] The authors are indebted to Sigrid Böge for this short proof.

write $M(a^{1/n})$ for the extension of M obtained by adjoining an nth root $a^{1/n}$ to M. Of course, there are n such roots, but all of the extensions $M(a^{1/n})$ are the same.

{Zgag}

Lemma 1.4. *Let M be a field of characteristic 0 that contains all roots of unity and let a be an element of M. Suppose that for each positive integer n the extension $M(a^{1/n})/M$ is cyclic of degree n. Then, $N := \bigcup_{n=1}^{\infty} M(a^{1/n})$ is a Galois extension of M and $\mathrm{Gal}(N/M) \cong \hat{\mathbb{Z}}$.*

Proof. We use a standard inverse limit argument.

For each positive integer n let

$$\rho_n\colon \mathrm{Gal}(M(a^{1/(n+1)!})/M) \to \mathrm{Gal}(M(a^{1/n!})/M)$$

be the restriction map. Then we use induction to construct a sequence of epimorphisms $\pi_n\colon \hat{\mathbb{Z}} \to \mathrm{Gal}(M(a^{1/n!})/M)$ such that $\rho_n \circ \pi_{n+1} = \pi_n$. Since $N = \bigcup_{n=1}^{\infty} M(a^{1/n!})$, there exists an epimorphism $\pi\colon \hat{\mathbb{Z}} \to \mathrm{Gal}(N/M)$ with $\mathrm{res}_n \circ \pi = \pi_n$ for each n, where res_n is the restriction map $\mathrm{Gal}(N/M) \to \mathrm{Gal}(M(a^{1/n!})/M)$.

By assumption, $\mathrm{Gal}(M(a^{1/n!})/M) \cong \mathbb{Z}/n!\mathbb{Z}$, hence $\mathrm{Ker}(\pi_n) = n!\hat{\mathbb{Z}}$ is the unique open subgroup of $\hat{\mathbb{Z}}$ of index $n!$ [FrJ08, p. 14, Lemma 1.4.4]. Therefore, by [FrJ08, p. 6, Remark 1.2.1(a)], $\mathrm{Ker}(\pi)$ is the trivial subgroup of $\hat{\mathbb{Z}}$. We conclude that π is an isomorphism, as desired. □

2 The field $\mathbb{Q}_{\mathrm{ab}}$

{Weber}

Next, we consider the field $\mathbb{Q}_{\mathrm{ab}}$ obtained from $\mathbb{Q}$ by adjoining all of the roots of unity in $\tilde{\mathbb{Q}}$. Obviously, $\mathbb{Q}_{\mathrm{ab}}$ is an abelian extension of $\mathbb{Q}$. Moreover, the Kronecker-Weber theorem says that $\mathbb{Q}_{\mathrm{ab}}$ is the maximal abelian extension of $\mathbb{Q}$ [Neu99, p. 324, Thm. 1.10].

{Abel}

Lemma 2.1. *For each positive integer n, the extension $\mathbb{Q}_{\mathrm{ab}}(2^{1/2n})/\mathbb{Q}_{\mathrm{ab}}$ is cyclic of degree n.*

Proof. Since $\mathbb{Q}_{\mathrm{ab}}$ contains all of the roots of unity, it follows from Lemma 1.3(b) that for each n, $\mathbb{Q}_{\mathrm{ab}}(2^{1/2n})$ is a cyclic extension of $\mathbb{Q}_{\mathrm{ab}}$ of degree dividing $2n$. Thus, it suffices to prove that the polynomial $X^n - \sqrt{2}$ is irreducible in $\mathbb{Q}_{\mathrm{ab}}[X]$.

To this end we observe that $\sqrt{2} \in \mathbb{Q}_{\mathrm{ab}} \cap \mathbb{R}$ and prove that both conditions of Lemma 1.1 hold with $(\sqrt{2}, \mathbb{Q}_{\mathrm{ab}})$ replacing (a, K).

Proof of the first condition: Let p be a prime number and assume toward contradiction that $\sqrt{2} \in \mathbb{Q}_{\mathrm{ab}}^p$. Thus, there exists $b \in \mathbb{Q}_{\mathrm{ab}}$ with $\sqrt{2} = b^p$, so $b^{2p} - 2 = 0$.

By Eisenstein's criterion [Lan93, p. 183, Thm. 3.1], the polynomial $X^{2p} - 2$ is irreducible in $\mathbb{Q}[X]$. Since, by the preceding paragraph, one of the roots of $X^{2p} - 2$ is in $\mathbb{Q}_{\mathrm{ab}}$ and since $\mathbb{Q}_{\mathrm{ab}}$ is a Galois extension of $\mathbb{Q}$, all of the roots of

$X^{2p}-2$ are in $\mathbb{Q}_{ab}$. In particular, the unique positive real root $2^{1/2p}$ of $X^{2p}-2$ is in $\mathbb{Q}_{ab}$. Since $\mathrm{Gal}(\mathbb{Q}_{ab}/\mathbb{Q})$ is abelian, $\mathbb{Q}(2^{1/2p})$ is a Galois extension of $\mathbb{Q}$. Hence, all of the roots of $X^{2p}-2$ are in $\mathbb{Q}(2^{1/2p})$. One of them is $\zeta_{2p}2^{1/2p}$, with ζ_{2p} being a nonreal root of unity of order $2p$. Thus, $\zeta_{2p}2^{1/2p}\in\mathbb{Q}(2^{1/2p})\subseteq\mathbb{R}$. Therefore, the nonreal complex number ζ_{2p} is in $\mathbb{R}$. This is a contradiction.

Proof of the second condition: Again, assume toward contradiction that $\sqrt{2}\in -4\mathbb{Q}_{ab}^4$. Thus, there exists $c\in\mathbb{Q}_{ab}$ such that $\sqrt{2}=-4c^4$, so $c^8=\frac{1}{8}$, hence c is a root of the polynomial $8X^8-1$. By Corollary 1.2, $8X^8-1$ is irreducible in $\mathbb{Z}[X]$. Hence, since $\mathbb{Q}_{ab}$ is Galois over $\mathbb{Q}$, each two of the roots of $8X^8-1$ generate the same field over $\mathbb{Q}$. One of these roots is the positive real 8th root $8^{-1/8}$ of $\frac{1}{8}$. Another one is $\zeta_8 8^{-1/8}$, with ζ_8 being a nonreal complex 8th root of 1. Hence, $\zeta_8 8^{-1/8}\in\mathbb{Q}(8^{-1/8})\subseteq\mathbb{R}$, so $\zeta_8\in\mathbb{R}$, which is a contradiction. □

{Lng}

Remark 2.2. Example 3 on page 270 of [Lan93] computes the structure of the Galois group of the polynomial X^4-2 over $\mathbb{Q}$. Among others, this group turns out to be non-abelian, in particular $X^2-\sqrt{2}$ is irreducible over $\mathbb{Q}_{ab}$. The first step in that example chooses a real root α of X^4-2 and notices that $\mathbb{Q}(\alpha)\cap\mathbb{Q}(\sqrt{-1})=\mathbb{Q}$, because otherwise $\sqrt{-1}$ would lie in $\mathbb{R}$.

We have used an analogous argument in each of the two parts of the proof of Lemma 2.1. ∎

Although we are mainly interested here in fields of characteristic 0, we nevertheless prove a result which holds in general. To this end we denote the separable algebraic closure of a field K by K_{sep} and let $\mathrm{Gal}(K)=\mathrm{Gal}(K_{\mathrm{sep}}/K)$ be the absolute Galois group of K. We also denote the fixed field in K_{sep} of an element $\sigma\in\mathrm{Gal}(K)$ by $K_{\mathrm{sep}}(\sigma)$.

Lemma 2.3. *Let K be a countable Hilbertian field and let N be a Galois extension of K with $\mathrm{Gal}(N/K)\cong\hat{\mathbb{Z}}$. Then, there exists $\sigma\in\mathrm{Gal}(K)$ such that $\mathrm{Gal}(K_{\mathrm{sep}}(\sigma))\cong\hat{\mathbb{Z}}$, $K_{\mathrm{sep}}(\sigma)$ is PAC, $N\cap K_{\mathrm{sep}}(\sigma)=K$, and $NK_{\mathrm{sep}}(\sigma)=K_{\mathrm{sep}}$.* {PAC}

Proof. We list the absolutely irreducible polynomials in $K[T,X]$ which are separable in X as $f_0,f_1,f_2,\ldots$ with $f_0(T,X)=X-T$. Let $(t_0,x_0)=(0,0)$. Inductively assume that we have constructed $(t_0,x_0),\ldots,(t_n,x_n)\in K\times K_{\mathrm{sep}}$ such that $f_i(t_i,x_i)=0$ for $i=0,\ldots,n$ and $N\cap K(x_0,\ldots,x_n)=K$. Let K_n be the Galois hull of $K(x_0,\ldots,x_n)/K$ and set $N_n=NK_n$. Then,

$$\mathrm{Gal}(N_n/K)\cong\{(\sigma,\tau)\in\mathrm{Gal}(N/K)\times\mathrm{Gal}(K_n/K)\mid\ \mathrm{res}_{N\cap K_n}\sigma=\mathrm{res}_{N\cap K_n}\tau\}$$

[FrJ08, p. 11, (2f)], so $\mathrm{Gal}(N_n/K)$ is a finitely generated profinite group. By [FrJ08, p. 328, Lemma 16.10.2], $\mathrm{Gal}(N_n/K)$ is a small profinite group in the sense of [FrJ08, p. 329]. Since $f_{n+1}(T,X)$ is absolutely irreducible, $f_{n+1}(T,X)$ is irreducible over N_n. Hence, by [FrJ08, p. 332, Prop. 16.11.1], there exists $t_{n+1}\in K$ such that $f_{n+1}(t_{n+1},X)$ is separable and irreducible over N_n, so also over $K(x_0,\ldots,x_n)$. Choose $x_{n+1}\in K_{\mathrm{sep}}$ with $f_{n+1}(t_{n+1},x_{n+1})=0$. Then,

N_n is linearly disjoint from $K(x_0, \ldots, x_n, x_{n+1})$ over $K(x_0, \ldots, x_n)$, hence $N_n \cap K(x_0, \ldots, x_n, x_{n+1}) = K(x_0, \ldots, x_n)$, so

$$\begin{aligned} N \cap K(x_0, \ldots, x_n, x_{n+1}) &= N \cap N_n \cap K(x_0, \ldots, x_n, x_{n+1}) \\ &= N \cap K(x_0, \ldots, x_n) = K. \end{aligned}$$

This completes the induction.

Having completed the induction, we write $K' = K(x_0, x_1, x_2, \ldots)$. Then, K' is linearly disjoint from N over K and every absolutely irreducible polynomial in $K[T, X]$ has a zero in K'. By [FrJ08, p. 195, Thm. 11.2.3], K' is PAC. Moreover, the restriction map $\mathrm{Gal}(K') \to \mathrm{Gal}(N/K)$ is surjective.

Since $\mathrm{Gal}(N/K) \cong \hat{\mathbb{Z}}$, there exists $\sigma \in \mathrm{Gal}(K')$ whose restriction to N generates $\mathrm{Gal}(N/K)$. Hence, $N \cap K_{\mathrm{sep}}(\sigma) = K$ and $\mathrm{Gal}(N \cdot K_{\mathrm{sep}}(\sigma)/K_{\mathrm{sep}}(\sigma)) \cong \hat{\mathbb{Z}}$. By [FrJ08, p. 331, Cor. 16.10.8], the restriction map $\mathrm{Gal}(K_{\mathrm{sep}}(\sigma)) \to \mathrm{Gal}(NK_{\mathrm{sep}}(\sigma)/K_{\mathrm{sep}}(\sigma))$ is an isomorphism, so $NK_{\mathrm{sep}}(\sigma) = K_{\mathrm{sep}}$ and $\mathrm{Gal}(K_{\mathrm{sep}}(\sigma))$ is isomorphic to $\hat{\mathbb{Z}}$. Since K' is PAC and $K' \subseteq K_{\mathrm{sep}}(\sigma)$, [FrJ08, p. 196, Cor. 11.2.5] implies that $K_{\mathrm{sep}}(\sigma)$ is PAC, as claimed. □

3 Main results

{Main}

We prove the existence of pseudo finite fields with the Laurent property that are algebraic over $\mathbb{Q}_{\mathrm{ab}}$ and use them to construct non-principal ultraproducts of the fields $\mathbb{F}_p$ with pseudo finite algebraic parts that have the Laurent property.

{Sigma}

Theorem 3.1. *There exists $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}})$ such that $\tilde{\mathbb{Q}}(\sigma)$ is pseudo finite and has the Laurent property.*

Proof. By Lemma 2.1, for each positive integer n the extension $\mathbb{Q}_{\mathrm{ab}}(2^{1/2n})/\mathbb{Q}_{\mathrm{ab}}$ is cyclic of degree n. Therefore, by Lemma 1.4, $N := \bigcup_{n=1}^{\infty} \mathbb{Q}_{\mathrm{ab}}(2^{1/2n})$ is a Galois extension of $\mathbb{Q}_{\mathrm{ab}}$ with Galois group $\hat{\mathbb{Z}}$.

Since $\mathbb{Q}$ is countable, so is $\mathbb{Q}_{\mathrm{ab}}$. By Kuyk's theorem, $\mathbb{Q}_{\mathrm{ab}}$ is Hilbertian [FrJ08, p. 333, Thm. 16.11.3]. Thus, taking into account that every subfield of $\tilde{\mathbb{Q}}$ is perfect, Lemma 2.3 supplies $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}})$ such that $\tilde{\mathbb{Q}}(\sigma)$ is pseudo finite, $N \cap \tilde{\mathbb{Q}}(\sigma) = \mathbb{Q}_{\mathrm{ab}}$, and $N\tilde{\mathbb{Q}}(\sigma) = \tilde{\mathbb{Q}}$. Hence, $\tilde{\mathbb{Q}} = \bigcup_{n=1}^{\infty} \tilde{\mathbb{Q}}(\sigma)((\sqrt{2})^{1/n})$. Moreover, for each n the polynomial $X^n - \sqrt{2}$ is irreducible over $\tilde{\mathbb{Q}}(\sigma)$ of degree n. We conclude that $\tilde{\mathbb{Q}}(\sigma)$ has the Laurent property, as desired. □

Theorem 3.1 leads to a partially explicit version of the Gismatullin-Tarasek theorem mentioned in the Introduction.

Given a field F of characteristic 0 we write $F_{\mathrm{alg}} = \tilde{\mathbb{Q}} \cap F$ for the algebraic part of F. Note that the right hand side of the latter equality depends, up to isomorphism, on an embedding of $\tilde{\mathbb{Q}}$ in $\tilde{F}$.

{UltPro}

Theorem 3.2. *There exists a non-principal ultraproduct F^* of the fields $\mathbb{F}_p$, where p ranges over all prime numbers, with the following properties:*

(a) *F^* is pseudo finite,*

(b) F^*_{alg} *is pseudo finite,*
(c) $\tilde{\mathbb{Q}} = \bigcup_{n=1}^{\infty} F^*_{\mathrm{alg}}((\sqrt{2})^{1/n})$ *and* $X^n - \sqrt{2}$ *is irreducible over* F^*_{alg} *for each* n*,*
(d) $\widetilde{F^*} = \bigcup_{n=1}^{\infty} F^*((\sqrt{2})^{1/n})$ *and* $X^n - \sqrt{2}$ *is irreducible over* F^* *for each* n*, so*
(e) F^*_{alg} *and* F^* *have the Laurent property.*

Proof. Theorem 3.1 provides an element $\sigma \in \mathrm{Gal}(\mathbb{Q}_{\mathrm{ab}})$ such that $\tilde{\mathbb{Q}}(\sigma)$ is pseudo finite, so $\mathrm{Gal}(\tilde{\mathbb{Q}}(\sigma)) \cong \hat{\mathbb{Z}}$. Moreover, $\mathrm{Gal}(\tilde{\mathbb{Q}}(\sigma)((\sqrt{2})^{1/n}))$ is the unique extension of $\tilde{\mathbb{Q}}(\sigma)$ of degree n and the union of these extensions is $\tilde{\mathbb{Q}}$, so $\tilde{\mathbb{Q}}(\sigma)$ has the Laurent property.

By [FrJ08, p. 451, Thm. 20.10.8(d)], there exists a non-principal ultraproduct F^* of the fields $\mathbb{F}_p$, where p ranges over all prime numbers, such that $F^*_{\mathrm{alg}} = \tilde{\mathbb{Q}} \cap F^* = \tilde{\mathbb{Q}}(\sigma)$. Together with the previous paragraph, this gives (b) and (c). Moreover, the restriction map ρ: $\mathrm{Gal}(F^*) \to \mathrm{Gal}(\tilde{\mathbb{Q}}(\sigma))$ is surjective. By [FrJ08, p. 451, Thm. 20.10.8(a)], F^* is pseudo finite (as stated in (a)), in particular $\mathrm{Gal}(F^*) \cong \hat{\mathbb{Z}}$. Hence, by [FrJ08, p. 331, Cor. 16.10.8], ρ is an isomorphism. Therefore, $\widetilde{F^*} = \bigcup_{n=1}^{\infty} F^*((\sqrt{2})^{1/n})$. Moreover, for each n the polynomial $X^n - \sqrt{2}$ is irreducible over F^* of degree n, as stated in (d). □

4 Concluding Remarks

{CNRM}

We notice that the set of all $\sigma \in \mathrm{Gal}(\mathbb{Q})$ such that $\tilde{\mathbb{Q}}(\sigma)$ is a Laurent field has Haar measure 0. Then we consider the set $\mathcal{Q}$ of all non-principal ultraproducts of finite fields, finitely many in each characteristic, and prove that it is "rare" for a field $F \in \mathcal{Q}$ to be a Laurent field.

Finally, we consider the set $\mathcal{P}$ of all non-principal ultraproducts of the fields $\mathbb{F}_p$ and prove, under the continuum hypothesis, that if $F, F' \in \mathcal{P}$ and F' is elementarily equivalent to F, then $F' \cong F$. Thus, since F has a Laurent element, so does F'.

For a field K we let μ_K be the unique Haar measure of $\mathrm{Gal}(K)$ with $\mu_K(\mathrm{Gal}(K)) = 1$ [FrJ08, p. 366, Prop. 18.2.1].

{ZERO}

Theorem 4.1. *Let* K *be a countable field of characteristic* 0*. Then the set of all* $\sigma \in \mathrm{Gal}(K)$ *such that* $\tilde{K}(\sigma)$ *has the Laurent property has* μ_K*-measure zero.*

Proof. Suppose that $\tilde{K}(\sigma)$ with $\sigma \in \mathrm{Gal}(K)$ has the Laurent property. In particular, the field $\tilde{K}(\sigma)$ has an element a such that $[\tilde{K}(\sigma)(a^{1/p}) : \tilde{K}(\sigma)] = p$ for each prime number p.

In particular, $M := K(a)$ contains a but $M(a^{1/p}) \not\subseteq \tilde{K}(\sigma)$. Hence, by Lemma 1.1, $[M(a^{1/p}) : M] = p$ and

$$\sigma \notin \mathrm{Gal}(M(a^{1/p})). \qquad (1)$$ {rar1}

Let $S_{K,a,p}$ be the set of all $\sigma \in \mathrm{Gal}(M)$ that satisfy (1), that is $S_{K,a,p} = \mathrm{Gal}(M) \smallsetminus \mathrm{Gal}(M(a^{1/p}))$. By [FrJ08, p. 364, Lemma 18.1.1(a)], $\mu_M(S_{K,a,p}) = 1 - \frac{1}{p}$. By [FrJ08, p. 374, Example 18.3.8], the profinite groups $\mathrm{Gal}(M(a^{1/p})) =$

$\mathrm{Gal}(M) \smallsetminus S_{K,a,p}$, with p ranging over all prime numbers are μ_M-independent. Therefore, by [FrJ08, p. 372, Lemma 18.3.4 and Example 18.3.3], the set $S_{K,a} := \bigcap_p S_{K,a,p}$ satisfies

$$\mu_M(S_{K,a}) = \prod_p \mu_M(S_{K,a,p}) = \prod_p \big(1 - \frac{1}{p}\big) = 0. \qquad (2)$$ {rar2}

Therefore, by [FrJ08, p. 370, Prop. 18.2.4], $\mu_K(S_{K,a}) = \frac{1}{[M:K]}\mu_M(S_{K,a}) = 0$.

Let $S := \bigcup_a S_{K,a}$, where a ranges over the countably many elements in $\tilde{K}$ that satisfy $[K(a)(a^{1/p}) : K(a)] = p$ for each prime number p. Then, S contains the set of all $\sigma \in \mathrm{Gal}(K)$ such that $\tilde{K}(\sigma)$ has the Laurent property. Since K is countable, we have by the sentence following (2) that $\mu_K(S) = 0$. Hence, the Haar measure of all $\sigma \in \mathrm{Gal}(K)$ such that $\tilde{K}(\sigma)$ has the Laurent property is 0. ∎

{RARE}

Remark 4.2. Let F be a field of characteristic 0 that has the Laurent property with a Laurent element a.

Suppose that $\mathrm{Gal}(F) \cong \hat{\mathbb{Z}}$. In particular, F has for each $n \in \mathbb{N}$ a unique extension F_n of degree n [FrJ08, p. 14, Lemma 1.4.4], F_n/F is Galois, and $\mathrm{Gal}(F_n/F) \cong \mathbb{Z}/n\mathbb{Z}$. By definition, $[F(a^{1/n}) : F] = n$, so $F_n = F(a^{1/n})$.

In particular, for every prime number p and with ζ_p being the primitive root of 1 of order p, we have $\zeta_p a^{1/p} \in F(a^{1/p})$, so $\zeta_p \in F(a^{1/p})$. Since $[F(\zeta_p) : F] | p-1$, we conclude that $\zeta_p \in F$. It follows that the compositum L of all fields $\mathbb{Q}(\zeta_p)$ with p ranging over all prime numbers is contained in F. Note that L is an infinite algebraic extension of $\mathbb{Q}$. ∎

{ULPR}

Example 4.3. Every non-principal ultraproduct F of distinct finite fields is psuedo finite [FrJ08, p. 449, Lemma. 20.10.1]. Moreover, if $F = \prod_{q\in\mathcal{Q}} \mathbb{F}_q/\mathcal{D}$, where $\mathcal{Q}$ is the set of all prime powers and $\{q \in \mathcal{Q} \mid\ p|q\}$ is finite for every prime number p, and where $\mathcal{D}$ is a non-principal ultrafilter on $\mathcal{Q}$, then $\mathrm{char}(F) = 0$, so $F \cap \tilde{\mathbb{Q}} = \tilde{\mathbb{Q}}(\sigma)$ for some $\sigma \in \mathrm{Gal}(\mathbb{Q})$. Hence, by Remark 4.2, F does not have the Laurent property, unless the field L introduced in Remark 4.2 is contained in F. Since $[L : \mathbb{Q}] = \infty$, it is "rare" for F to have the Laurent property. In particular, the example of Gismatullin and Tarasek for an ultraproduct of finite fields having the Laurent property mentioned in the Introduction is "rare". ∎

We end our note with a discussion of the Laurent property among the set $\mathcal{P}$ of all non-principal ultraproducts of the fields $\mathbb{F}_p$, with p ranging on all prime numbers.

{SATR}

Remark 4.4. Suppose that F and F' are elementarily equivalent fields in the language of rings with F being a Laurent field. Then, it is not clear whether F' is also a Laurent field.

However, if $F, F' \in \mathcal{P}$, then by [FrJ08, p. 143, Lemma 7.7.4], both F and F' are $\aleph_1$-saturated. In addition, their cardinality is $2^{\aleph_0}$. Assuming the continuum hypothesis $2^{\aleph_0} = \aleph_1$, we may conclude from [Pil02, p. 39, Prop. 4.5] that $F \cong F'$. Alternatively, we may apply [Mar02, p. 144, Thm. 4.3.20] to the complete theory $T := \mathrm{Th}(F) = \mathrm{Th}(F')$ to achieve the same conclusion.

Since F has a Laurent element, so does F'. Hence, F' has the Laurent propery. ∎

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