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GALOIS STRATIFICATION OVER e-FOLD ORDERED FROBENIUS FIELDS

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ABSTRACT

An e-fold ordered field is said to be Frobenius if it is a PRCe field which has the embedding property. By means of a Galois stratification procedure we prove that the theory of e-fold ordered Frobenius fields is decidable.

Introduction

A field M is said to be **pseudo algebraically closed** (PAC) if every absolutely irreducible variety over M has an M-rational point. A **Frobenius** field is a PAC field with the embedding property. Developing the method of Galois stratification introduced in [FS], M. Fried, M. Jarden, and the first author established a decision procedure for Frobenius fields [FHJ1].

The analogue of PAC, in the case of ordered fields, is **pseudo real closed (PRC)**. A field M is PRC if every absolutely irreducible variety over M has an M-rational point provided it has an $\overline{M}$-rational simple point for each real

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closure $\overline{M}$ of M [P2, Theorem 1.2]. A **PRC***e* field is a PRC field with exactly e orderings.

The main goal of this paper is to establish an equivalent of Frobenius fields in the class of PRC*e* fields. To achieve this we use a technical tool: "e-structures". An e-**structure** is an $(e+1)$-tuple $\mathbf{G} = (G; \mathcal{E}_1, \ldots, \mathcal{E}_e)$, where G is a profinite group and the $\mathcal{E}_j$ are conjugacy classes of involutions. A typical example is $\mathbf{G} = \mathbf{G}(\widetilde{M}/M, \mathbf{P})$, where G is the absolute Galois group of an e-fold ordered field $(M, \mathbf{P}) = (M, P_1, \ldots, P_e)$, and $\mathcal{E}_j$ is the set of the involutions in G such that P_j extends to their fixed fields.

A PRC*e* field $(M, \mathbf{P})$ is said to be **Frobenius** if $\mathbf{G}(\widetilde{M}/M, \mathbf{P})$ has the embedding property (in the category of e-structures). Geyer fields and v.d. Dries fields are shown to be Frobenius. Along the lines of treatment in [FHJ1] we find a decision procedure for the theory of Frobenius fields in the language of e-fold ordered fields.

A theorem of M. Knebusch allows us to extend the notion of decomposition group to the case of e-fold ordered fields. This, together with the use of non-singular basic sets are the main new ingredients to obtain this result.

We shall assume, for simplicity, that all our fields are of characteristic 0. This partly excludes the case $e = 0$, dealt with in [FHJ1].

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1. Ordered fields and effectiveness

An **ordering** on a field F is a set $P \subseteq F$ such that $P+P \subseteq P$, $P \cdot P \subseteq P$, $P \cap -P = \{0\}$ and $P \cup -P = F$. Let X_F denote the set of orderings of F. The **Harrison topology** on X_F is defined by the basis $\{H_F(\alpha_1, \ldots, \alpha_n) | \; \alpha_1, \ldots, \alpha_n \in F\}$, where the **Harrison set** $H_F(\alpha_1, \ldots, \alpha_n)$ is $\{P \in X_F | \; \alpha_1, \ldots, \alpha_n \in P\}$. This topology makes X_F a Boolean space [P1, Theorem 6.5]. Let K/F be an arbitrary field extension. The map $\mathrm{res}_F \colon X_K \to X_F$ defined by $\mathrm{res}_F(P) = P \cap F$ is continuous. If the extension K/F is finitely generated, then res_F is also open [ELW, 4.bis]. We call K/F **totally real** if res_F is surjective.

The **real closure** $(\overline{K}, \overline{P})$ of an ordered field (K, P) is a maximal ordered algebraic extension of (K, P). It exists, is unique up to a K-isomorphism, and $\overline{P} = \overline{K}^2$.

Definition 1.1: Let $\varphi \colon R \to K$ be a homomorphism from a domain R into a field K. Let P be an ordering on K. An ordering Q on the quotient field of R is

φ-**compatible** with P if $a \in R \cap Q$ implies $\varphi(a) \in P$. ∎

The following theorem guarantees the extension of compatible orderings.

PROPOSITION 1.2 (Knebusch): *Let $R \subseteq S$ be regular domains, and let $E \subseteq F$ be their quotient fields. Let (K, P) be an ordered field, and let $\varphi\colon R \to K$ be a homomorphism that extends to a homomorphism $\psi\colon S \to K$. Then every ordering on R that is φ-compatible with P extends to an ordering on S that is ψ-compatible with P. In particular, there exists an ordering on S that is ψ-compatible with P.*

Proof: We may assume that (K, P) is real closed, otherwise replace it by its real closure. As S is regular (i.e., its localization at each prime is a regular local ring), ψ extends to a place $\psi'\colon F \to K \cup \{\infty\}$ [JR, Appendix A]. Its restriction to E is a place $\varphi'\colon E \to K \cup \{\infty\}$ that extends φ. By [K, Theorem 2.6] every φ'-compatible ordering on E extends to a ψ'-compatible ordering on F.

The last assertion of the theorem follows by replacing φ by its restriction $\varphi_0\colon \mathbb{Z} \to K$, since the unique ordering on $\mathbb{Q}$ is φ_0-compatible with P. ∎

Remark 1.3: Note that if R is a finitely generated ring over a field K then there is $0 \neq d \in R$ such that $R[d^{-1}]$ is regular. Indeed, write R as $K[\mathbf{x}, g(\mathbf{x})^{-1}]$, where $\mathbf{x} = (x_1, \ldots, x_n)$ is a generic point of a K-irreducible set V over K, and let $A = V \smallsetminus V(g)$. Put $k = n - \dim_K V$. Since $\mathbf{x}$ is a non-singular point of V, there exists a $(k \times k)$-submatrix of the Jacobian matrix $(\partial f_i / \partial X_j)$ with determinant $d(\mathbf{x}) \neq 0$. The open subset $A' = A \smallsetminus V(d)$ of A is non-singular, and hence $K[A'] = K[\mathbf{x}, (gd)(\mathbf{x})^{-1}] = R[d^{-1}]$ is a regular ring [N, Theorem 46.3, Corollary 14.6]. ∎

The primitive recursiveness of a decision procedure for fields requires that all operations involved are computable in a primitive recursive way. We supplement the extensive treatment of [FJ, §17] by discussion of orderings and inequalities.

Let K be a presented field [FJ, Definition 17.1]. An ordering P on K is **presented** if "$\in P$" or, equivalently, "$>_P 0$" is a primitive recursive relation. An e-fold ordered field $(K, P_1, \ldots, P_e)$ is **presented** if K is a presented field and $P_1, \ldots, P_e$ are presented orderings.

A **polynomial relation** (or a quantifier free formula in the language of ordered fields) is a Boolean combination of relations of the form $p(x_1, \ldots, x_n) > 0$, where p is a polynomial with integral coefficients.

P.J. Cohen [C, §1, Theorems A_n ,B_n] proves the following version of Tarski's Principle:

PROPOSITION 1.4: *Let $n \geq 1$. For each polynomial relation $A(X_1, \ldots, X_n)$ we can find by a primitive recursive procedure a polynomial relation $B(X_2, \ldots, X_n)$ such that*

$$(\exists X_1)\ A(X_1, \ldots, X_n) \longleftrightarrow B(X_2, \ldots, X_n)$$

holds over every real closed field.

We will need the following criterion (cf. [P2, (0.4)] and [L, XI, §3]).

COROLLARY 1.5: *Let (K, P) be an ordered field and let $(\overline{K}, \overline{P})$ be its real closure. Let $V \subseteq \mathbb{A}^n$ be an affine K-variety with generic point $\mathbf{x}$ over K and let $F = K(\mathbf{x})$ be its function field. Let $h_1, \ldots, h_r \in K[X_1, \ldots, X_n]$ be polynomials not vanishing on V. Then P extends to an ordering on F contained in $H_F = H_F(h_1(\mathbf{x}), \ldots, h_r(\mathbf{x}))$ if and only if there is a nonsingular point $\mathbf{a} \in V(\overline{K})$ such that $h_1(\mathbf{a}), \ldots, h_r(\mathbf{a}) > 0$ with respect to $\overline{P}$.*

Proof: Write V as $V(f_1, \ldots, f_m)$, with $f_i \in K[\mathbf{X}]$. If P extends to an ordering Q on F contained in H_F, then the substitution $\mathbf{X} \to \mathbf{x}$ shows that the sentence

$$(\exists \mathbf{X}) \bigwedge_{i=1}^{m} f_i(\mathbf{X}) = 0 \wedge \bigwedge_{\nu=1}^{r} h_\nu(\mathbf{X}) > 0 \wedge \operatorname{rank}(\partial f_i / \partial X_j) = n - \dim V$$

holds in (F, Q), and hence also in its real closure. It immediately follows from Proposition 1.4 that the sentence also holds in $(\overline{K}, \overline{P})$. This produces the desired nonsingular point $\mathbf{a} \in V(\overline{K})$.

Conversely, let $\mathbf{a} \in V(\overline{K})$ be a nonsingular point with $h_1(\mathbf{a}), \ldots, h_r(\mathbf{a}) > 0$. The local ring S of $\mathbf{a}$ on V is regular and $\mathbf{x} \to \mathbf{a}$ defines a K-homomorphism $\psi\colon S \to \overline{K}$. Proposition 1.2, applied to the rings $K \subseteq S$, produces an extension $Q \in X_F$ of P that is ψ-compatible with $\overline{P}$. We have $h_\nu(\mathbf{x}) \in Q$ for each ν, otherwise $-h_\nu(\mathbf{x}) \in Q$, and hence the ψ-compatability gives $-h_\nu(\mathbf{a}) = \psi(h_\nu(\mathbf{x})) \geq 0$, a contradiction. ∎

Let K be a presented field. A union $\bigcup_{i=1}^{k} H_K(h_{i1}, \ldots, h_{ir_i})$ of Harrison sets is **presented** if all the $h_{ij} \in K$ and the numbers $k, r_1, \ldots, r_k$ are explicitly given.

The intersection, union, and complement are effective operations on the Harrison topology, i.e., the result of the operation on presented sets is presented. Furthermore:

LEMMA 1.6: *Let K be a presented field, and let F be a field finitely generated and presented over K. Let H_F be a presented Harrison set. Then $H_K = \mathrm{res}_K H_F$ can be effectively computed.*

Moreover, let R be a presented subring of K, and let $f_1, \dots, f_m, h_1, \dots, h_r \in R[X_1, \dots, X_n]$ be given polynomials such that F is the function field of the K-variety $V = V(f_1, \dots, f_m)$. Let $\mathbf{x}$ be the generic point of V over K. Then we can compute a finite subset $\{q_{ij} \mid i \in I,\ j \in J(i)\}$ of R and $0 \neq p \in R$ such that

(a) $\mathrm{res}_K H_F(h_1(\mathbf{x}), \dots, h_r(\mathbf{x})) = \bigcup_{i\in I} \bigcap_{j\in J(i)} H_K(q_{ij})$.

(b) *Let $\varphi: R \to K'$ be a homomorphism into a field K' (write φ as $a \mapsto a'$ and extend it to polynomials) such that $p' \neq 0$ and $V' = V(f_1', \dots, f_m')$ is a K'-variety. Let $\mathbf{x}'$ be the generic point of V' over K' and let $F' = K'(\mathbf{x}')$. Then $\mathrm{res}_{K'} H_{F'}(h_1'(\mathbf{x}'), \dots, h_r'(\mathbf{x}')) = \bigcup_{i\in I} \bigcap_{j\in J(i)} H_{K'}(q_{ij}')$.*

Proof: Let N be a bound on the total degrees of the f_i and h_j. It follows from Corollary 1.5 that there is a formula $\theta(\mathbf{Y})$ in the language of rings with the predicate $>$, that depends only on n, N, m, and r, with the following property. Let $(\overline{K}', \overline{P}')$ be the real closure of an ordered field (K', P'). Let $f_1', \dots, f_m', h_1', \dots, h_r' \in K'[\mathbf{X}]$ be of total degree $\leq N$, and let $\mathbf{c}'$ be the sequence of their coefficients. Assume that $V' = V(f_1', \dots, f_m')$ is a K'-variety, let $\mathbf{x}'$ be its generic point over K', and let $F' = K'(V')$. Then $P' \in \mathrm{res}_{K'} H_{F'}(h_1', \dots, h_r')$ if and only if $(\overline{K}', \overline{P}') \models \theta(\mathbf{c}')$.

By Proposition 1.4 we may assume that θ is quantifier free. After some trivial identifications we can write it as $\bigvee_{i=1}^k p_i(\mathbf{Y}) \neq 0 \wedge \bigwedge_{j\in J(i)} q_{ij}(\mathbf{Y}) \geq 0$, for suitable $p_i, q_{ij} \in \mathbb{Z}[\mathbf{Y}]$. Furthermore, $(K', P') \models \theta(\mathbf{c}')$ if and only if $(\overline{K}', \overline{P}') \models \theta(\mathbf{c}')$. Finally, the clause '$p_i(\mathbf{c}') \neq 0$' does not depend on P'. Therefore

$$(1.7) \quad P' \in \mathrm{res}_{K'} H_{F'}(h_1'(\mathbf{x}'), \dots, h_r'(\mathbf{x}')) \quad \Leftrightarrow \quad P' \in \bigcup_{i\in I(\mathbf{c}')} \bigcap_{j\in J(i)} H_K(q_{ij}(\mathbf{c}')),$$

where $I(\mathbf{c}') = \{1 \leq i \leq k \mid p_i(\mathbf{c}') \neq 0\}$.

Let $\mathbf{c}$ be the sequence of coefficients of the f_i and h_j. Assertion (a) follows from (1.7) with $I = I(\mathbf{c})$ and $q_{ij} = q_{ij}(\mathbf{c})$.

Furthermore, put $p = \prod_{i\in I} p_i(\mathbf{c})$. Let $\varphi: R \to K'$ be as in (b). Then $\mathbf{c}' = \varphi(\mathbf{c})$ is the sequence of coefficients of the f_i' and h_j'. If $p' \neq 0$, then $I(\mathbf{c}') = I(\mathbf{c})$. Therefore assertion (b) follows from (1.7). ∎

2. e-Structures

Definition 2.1: An e-**structure** $\mathbf{G}$ is a system $\mathbf{G} = (G; \mathcal{E}_1, \ldots, \mathcal{E}_e)$, where G is a profinite group and $\mathcal{E}, \ldots, \mathcal{E}_e$ are conjugacy classes of involutions in G. If G is a pro-2 group, then $\mathbf{G}$ is said to be a **pro-2** e-structure. For an e-structure $\mathbf{G}$ we refer to $\mathcal{E}_j$ as $\mathcal{E}_j(\mathbf{G})$, and to the underlying group as G. We put $\mathcal{E}(\mathbf{G}) = \bigcup_{j=1}^{e} \mathcal{E}_j(\mathbf{G})$. A **morphism (epimorphism)** $\varphi\colon \mathbf{G} \to \mathbf{H}$ between two e-structures is a morphism (epimorphism) $\varphi\colon G \to H$ that maps $\mathcal{E}_j(\mathbf{G})$ into (onto, a fortiori) $\mathcal{E}_j(\mathbf{H})$.

We say that $\mathbf{G}$ is a **substructure** of $\mathbf{H}$ if $G \leq H$ and $\mathcal{E}_j(\mathbf{G}) \subseteq \mathcal{E}_j(\mathbf{H})$ for each j.

Let $n \geq 0$. For a sequence $(\varepsilon; x) = (\varepsilon_1, \ldots, \varepsilon_e; x_1, \ldots, x_n)$ of elements of G we write $(\varepsilon; x) \in \mathbf{G}^{(e;n)}$, if $\varepsilon_j \in \mathcal{E}_j(\mathbf{G})$ for $j = 1, \ldots, e$. We say that $(\varepsilon; x) \in \mathbf{G}^{(e;n)}$ **generates** $\mathbf{G}$ if $G = \langle \varepsilon_1, \ldots, \varepsilon_e, x_1, \ldots, x_n \rangle$. ∎

Example 2.2: Let $(E, \mathbf{Q}) = (E, Q_1, \ldots, Q_e)$ be an e-fold ordered field, and let F/E be a Galois extension such that F is not formally real. Let $G = G(F/E)$ be the Galois group of F/E. Denote by $\mathcal{E}_j = \mathcal{E}_j(F/E, \mathbf{Q})$ the set of involutions ε in G such that Q_j extends to an ordering of $F(\varepsilon)$. This is a conjugacy class in G [HJ1, Proposition 2.1]. The e-structure $\mathbf{G}(F/E, \mathbf{Q}) = (G; \mathcal{E}_1, \ldots, \mathcal{E}_e)$ is called a **Galois** e-structure.

The **absolute** Galois e-structure of $(K, \mathbf{Q})$ is $\mathbf{G}(K, \mathbf{Q}) = \mathbf{G}(\widetilde{K}/K, \mathbf{Q})$, where $\widetilde{K}$ is the algebraic closure of K.

Let $\mathbf{G}(L/K, \mathbf{P})$ be another Galois e-structure such that $(K, \mathbf{P}) \subseteq (E, \mathbf{Q})$, E and L are linearly disjoint over K, and $L \subseteq F$. Then the restriction map res: $\mathbf{G}(F/E, \mathbf{Q}) \to \mathbf{G}(L/K, \mathbf{P})$ is an epimorphism [HJ1, Lemma 3.5]. ∎

Definition 2.3: Let $(K, \mathbf{P})$ be an e-fold ordered field, and let F/E be a Galois extension with $K \subseteq E$ and F not formally real. Denote

$$\mathrm{Sub}[F/E, \mathbf{P}] = \{\mathbf{G}(F/E', \mathbf{Q}') |\ E \subseteq E' \subseteq F,\ (K, \mathbf{P}) \subseteq (E', \mathbf{Q}')\}$$

This is the collection of all e-structures $\mathbf{H}$ such that $H \leq G(F/E)$ and P_j extends to the fixed field of $\varepsilon \in \mathcal{E}_j(\mathbf{H})$, for each $1 \leq j \leq e$. ∎

Definition 2.4: A (a pro-2) e-structure $\mathbf{G}$ is **free** if there is $n \geq 0$ and $(\varepsilon; x) \in \mathbf{G}^{(e;n)}$ with the following property. Given a (pro-2) e-structure $\mathbf{A}$ and $(\delta; a) \in \mathbf{A}^{(e;n)}$, there exists a unique morphism $\varphi\colon \mathbf{G} \to \mathbf{A}$ that maps $(\varepsilon; x)$ on $(\delta; a)$. We then call $(\varepsilon; x)$ a **basis** of the structure $\mathbf{G}$. ∎

Example 2.5: Let $\widehat{D}_{e,n}$ be the real free group with basis $(\varepsilon_1, \ldots, \varepsilon_e; x_1, \ldots, x_n)$ [HJ2, p. 157]. Let $\mathcal{E}_j$ be the conjugacy class of the involution ε_j in $\widehat{D}_{e,n}$. Then the e-structure $\hat{\mathbf{D}}_{e,n} = (\widehat{D}_{e,n}; \mathcal{E}_1, \ldots, \mathcal{E}_e)$ is free.

The corresponding example in the category of pro-2 e-structures will be denoted $\hat{\mathbf{D}}_{e,n}(2)$. ∎

3. Projective and superprojective e-structures

An **embedding problem** for an e-structure $\mathbf{G}$ is a diagram

(3.1)
$$\begin{array}{ccc} & & \mathbf{G} \\ & & \downarrow{\scriptstyle\rho} \\ \mathbf{B} & \xrightarrow{\pi} & \mathbf{A} \end{array}$$

in which π is an epimorphism and ρ is a morphism of e-structures. The problem is **finite** if $\mathbf{B}$ is finite. The problem is **proper** if ρ is an epimorphism. A morphism (epimorphism) λ: $\mathbf{G} \to \mathbf{B}$ such that $\pi \circ \lambda = \rho$ is called a **solution** (**proper solution**) to the embedding problem (proper embedding problem).

Let $\operatorname{Im}\mathbf{G}$ be the set of finite e-structures $\mathbf{B}$ for which there exists an epimorphism $\mathbf{G} \to \mathbf{B}$.

Definition 3.2: An e-structure $\mathbf{G}$ is **superprojective** if

(i) $\mathbf{G}$ is **projective**, i.e., every finite embedding problem (3.1) for $\mathbf{G}$ is solvable. (Replacing $\mathbf{A}$ by $\rho(\mathbf{G})$ and B by $\pi^{-1}(A)$ we may assume that (3.1) is proper.)

(ii) $\mathbf{G}$ has the **embedding property**, i.e., every finite proper embedding problem (3.1) with $\mathbf{B} \in \operatorname{Im}\mathbf{G}$ has a proper solution. ∎

To prove that free e-structures are superprojective we need the following analogue of Gaschütz' lemma [FJ, Lemma 15.30 and J1, Lemma 5.3].

LEMMA 3.3: *Let ρ: $\mathbf{G} \to \mathbf{A}$ be an epimorphism of e-structures. Assume that $(\varepsilon; a_n) \in \mathbf{A}^{(e;n)}$ generates $\mathbf{A}$ and that an element of $\mathbf{G}^{(e;n)}$ generates $\mathbf{G}$. Then there exists a system of generators $(\delta; g) \in \mathbf{G}^{(e;n)}$ of $\mathbf{G}$ such that $\rho(\delta; g) = (\varepsilon; a_n)$.*

Proof: Since the epimorphism ρ can be represented by an inverse limit of epimorphisms between finite structures and since an inverse limit of finite nonempty sets is not empty, we may assume that $\mathbf{G}$ is a finite e-structure.

Let $\mathbf{C}$ be an e-substructure of $\mathbf{G}$ such that $\rho(\mathbf{C}) = \mathbf{A}$. For every $(\varepsilon; a) \in \mathbf{A}^{(e;n)}$ that generates $\mathbf{A}$ let $\Psi_{\mathbf{C}}(\varepsilon; a)$ be the set of $(\delta; g) \in \mathbf{C}^{(e;n)}$ that satisfy $\rho(\delta; g) =$

$(\varepsilon; a)$, and let $\Phi_{\mathbf{C}}(\varepsilon; a)$ be the set of those $(\delta; g) \in \Psi_{\mathbf{C}}(\varepsilon; a)$ that generate $\mathbf{C}$. We show by induction on $|C|$ that $|\Phi_{\mathbf{C}}(\varepsilon; a)|$ is independent of $(\varepsilon; a)$.

First notice that $|\Psi_{\mathbf{C}}(\varepsilon; a)|$ is independent of $(\varepsilon; a)$. Indeed,

$$|\{g_j \in C|\ \rho(g_j) = a_j\}| = |\ker \mathrm{res}_C \rho|,$$

and if $\varepsilon_i, \varepsilon_i' \in \mathcal{E}_i(\mathbf{A})$ then $\{\delta_i \in \mathcal{E}_i(\mathbf{C})|\ \rho(\delta_i) = \varepsilon_i\}$ and $\{\delta_i \in \mathcal{E}_i(\mathbf{C})|\ \rho(\delta_i) = \varepsilon_i'\}$ are conjugate in C, and hence have the same number of elements. Since every $(\delta; g) \in \Psi_{\mathbf{C}}(\varepsilon; a)$ generates an e-substructure $\mathbf{B}$ of $\mathbf{C}$ with $\rho(\mathbf{B}) = \mathbf{A}$, we have

$$|\Psi_{\mathbf{C}}(\varepsilon; a)| = |\Phi_{\mathbf{C}}(\varepsilon; a)| + \sum_{\substack{\mathbf{B}<\mathbf{C} \\ \rho(\mathbf{B})=\mathbf{A}}} |\Phi_{\mathbf{B}}(\varepsilon; a)|.$$

By the induction hypothesis the $|\Phi_{\mathbf{B}}(\varepsilon; a)|$ are independent of $(\varepsilon; a)$. Therefore so is $|\Phi_{\mathbf{C}}(\varepsilon; a)|$.

Let $(\delta'; g') \in \mathbf{G}^{(e;n)}$ generate $\mathbf{G}$. Then $(\rho(\delta'); \rho(g'))$ generates $\mathbf{A}$, and hence $|\Phi_{\mathbf{G}}(\varepsilon; a)| = |\Phi_{\mathbf{G}}(\rho(\delta'); \rho(g'))| \geq 1$. ∎

COROLLARY 3.4:
(i) *The free e-structure* $\hat{\mathbf{D}}_{e,n}$ *is superprojective.*
(ii) *The free pro-2 e-structure* $\hat{\mathbf{D}}_{e,n}(2)$ *is superprojective.*

Proof: Consider an embedding problem (3.1) for $\mathbf{G} = \hat{\mathbf{D}}_{e,n}$. Let $(\varepsilon; a) \in \mathbf{G}^{(e;n)}$ be a basis for $\mathbf{G}$. As $\pi(\mathbf{B}) = \mathbf{A}$, there is $(\delta; g) \in \mathbf{B}^{(e;n)}$ such that $\pi(\delta; g) = \rho(\varepsilon; a)$. The map $(\varepsilon; a) \to (\delta; g)$ extends to a solution $\lambda\colon \mathbf{G} \to \mathbf{B}$ of (3.1).

Assume that (3.1) is proper and that $\mathbf{B} \in \mathrm{Im}\,\hat{\mathbf{D}}_{e,n}$. Then $\rho(\varepsilon; a)$ generates $\mathbf{A}$. By Lemma 3.3 we may assume that $(\delta; g)$ generates $\mathbf{B}$. Therefore λ is an epimorphism.

The superprojectivity of $\hat{\mathbf{D}}_{e,n}(2)$ is proved analogously. We only remark that in the embedding problem (3.1) for $\mathbf{G} = \hat{\mathbf{D}}_{e,n}(2)$ we may replace A and B by their 2-Sylow subgroups to assume that $\mathbf{A}$ and $\mathbf{B}$ are pro-2 e-structures. ∎

We conclude this section with some results on projective e-structures.

LEMMA 3.5: *Let* $\mathbf{G}$ *be an e-structure.*
(a) *If* $\mathbf{G}$ *is projective, then* $\mathcal{E}(\mathbf{G})$ *is the set of all involutions in* G, *the* $\mathcal{E}_j(\mathbf{G})$ *are disjoint (i.e., pairwise distinct), and there is an open subgroup* G' *of* G *of index* ≤ 2 *that does not meet* $\mathcal{E}(\mathbf{G})$.

(b) *Let N be an open subgroup of G. Assume that every finite proper embedding problem (3.1) for $\mathbf{G}$ with $\ker(\rho) \leq N$ is solvable. Then $\mathbf{G}$ is projective.*

Proof: (a) Let $\varepsilon \in G$ be an involution. If $\varepsilon \notin \mathcal{E}(\mathbf{G})$, then there is an epimorphism ρ: $\mathbf{G} \to \mathbf{A}$ onto a finite quotient of $\mathbf{G}$ such that $\rho(\varepsilon) \notin \mathcal{E}(\mathbf{A})$. By [HJ1, Corollary 6.2 with $I = \mathcal{E}(\mathbf{A})$] there is an epimorphism π: $\mathbf{B} \to \mathbf{A}$ of finite e-structures that maps the involutions of B into $\{1\} \cup \mathcal{E}(\mathbf{A})$. Let λ: $\mathbf{G} \to \mathbf{B}$ be a solution to this embedding problem. Then π maps the involution $\lambda(\varepsilon)$ onto $\rho(\varepsilon)$, and hence $\rho(\varepsilon) \in \mathcal{E}(\mathbf{A})$. Thus $\varepsilon \in \mathcal{E}(\mathbf{G})$.

Again, let ρ: $\mathbf{G} \to \mathbf{A}$ be an epimorphism onto a finite quotient of $\mathbf{G}$. There is another finite e-structure $\mathbf{B}$ and another epimorphism π: $\mathbf{B} \to \mathbf{A}$ such that the $\mathcal{E}_j(\mathbf{B})$ are disjoint and there is an open subgroup B' of B of index ≤ 2 that does not meet $\mathcal{E}(\mathbf{B})$. (E.g., let $\mathbf{B}$ be a sufficiently large quotient of $\hat{\mathbf{D}}_{e,n}$ with a homomorphism β: $B \to \mathbb{Z}/2\mathbb{Z}$ that maps $\mathcal{E}(\mathbf{B})$ on the generator of $\mathbb{Z}/2\mathbb{Z}$ and let $B' \ker(\beta)$.) The existence of a solution λ: $\mathbf{G} \to \mathbf{B}$ to this embedding problem shows that the $\mathcal{E}_j(\mathbf{G})$ are disjoint and $\lambda^{-1}(B')$ does not meet $\mathcal{E}(\mathbf{G})$.

(b) Let (3.1) be a finite proper embedding problem for $\mathbf{G}$. Let ρ_1: $\mathbf{G} \to \mathbf{A}_1$ be an epimorphism onto a finite quotient $\mathbf{A}_1$ of $\mathbf{G}$ with $\ker(\rho_1) \leq N \cap \ker(\rho)$. Then ρ factors into ρ_1 and a morphism ρ_2: $\mathbf{A}_1 \to \mathbf{A}$. For each j choose $\varepsilon_j \in \mathcal{E}_j(\mathbf{G})$ and $\delta_j \in \mathcal{E}(\mathbf{B})$ such that $\pi(\delta_j) = \rho(\varepsilon_j)$. This yields a commutative diagram of epimorphisms

$$\begin{array}{ccc} & & \mathbf{G} \\ & & \downarrow{\scriptstyle \rho_1} \\ \mathbf{B}_1 & \xrightarrow{\pi_1} & \mathbf{A}_1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \rho_2} \\ \mathbf{B} & \xrightarrow{\pi} & \mathbf{A} \end{array}$$

in which $B_1 = B \times_A A_1$, and $\mathcal{E}_j(\mathbf{B}_1)$ is the conjugacy class of $(\delta_j, \varepsilon_j)$. By assumption there is λ_1: $\mathbf{G} \to \mathbf{B}_1$ such that $\pi_1 \circ \lambda = \rho_1$. Clearly, $p \circ \lambda_1$ solves (3.1). ∎

The e-structures are closely related to Artin-Schreier structures of [HJ1]. To explain and use this, we first introduce a convenient link between them.

A **weak structure** is a system $\mathfrak{G} = \langle G, G', X \rangle$, where G is a profinite group, G' is a subgroup of index ≤ 2 in G, and $X \subseteq G \smallsetminus G'$ is a closed set of involutions, closed under conjugation in G. The canonical example is $\mathfrak{G}(L/K) =$

$\langle G(L/K), G(L/K(\sqrt{-1})), X(L/K)\rangle$, where L/K is a Galois extension of fields with $\sqrt{-1} \in L$, and $X(L/K)$ the set of real involutions in $G(L/K)$. We usually write the underlying group, the underlying subgroup, and the set of involutions of a weak structure $\mathfrak{A}$ as A, A', and $X(\mathfrak{A})$. Analogously for $\mathfrak{B}$, $\mathfrak{G}$, $\mathfrak{H}$, etc. A **morphism** of weak structures $\varphi\colon \mathfrak{H} \to \mathfrak{G}$ is a continuous homomorphism $\varphi\colon H \to G$ with $\varphi^{-1}(G') = H'$ that maps $X(\mathfrak{H})$ into $X(\mathfrak{G})$. It is an **epimorphism** if $\varphi(H) = G$ and $\varphi(X(\mathfrak{H})) = X(\mathfrak{G})$.

In the language of [HJ1, Definition 3.1] our weak structure is a 'weak Artin-Schreier structure in which the forgetful map is the inclusion'. In fact, $\mathfrak{G}$ is an Artin-Schreier structure if and only if for each $x \in X$ the centralizer of x in G is $\{1, x\}$. Thus $\mathfrak{G}(K) = \mathfrak{G}(\tilde{K}/K)$ is an Artin-Schreier structure [HJ1, Remark b) on p. 470].

More precisely, an **Artin-Schreier structure** is a system $\langle G, G', X, d\rangle$, where G and G' are as above, X is a Boolean space on which G continuously acts from the right, and $d\colon X \to G$ is a continuous map into the set of involutions in $G \smallsetminus G'$ such that $\{\sigma \in G \mid x^\sigma = x\} = \{1, d(x)\}$ for all $x \in X$. The standard example is $\langle G(L/K), G(L/K(\sqrt{-1})), X(L/K), d\rangle$, where L/K is a Galois extension of fields with $\sqrt{-1} \in L$, and $X(L/K)$ the space of maximal ordered subfields of L containing K; each $(L', Q) \in X(L/K)$ is the fixed field of an involution $\varepsilon \in G(L/K)$, and d is the map $(L', Q) \mapsto \varepsilon$ [HJ1, Example 3.2].

A **morphism** of Artin-Schreier structures $\langle H, H', Y, d\rangle \to \langle G, G', X, d\rangle$ consists of a group homomorphism $\varphi\colon H \to G$ and a continuous map $\varphi\colon Y \to X$ such that $\varphi^{-1}(G') = H'$, $d \circ \varphi = \varphi \circ d$, and $\varphi(y^\sigma) = \varphi(y)^{\varphi(\sigma)}$, for all $y \in Y$ and $\sigma \in H$. It is an **epimorphism** if $\varphi(H) = G$ and $\varphi(Y) = X$.

It follows that $\langle G, G', X, d\rangle \mapsto \langle G, G', d(X)\rangle$ is a functor from the category of Artin-Schreier structures into the category of weak structures that maps epimorphisms onto epimorphisms. This functor translates the results about Artin-Schreier structures to results about the corresponding weak structures.

[HJ1, Lemma 7.5] states – and we may take it here as the definition – that a **projective Artin-Schreier structure** is a weak (!) structure $\mathfrak{G}$ that satisfies the following condition. Let $\pi\colon \mathfrak{B} \to \mathfrak{A}$ be an epimorphism of finite weak structures, and let $\rho\colon \mathfrak{G} \to \mathfrak{A}$ be a morphism. Then there exists a morphism $\lambda\colon \mathfrak{G} \to \mathfrak{B}$ such that $\pi \circ \lambda = \rho$.

LEMMA 3.6: *Let* $\mathbf{G}$ *be an e-structure, and let* $G' \leq G$ *be an open subgroup of index* ≤ 2 *that does not meet* $\mathcal{E}(\mathbf{G})$. *Then* $\mathbf{G}$ *is a projective e-structure if and* *only if* $\mathfrak{G} = \langle G, G', \mathcal{E}(\mathbf{G})\rangle$ *is a projective Artin-Schreier structure with the* $\mathcal{E}_j(\mathbf{G})$ *disjoint.*

Proof: Assume that $\mathfrak{G}$ is a projective Artin-Schreier structure with the $\mathcal{E}_j(\mathbf{G})$ disjoint. We have to solve a finite proper embedding problem (3.1) for $\mathbf{G}$. Put $A' = \rho(G')$ and $B' = \pi^{-1}(A')$. We have $\mathcal{E}_j(\mathbf{A}) = \rho(\mathcal{E}_j(\mathbf{G}))$. By Lemma 3.5(b) we may assume that $\ker(\rho)$ is so small that $A', \mathcal{E}_1(\mathbf{A}), \ldots, \mathcal{E}_e(\mathbf{A})$ are disjoint. Then also $B', \mathcal{E}_1(\mathbf{B}), \ldots, \mathcal{E}_e(\mathbf{B})$ are disjoint.

Let $\mathfrak{A} = \langle A, A', \mathcal{E}(\mathbf{A})\rangle$, and $\mathfrak{B} = \langle B, B', \mathcal{E}(\mathbf{B})\rangle$. These are weak structures, and ρ and π induce in an obvious way a morphism $\rho\colon \mathfrak{G} \to \mathfrak{A}$ and an epimorphism $\pi\colon \mathfrak{B} \to \mathfrak{A}$. By assumption there is a morphism $\lambda\colon \mathfrak{G} \to \mathfrak{B}$ such that $\pi \circ \lambda = \rho$. Thus the group homomorphism $\lambda\colon G \to B$ maps $\mathcal{E}(\mathbf{G})$ into $\mathcal{E}(\mathbf{B})$. But $\lambda(\mathcal{E}_j(\mathbf{G}))$ does not meet $\mathcal{E}_i(\mathbf{B})$ for $i \neq j$, because $\pi(\lambda(\mathcal{E}_j(\mathbf{G})) \cap \mathcal{E}_i(\mathbf{B})) \subseteq \mathcal{E}_j(\mathbf{A}) \cap \mathcal{E}_i(\mathbf{A}) = \emptyset$. Therefore $\lambda(\mathcal{E}_j(\mathbf{G})) \subseteq \mathcal{E}_j(\mathbf{B})$, and hence λ solves (3.1).

Conversely, let $\mathbf{G}$ be a projective e-structure. Let $\pi\colon \mathfrak{B} \to \mathfrak{A}$ be an epimorphism of finite weak structures, and let $\rho\colon \mathfrak{G} \to \mathfrak{A}$ be a morphism. Extend the groups A of $\mathfrak{A}$ and B of $\mathfrak{B}$ to finite e-structures $\mathbf{A}$ and $\mathbf{B}$ by letting $\mathcal{E}_j(\mathbf{A})$ be the conjugacy class of $\rho(\mathcal{E}_j(\mathbf{G}))$ in A and $\mathcal{E}_j(\mathbf{B}) \subseteq X(\mathfrak{B})$ be a conjugacy class in B mapped by π onto $\mathcal{E}_j(\mathbf{A})$. Then there is $\lambda\colon \mathbf{G} \to \mathbf{B}$ such that $\pi \circ \lambda = \rho$. In particular, the group homomorphism $\lambda\colon G \to B$ satisfies $\lambda(\mathcal{E}_j(\mathbf{G})) \subseteq \mathcal{E}_j(\mathbf{B})$ for each j, and hence it maps $X(\mathfrak{G}) = \mathcal{E}(\mathbf{G})$ into $\mathcal{E}(\mathbf{B}) \subseteq X(\mathfrak{B})$. Thus λ is a morphism of weak structures. ∎

4. Galois covers and decomposition structures

Let S/R be a **Galois ring cover** and let F/E be the corresponding field cover.

This means [FJ, p. 57] that $R \subseteq S$ are integrally closed domains with $E \subseteq F$ their respective quotient fields, F/E is a finite Galois extension, and $S = R[z]$, where z, a **primitive element** for the cover, is integral over R and its discriminant over E is a unit of R. Thus S/R is étale (actually, "standard étale" [R]).

Remark 4.1: [FHJ2, Section 1]. Let M be a field, and let $\varphi_0\colon R \to M$ be a homomorphism. Then φ_0 extends to a homomorphism $\varphi\colon S \to \widetilde{M}$. Furthermore, $M(\varphi(S))$ is a finite Galois extension of M.

(a) Let N/M be a Galois extension such that $\varphi(S) \subseteq N$. Then φ induces a homomorphism $\varphi^*\colon G(N/M) \to G(F/E)$ implicitly defined by the formula

$$\varphi(\varphi^*(\sigma)(s)) = \sigma(\varphi(s)), \qquad \text{for} \quad \sigma \in G(N/M) \quad \text{and} \quad s \in S.$$

(b) If N_1/M is a Galois extension such that $N \subseteq N_1$, and $\varphi_1^*\colon G(N_1/M) \to G(F/E)$ is the induced homomorphism by φ_1, then, by (a), $\varphi^* = \varphi_1^* \circ \mathrm{res}_N$. Therefore, unless stated otherwise, we will take N to be $\widetilde{M}$.

(c) If $N = M(\varphi(S))$, then φ^* is injective.

(d) If φ is an inclusion of rings, then φ^* is the restriction to F.

(e) Let S'/R' be another Galois cover and let F'/E' be the corresponding extension of quotient fields. Let $\rho\colon S \to S'$ and $\varphi'\colon S' \to N$ be homomorphisms such that $\rho(R) \subseteq R'$ and $\varphi'(R') \subseteq M$. Consider the induced homomorphisms $\varphi^*\colon G(M) \to G(F/E)$, $\varphi'^*\colon G(M) \to G(F'/E')$, and $\rho^*\colon G(F'/E') \to G(F/E)$. If $\varphi = \varphi' \circ \rho$, then $\varphi^* = \rho^* \circ \varphi'^*$. In particular, if $R \subseteq R'$ and $S \subseteq S'$ and φ' extends φ then $\varphi^* = \mathrm{res}_F \varphi'^*$.

(f) Let $\tau \in \mathcal{G}(F/E)$. Then $\varphi \circ \tau\colon S \to N$ also extends φ_0, and every extension of φ_0 to S is of this form. Furthermore, $(\varphi \circ \tau)^*(\sigma) = \tau^{-1}\varphi^*(\sigma)\tau$ for all $\sigma \in \mathcal{G}(N/M)$. ∎

Let K be a subfield of R and L the algebraic closure of K in F.

Definition 4.2: (a) S/R is **regular over** K, if the extension E/K is regular. In that case L/K is a finite Galois extension.

(b) S/R is **finitely generated** over K, if R and S are finitely generated rings over K.

(c) S/R is **real** if R is a regular ring and F is not formally real. (In this case S and the integral closures of R in the intermediate fields of F/E are also regular rings [R, p. 75].)

(d) F/E is **amply real** over K if E/K is a regular extension, the algebraic closure L of K in F is not formally real, and the extension $F(\varepsilon)/L(\varepsilon)$ is totally real for every real involution $\varepsilon \in G(F/E)$. ∎

Assume for the rest of this section that S/R is real. Add to the preceding discussion e-tuples $\mathbf{P}_0$ and $\mathbf{P}$ of orderings on K and M, respectively, such that $(K, \mathbf{P}_0) \subseteq (M, \mathbf{P})$. Let $\varphi^*\colon G(M) \to G(F/E)$ be the induced homomorphism.

Definition 4.3: The e-structure $\varphi^*(G(\widetilde{M}/M, \mathbf{P}))$ is called the **decomposition structure** of φ. We denote it by $\mathbf{Ar}(S/R, M, \mathbf{P}, \varphi)$, or, by abuse of notation, $\mathbf{Ar}\,\varphi$. It satisfies $\mathbf{Ar}\,\varphi \in \mathrm{Sub}[F/E, \mathbf{P}_0]$. ∎

We explain the last assertion. Fix $\varepsilon_j \in \mathcal{E}_j(\widetilde{M}/M, \mathbf{P})$. Then $\widetilde{M}(\varepsilon_j)$ is a real closure of (M, P_j). It follows from the formula of Remark 4.1(a) that φ maps $S \cap F(\varphi^*(\varepsilon_j))$ into $\widetilde{M}(\varepsilon_j)$. Knebusch' Proposition 1.2, applied to the ring extension $K \subseteq S \cap F(\varphi^*(\varepsilon_j))$, asserts that P_{0j} extends to a φ-compatible ordering on $F(\varphi^*(\varepsilon_j))$. ∎

The following lemma shows how to make field covers amply real.

LEMMA 4.4: *Let F/E be real, regular, and finitely generated over K. Assume that the algebraic closure L of K in F is not formally real.*

(a) *For each real involution $\varepsilon \in G(F/E)$ there are finitely many $a_{\varepsilon ik} \in L(\varepsilon)$ such that $\mathrm{res}_{L(\varepsilon)} X_{F(\varepsilon)} = \bigcup_i \left(\bigcap_k H(a_{\varepsilon ik})\right)$.*

(b) *Let L' be a finite Galois extension of K that contains L and all $\sqrt{a_{\varepsilon ik}}$. Put $F' = FL'$. Then F'/E is amply real over K. Moreover, an involution $\varepsilon' \in G(F'/E)$ is real if and only if $\mathrm{res}_F \varepsilon'$ and $\mathrm{res}_{L'} \varepsilon'$ are real and there is i such that*

$$\sqrt{a_{\varepsilon ik}} \in L'(\varepsilon') \quad \text{for all } k. \tag{4.6}$$

Proof: (a) merely says that $\mathrm{res}_{L(\varepsilon)} X_{F(\varepsilon)}$ is clopen in $X_{L(\varepsilon)}$ (Section 1).

(b) Let $\varepsilon = \mathrm{res}_F \varepsilon'$. If ε' is real, there is an ordering on $F'(\varepsilon')$. Its restriction P_0 to $L(\varepsilon)$ is in $\mathrm{res}_{L(\varepsilon)} X_{F(\varepsilon)}$ and extends to $L'(\varepsilon')$. By (a) there is i such that $a_{\varepsilon ik} \in P_0$ for all k. Put $L_i = L(\varepsilon)(\sqrt{a_{\varepsilon ik}} |\ k)$. Then P_0 extends to L_i, and therefore $L_i \subseteq L'(\varepsilon'')$ for some real involution ε'' of $G(L'/L(\varepsilon))$, which is conjugate to ε' over $L(\varepsilon)$. As $L_i/L(\varepsilon)$ is Galois, we have $L_i \subseteq L'(\varepsilon')$. This gives (4.6).

Conversely, assume that ε and $\mathrm{res}_{L'} \varepsilon'$ are real and (4.6) holds with some i. Clearly L' is the algebraic closure of K in F'. It remains to show that $F'(\varepsilon')/L'(\varepsilon')$ is totally real. Let P' be an ordering on $L'(\varepsilon')$. By (4.6), $a_{\varepsilon ik} \in P'$ for all k. By (a), $\mathrm{res}_{L(\varepsilon)} P'$ extends to an ordering of $F(\varepsilon)$, say Q. As $L' \cap F = L$, the fields $L'(\varepsilon')$ and $F(\varepsilon)$ are linearly disjoint over $L(\varepsilon)$, and $F'(\varepsilon')$ is their compositum. Therefore P' and Q extend to an ordering of $F'(\varepsilon')$ [J1, p. 241]. ∎

5. e-fold ordered Frobenius fields

We extend the definition of Frobenius field [FHJ1, §1 and FJ, Definition 23.1] to the class of e-fold ordered fields. The results of this section generalize [FHJ1, Theorem 1.2 and FJ, Propositions 23.2-3].

Definition 5.1: An e-fold ordered field $(M, \mathbf{P}) = (M, P_1, \ldots, P_e)$ is a **PRCe** field if $P_1, \ldots, P_e$ are distinct and every absolutely irreducible variety V over M has an M-rational point, provided that $P_1, \ldots, P_e$ extend to the function field of V over M.

For such a field condition C_M^Z of [P2, p.136] holds with $Z = \{P_1, \ldots, P_e\}$. By [P2, Proposition 1.6], $P_1, \ldots, P_e$ are all the orderings on M and they induce different topologies on M. Therefore $(M, \mathbf{P})$ is PRCe if and only if $(M, \mathbf{P})$ is existentially closed (in the language of fields augmented by e predicates for the orderings) in an extension $(F, \mathbf{Q})$ such that F/M is regular [P2, Theorem 1.7]. ∎

Example 5.2: (a) Let K be a countable Hilbertian field, and let $\overline{K}_1, \ldots, \overline{K}_e$ be fixed real closures of K. For $\sigma \in G(K)^{e+n}$ let

$$(K_\sigma, \mathbf{P}_\sigma) = (\overline{K}_1^{\sigma_1} \cap \cdots \cap \overline{K}_e^{\sigma_e} \cap \widetilde{K}(\sigma_{e+1}) \cap \cdots \cap \widetilde{K}(\sigma_{e+n}); P_{\sigma_1}, \ldots, P_{\sigma_e}),$$

where P_{σ_j} is the ordering induced by $\overline{K}_j^{\sigma_j}$. Then, for almost all (in the sense of the Haar measure) $\sigma \in G(K)^{e+n}$ the field $(K_\sigma, \mathbf{P}_\sigma)$ is PRCe and $\mathbf{G}(K_\sigma, \mathbf{P}_\sigma) \cong \hat{\mathbf{D}}_{e,n}$ [HJ1, Proposition 5.6]. For $n = 0$ these fields are called **Geyer fields** of corank e [J1, Theorem 6.7] and $\hat{\mathbf{D}}_{e,0}$ is denoted by $\hat{\mathbf{D}}_e$.

(b) A maximal algebraic extension $(M, \mathbf{P})$ of a Geyer field is a PRCe field, and $G(M, \mathbf{P}) \cong \hat{\mathbf{D}}_e(2)$ [J2, Lemma 2.3 and Proposition 4.1]. Such a field is called a **v.d. Dries field** of corank e.

The absolute Galois structures of Geyer and v.d. Dries fields have the embedding property (Corollary 3.4). ∎

Definition 5.3: An e-fold ordered field $(M, \mathbf{P})$ with $P_1, \ldots, P_e$ distinct is said to be **Frobenius** if it satisfies the following condition. Let S/R be a real Galois ring cover, regular and finitely generated over M. Let the corresponding field cover F/E be amply real over M, and let N be the algebraic closure of M in F. Let $\mathbf{H} \in \mathrm{Sub}[F/E, \mathbf{P}]$ such that $\mathbf{H} \in \mathrm{Im}\, \mathbf{G}(M, \mathbf{P})$, and $\mathrm{res}_N \mathbf{H} = \mathbf{G}(N/M, \mathbf{P})$. Then there exists an M-homomorphism φ: $S \to \widetilde{M}$ with $\varphi(R) \subseteq M$ such that $\mathbf{Ar}\, \varphi = \mathbf{H}$. ∎

LEMMA 5.4: *Let L/K be a finite Galois extension, L not formally real, and let π: $G \to G(L/K)$ be an epimorphism of finite groups. There exists a totally real finitely generated regular extension E of K, a Galois extension F/E such that L is the algebraic closure of K in F and $F(\varepsilon)/L(\varepsilon)$ is totally real for every* *involution* $\varepsilon \in G(F/E)$ *with* $\mathrm{res}_L\varepsilon$ *real, and an isomorphism* $\theta\colon G \to G(F/E)$ *such that the following diagram commutes.*

$$\begin{array}{ccc} G & \xrightarrow{\ \theta\ } & G(F/E) \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle \mathrm{res}_L} & \\ & G(L/K) & \end{array}$$

Proof: If $\sqrt{-1} \in L$, this is shown in Parts I and II of the proof of [HJ1, Lemma 9.4]. (In fact, the extension F/L constructed there is purely transcendental, and so is $F(\varepsilon)/L(\varepsilon)$, for each involution $\varepsilon \in G(F/E)$ with $\mathrm{res}_L\varepsilon$ real.)

In the general case let $L' = L(\sqrt{-1})$. Let $G' = G \times_G G(L'/K)$, and let $\pi'\colon G' \to G(L'/K)$ and $\rho\colon G' \to G$ be the coordinate projections. Assume that we have constructed a Galois cover F'/E of fields, regular and finitely generated over K such that L' is the algebraic closure of K in F' and $F'(\varepsilon')/L'(\varepsilon')$ is totally real for every involution $\varepsilon' \in G(F'/E)$ with $\mathrm{res}_{L'}\varepsilon'$ real, and an isomorphism $\theta'\colon G' \to G(F'/E)$ such that $\mathrm{res}_{L'} \circ \theta' = \pi'$. Let F be the fixed field of $\theta'(\ker(\rho))$ in F'. Then θ' induces an isomorphism $\theta\colon G \to G(F/E)$ that satisfies the requirements of the Lemma.

Indeed, $G(F'/E) = G(F/E) \times_{G(L/K)} G(L'/K)$, and hence F and L' are linearly disjoint over L, and $F' = FL'$, whence L' is the algebraic closure of K in F'. If $\varepsilon \in G(F/E)$ is an involution with $\mathrm{res}_L\varepsilon$ real, and P is an ordering on $L(\varepsilon)$, there is an involution $\varepsilon' \in G(L'/K)$ such that $\mathrm{res}_L\varepsilon = \mathrm{res}_L\varepsilon'$ and P extends to an ordering P' on $L'(\varepsilon')$. Let $\varepsilon'' = (\varepsilon, \varepsilon') \in G(F'/E)$. By assumption P' extends to an ordering Q' on $F(\varepsilon'')$, and so $\mathrm{res}_{F(\varepsilon)}Q'$ extends P. ∎

LEMMA 5.5: *Let* $(M, \mathbf{P})$ *be an e-fold ordered Frobenius field. Then* $(M, \mathbf{P})$ *is PRCe and* $\mathbf{G} = \mathbf{G}(M, \mathbf{P})$ *is superprojective.*

Proof: We first show that $(M, \mathbf{P})$ is PRCe. Let V be an absolutely irreducible variety over M, and let $E = M(V)$ be its function field. Then E/M is a regular extension. Put $F = E(\sqrt{-1})$. Assume that $\mathbf{P}$ extends to an e-tuple of orderings $\mathbf{Q}$ on E. Thus E/M is totally real and therefore F/E is amply real. Let R be a regular ring with quotient field E that contains the coordinate ring $M[V]$ (Remark 1.3), and let $S = R[\sqrt{-1}]$. Apply Definition 5.3 to $\mathbf{H} = \mathbf{G}(F/E, \mathbf{Q})$, and find an M-homomorphism $\varphi\colon S \to \widetilde{M}$ such that $\varphi(R) \subseteq M$. This gives an M-rational point on V.

Secondly, we show that each finite proper embedding problem (3.1) for $\mathbf{G}$ is (properly) solvable. We may identify $\rho\colon G \to A$ with the restriction map

$G \to G(N/M)$, where N is a finite Galois extension of M. Then $\mathrm{res}_N \varepsilon \neq 1$ for each $\varepsilon \in \mathcal{E}_j(\mathbf{G})$, and hence P_j does not extend to N. As $(M, \mathbf{P})$ is a PRCe field, $P_1, \ldots, P_e$ are all its (distinct) orderings, and hence N is not formally real. Thus the Galois e-structure $\mathbf{G}(N/M, \mathbf{P})$ is well defined, and $\rho: \mathbf{G} \to \mathbf{A}$ identifies with the restriction map $\mathbf{G} \to \mathbf{G}(N/M, \mathbf{P})$.

By Lemma 5.4 we may identify the epimorphism $\pi: B \to G(N/M)$ with the restriction map $\mathrm{res}_N: G(F/E) \to G(N/M)$, where F/E is a Galois extension such that N is the algebraic closure of M in F and $F(\varepsilon)/N(\varepsilon)$ is totally real for every involution $\varepsilon \in G(F/E)$ with $\mathrm{res}_N \varepsilon$ real. As $\mathrm{res}_N \mathbf{B} = \pi(\mathbf{B}) = \mathbf{G}(N/M, \mathbf{P})$, the latter is in particular true for every $\varepsilon \in \mathcal{E}_j(\mathbf{B})$, and hence $\mathbf{B} \in \mathrm{Sub}[F/E, \mathbf{P}]$.

Assume that $\mathbf{B} \in \mathrm{Im}\, \mathbf{G}$. There is a real Galois ring cover S/R, finitely generated over M, such that F/E is the corresponding field cover. As $(M, \mathbf{P})$ is Frobenius, there exists an M-map $\varphi: S \to \widetilde{M}$ with $\varphi(R) = M$ such that $\mathbf{Ar}\, \varphi = \mathbf{B}$. Thus $\varphi^*: G(M) \to B$ maps $\mathbf{G}(M, \mathbf{P})$ onto $\mathbf{B}$. The restriction of φ to N is an M-automorphism, so there is $\tau \in G(F/E)$ such that $\mathrm{res}_N \varphi = \mathrm{res}_N \tau^{-1}$. By Remark 4.1(f), $\varphi^*(\mathbf{G}(M, \mathbf{P})) = (\varphi \circ \tau)^*(\mathbf{G}(M, \mathbf{P}))$, and hence we may assume that $\mathrm{res}_N \varphi$ is the identity. It follows from the equation of Remark 4.1(a) that $\pi \circ \varphi^* = \rho$.

It remains to show that $\mathbf{G}$ is projective. We have already remarked that $P_1, \ldots, P_e$ are all the distinct orderings on M. Therefore $\mathcal{E}(\mathbf{G})$ is the set of all involutions in G, and the $\mathcal{E}_j(\mathbf{G})$ are distinct. By [HJ1, Theorem 10.1(b)] the Artin-Schreier structure $\mathfrak{G}(M) = \langle G(M), G(M(\sqrt{-1})), \mathcal{E}(\mathbf{G})\rangle$ is projective. By Lemma 3.6, $\mathbf{G}$ is projective. ∎

Proposition 5.6: *Let $(M, \mathbf{P})$ be a PRCe field with $\mathbf{G}(M, \mathbf{P})$ superprojective. Then $(M, \mathbf{P})$ is Frobenius.*

Proof: Let S/R, F/E, N, and $\mathbf{H}$ be as in Definition 5.3.

The superprojectivity of $\mathbf{G}(M, \mathbf{P})$ yields a Galois extension N' of M that contains N and an isomorphism $h: \mathbf{G}(N'/M, \mathbf{P}) \to \mathbf{H}$ such that

$$\begin{array}{ccc} \mathbf{G}(N'/M, \mathbf{P}) & \xrightarrow{\quad h \quad} & \mathbf{H} \\ & {\scriptstyle \mathrm{res}_N} \searrow \quad \swarrow {\scriptstyle \mathrm{res}_N} & \\ & \mathbf{G}(N/M, \mathbf{P}) & \end{array}$$

commutes. Let $F' = N'F$. Then

$$G(F'/E) = G(N'/M) \times_{G(N/M)} G(F/E).$$

Let $\Delta = \{(\delta, h(\delta)) | \delta \in G(N'/M)\}$. The fixed field D of Δ in F' is regular over M [FJ, p. 354]. Furthermore, $\mathbf{P}$ extends to D. Indeed, let $\varepsilon' \in \mathcal{E}_j(N'/M, \mathbf{P})$, and put $\varepsilon'' = (\varepsilon', h(\varepsilon')) \in G(F'/E)$. Then P_j extends to an ordering P_j' on $N'(\varepsilon')$. Let $\varepsilon = \mathrm{res}_F \varepsilon'' = h(\varepsilon')$. Since F/E is amply real over M, we may take $a_{\varepsilon jk} = \{1\})$ in Lemma 4.4. Furthermore, $\varepsilon \in \mathcal{E}_j(\mathbf{H})$ and $\mathbf{H} \in \mathrm{Sub}[F/E, \mathbf{P}]$, and hence ε is real. It follows that F'/E is amply real over M and ε'' is a real involution. Thus $F'(\varepsilon'')/N'(\varepsilon')$ is totally real. In particular, P_j' extends to an ordering on $F'(\varepsilon'')$. Its restriction to D extends P_j.

The integral closure U of R in D is finitely generated over M [FJ, p. 354], and hence U is the coordinate ring of an absolutely irreducible variety V defined over M. As $(M, \mathbf{P})$ is PRCe, there exists an M-homomorphism ψ_0: $U \to M$. It extends to the integral closure of U in F', and its restriction φ to S satisfies $\varphi(R) \subseteq M$. In fact, we may assume that $\varphi^* = h$ [FHJ1, Remark on p. 9]. Hence $\mathbf{Ar}\,\varphi = \mathbf{H}$. ∎

Example 5.7: Both Geyer fields and v.d. Dries fields (Example 5.2) are Frobenius fields. ∎

6. The Artin symbol

Let $(K, \mathbf{P}_0)$ be an e-fold ordered field. Recall [FJ, p. 244] that a **basic set over** K is a set of the form $A = V \setminus V(g)$, where V is a closed K-irreducible subset of an affine space $\mathbb{A}^n$ and $g \in K[X_1, \ldots, X_n]$ does not vanish on V. Let x_i be the restriction to V of the projection on the i-th coordinate. Then $\mathbf{x} = (x_1, \ldots, x_n)$ is a generic point of V over K. We put $K[A] = K[\mathbf{x}, g(\mathbf{x})^{-1}]$ and $K(A) = K(\mathbf{x})$. The **dimension** $\dim A$ of A is the transcendence degree of $K(A)$ over K. There is $0 \neq d \in K[\mathbf{X}]$ such that $A' = A \setminus V(d)$ is non-singular, that is, $K[A]$ is regular (Remark 1.3).

A Galois ring cover $C/K[A]$ is called a **Galois (ring/set) cover**, and is denoted by C/A. We write C/A for $C/K[A]$, let $K(C)$ be the quotient field of C, and write $G(C/A)$ for $G(K(C)/K(A))$. Thus $\mathrm{Sub}[C/A, \mathbf{P}_0]$ stands for $\mathrm{Sub}[K(C)/K(A), \mathbf{P}_0]$ (Definition 2.3).

Let C/A be a real Galois ring/set cover over K (Definition 4.2).

Notice that $G(C/A)$ acts by conjugation on $\mathrm{Sub}[C/A, \mathbf{P}_0]$. A **conjugacy domain** in $\mathrm{Sub}[C/A, \mathbf{P}_0]$ is a subset of $\mathrm{Sub}[C/A, \mathbf{P}_0]$ closed under conjugation. A **conjugacy class** in $\mathrm{Sub}[C/A, \mathbf{P}_0]$ is a minimal nonempty conjugacy domain. It is necessarily of the form $\{\mathbf{H}^\sigma | \sigma \in G(C/A)\}$, where $\mathbf{H} \in \mathrm{Sub}[C/A, \mathbf{P}_0]$.

Let $(M, \mathbf{P})$ be an e-fold ordered extension of $(K, \mathbf{P}_0)$. Each M-rational point $\mathbf{a} = (a_1, \ldots, a_n) \in A$ defines a K-homomorphism φ_0: $K[A] \to M$ (by $x_i \mapsto a_i$), which extends to φ: $K[C] \to \widetilde{M}$. As φ ranges over all possible extensions of φ_0 to $K[C]$, by Remark 4.1(f) the structure $\mathbf{Ar}\,\varphi$ ranges over a conjugacy class in $\mathrm{Sub}[C/A, \mathbf{P}_0]$. Denote this conjugacy class by $\mathbf{Ar}(C/A, M, \mathbf{P}, \mathbf{a})$, or $\mathbf{Ar}(C/A, \mathbf{a})$, for short, and call it the **Artin symbol** of $\mathbf{a}$.

This symbol is an enrichment of the Artin symbol $\mathrm{Ar}(C/A, M, \mathbf{a})$ of [FJ, Section 25.1], and therefore it has properties similar to those proved there:

Property 6.1: If D/A is another real Galois cover, with $C \subseteq D$, and $\mathbf{a} \in A(M)$, then, by Remark 4.1(e)), $\mathrm{res}_{K(C)}\,\mathbf{Ar}(D/A, \mathbf{a}) = \mathbf{Ar}(C/A, \mathbf{a})$. Thus we usually omit the reference to the cover and write $\mathbf{Ar}(A, \mathbf{a})$.

Furthermore, let $\mathrm{Con}(C/A)$ be a conjugacy domain in $\mathrm{Sub}[C/A, \mathbf{P}_0]$, and let $\mathcal{S}$ be a set of (isomorphism types of) e-structures. Define

$$\mathrm{Con}(D/A) = \{\mathbf{H} \in \mathrm{Sub}[D/A, \mathbf{P}_0] \mid \mathbf{H} \in \mathcal{S},\ \mathrm{res}_{K(C)}\mathbf{H} \in \mathrm{Con}(C/A)\}. \tag{6.2}$$

Assume that

(*) $\mathrm{Im}\,\mathbf{G}(M, \mathbf{P}) \cap \mathrm{Sub}[D/A, \mathbf{P}_0] = \mathcal{S}$.

Then $\mathbf{Ar}(C/A, \mathbf{a}) \subseteq \mathrm{Con}(C/A)$ if and only if $\mathbf{Ar}(D/A, \mathbf{a}) \subseteq \mathrm{Con}(D/A)$.

Indeed, if $\mathbf{Ar}(D/A, \mathbf{a}) \subseteq \mathrm{Con}(D/A)$, then definition (6.2) gives $\mathbf{Ar}(C/A, \mathbf{a}) = \mathrm{res}_{K(C)}\,\mathbf{Ar}(D/A, \mathbf{a}) \subseteq \mathrm{Con}(C/A)$. Conversely, let $\mathbf{Ar}(C/A, \mathbf{a}) \subseteq \mathrm{Con}(C/A)$. As $\mathbf{Ar}(D/A, \mathbf{a}) \subseteq \mathrm{Im}\,\mathbf{G}(M, \mathbf{P})$, it follows from (*) that $\mathbf{Ar}(D/A, \mathbf{a}) \subseteq \mathcal{S}$. Therefore $\mathbf{Ar}(D/A, \mathbf{a}) \subseteq \mathrm{Con}(D/A)$. ∎

Property 6.3: Replacing A by an open subset A' does not affect the Artin symbol, that is, $\mathbf{Ar}(A', \mathbf{a}) = \mathbf{Ar}(A, \mathbf{a})$, for each $\mathbf{a} \in A'(M)$. ∎

Property 6.4: Let C'/A' be a real Galois cover **induced** by C/A. I.e., A' is a nonsingular basic set contained in A, and the homomorphism $K[A] \to K[A']$ induced from the inclusion $A' \subseteq A$ extends to a homomorphism ρ: $C \to C'$ that maps a primitive element z of C/A onto a primitive element z' of C'/A'. Thus $C = K[A][z]$ and $C' = K[A'][z']$.

By Remark 4.1(c), ρ induces an embedding ρ^*: $G(C'/A') \to G(C/A)$. For each $\mathbf{a} \in A'(M)$ we have, by Remark 4.1(e), $\rho^*\,\mathbf{Ar}(A', \mathbf{a}) \subseteq \mathbf{Ar}(A, \mathbf{a})$.

For a conjugacy domain $\mathrm{Con}(A)$ in $\mathrm{Sub}[C/A, \mathbf{P}_0]$ let

$$\mathrm{Con}(A') = \{\mathbf{H} \in \mathrm{Sub}[C'/A', \mathbf{P}_0] \mid \rho^*(\mathbf{H}) \in \mathrm{Con}(A)\}$$

be the **induced conjugacy domain** in $\mathrm{Sub}[C'/A', \mathbf{P}_0]$. For $\mathbf{a} \in A'(M)$ we have $\mathbf{Ar}(A', \mathbf{a}) \subseteq \mathrm{Con}(A')$ if and only if $\mathbf{Ar}(A, \mathbf{a}) \subseteq \mathrm{Con(A)}$. ∎

7. Projections of conjugacy domains

For $n \geq 0$ let $\pi\colon \mathbb{A}^{n+1} \to \mathbb{A}^n$ be the projection on the first n coordinates. Let $A \subseteq \mathbb{A}^{n+1}$ and $B \subseteq \mathbb{A}^n$ be two non-singular basic sets such that $\pi(A) = B$. Then $K[B] \subseteq K[A]$. Let $\mathbf{x}$ and $(\mathbf{x}, y)$ be generic points of B and A, respectively. Then $K(A) = K(B)(y)$. Furthermore, let C/A and D/B be real Galois covers such that $K(D)$ contains the algebraic closure of $K(B)$ in $K(C)$.

Under these assumptions we define the projection of conjugacy domains associated with C/A. There are two cases: $\dim A = \dim B + 1$ (Lemma 7.2) and $\dim A = \dim B$ (Lemma 7.4).

Definition 7.1: Let M be an extension of K. An M-**specialization** of the pair $(C/A, D/B)$ is a K-homomorphism φ from C into an overfield of M such that $\varphi(K[B]) \subseteq M$ and, if y is transcendental over $K(B)$, then $\varphi(y)$ is transcendental over M.

For such a specialization put $y' = \varphi(y)$, $N = M[\varphi(D)]$, $R = M[\varphi(K[A])]$, $E = M(y')$ (the quotient field of R), $S = M[\varphi(C)]$, and $F = E[\varphi(C)]$ (the quotient field of S), Then φ induces an embedding $\varphi^*\colon G(F/E) \to G(C/A)$ (Remark 4.1(c)).

Assume that $\dim A = \dim B + 1$. The pair $(C/A, D/B)$ is said to be **specialization compatible** if

(i) $K(D)$ is the algebraic closure of $K(B)$ in $K(C)$,

and for every M and each M-specialization φ as above

(ii) $[K(C) : K(D)(y)] = [F : N(y')]$,

(iii) the cover $K(C)/K(A)$ is amply real over $K(B)$, and

(iv) for each involution $\varepsilon \in G(F/E)$ with $\varphi^*(\varepsilon)$ real the extension $F(\varepsilon)/N(\varepsilon)$ is totally real.

Assume that $\dim A = \dim B$. The pair $(C/A, D/B)$ is said to be **specialization compatible** if $K[A]$ is integral over $K[B]$ and $C = D$. ∎

LEMMA 7.2: *Assume that* $\dim A = \dim B + 1$ *and that* $(C/A, D/B)$ *is specialization compatible. Let* $\mathrm{Con}(A)$ *be a conjugacy domain in* $\mathrm{Sub}[C/A, \mathbf{P}_0]$, *and let* $\mathcal{S}$ *be a a set of (isomorphism types of) e-structures. Define* $\mathrm{Con}(B) = \mathrm{Con}(B, \mathcal{S}) =$

$\mathrm{res}_{K(D)}(\mathcal{S}\cap\mathrm{Con}(A))$. *Let* $(M,\mathbf{P})$ *be a Frobenius field that contains* $(K,\mathbf{P}_0)$, *and assume that*

(*) $\mathrm{Im}\,\mathbf{G}(M,\mathbf{P})\cap\mathrm{Sub}[C/A,\mathbf{P}_0]=\mathcal{S}$.

Then each $\mathbf{b}\in B(M)$ *satisfies:*

(7.3) $\mathbf{Ar}(B,\mathbf{b})\subseteq\mathrm{Con}(B)$ *if and only if there exists* $\mathbf{a}\in A(M)$ *such that* $\pi(\mathbf{a})=\mathbf{b}$ *and* $\mathbf{Ar}(A,\mathbf{a})\subseteq\mathrm{Con}(A)$.

Proof: Let $\mathbf{b}\in B(M)$. Extend $\mathbf{x}\to\mathbf{b}$ to an M-specialization φ of $(C/A,D/B)$, and let φ_0 be its restriction to D.

Assume that $\mathbf{Ar}(B,\mathbf{b})\subseteq\mathrm{Con}(B)$. Then $\mathbf{Ar}\,\varphi_0\in\mathrm{Con}(B)$, and hence there is $\mathbf{H}\in\mathcal{S}\cap\mathrm{Con}(A)$ such that $\mathrm{res}_{K(D)}\mathbf{H}=\mathbf{Ar}\,\varphi_0$. In particular, $\mathrm{res}_{K(D)}H=\varphi_0^*(G(N/M))$. By (*), $\mathbf{H}\in\mathrm{Im}\,\mathbf{G}(M,\mathbf{P})$. A diagram chasing on the following commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G(K(C)/K(D)(y)) & \longrightarrow & G(C/A) & \longrightarrow & G(D/B) & \longrightarrow & 1 \\ & & \uparrow{\scriptstyle\varphi^*} & & \uparrow{\scriptstyle\varphi^*} & & \uparrow{\scriptstyle\varphi_0^*} & & \\ 1 & \longrightarrow & G(F/NE) & \longrightarrow & G(F/E) & \longrightarrow & G(N/M) & \longrightarrow & 1\ , \end{array}$$

in which the left vertical arrow is an isomorphism by (ii), shows that the subgroup $H_0=\{\sigma\in G(F/E)|\ \varphi^*(\sigma)\in H\}$ of $G(F/E)$ satisfies $\varphi^*(H_0)=H$ and $\mathrm{res}_N H_0=G(N/M)$.

Expand H_0 to an e-structure $\mathbf{H}_0$ such that the isomorphism φ^*: $H_0\to H$ of groups extends to an isomorphism φ^*: $\mathbf{H}_0\to\mathbf{H}$ of e-structures. As the embedding φ_0^* maps $\mathrm{res}_N\mathbf{H}_0$ onto $\mathrm{res}_{K(D)}\mathbf{H}=\mathbf{Ar}\,\varphi_0=\varphi_0^*(\mathbf{G}(N/M,\mathbf{P}))$, we have $\mathrm{res}_N\mathbf{H}_0=\mathbf{G}(N/M,\mathbf{P})$. Moreover, let $\varepsilon\in\mathcal{E}_j(\mathbf{H}_0)$. As $\mathrm{res}_N\varepsilon\in\mathcal{E}_j(N/M,\mathbf{P})$, the ordering P_j extends to an ordering P_j' on $N(\varepsilon)$. But $\varphi^*(\varepsilon)\in\mathcal{E}_j(\mathbf{H})$ and $\mathbf{H}\in\mathrm{Sub}[C/A,\mathbf{P}_0]$, hence $\varphi^*(\varepsilon)$ is real. By (iv), P_j' extends to $F(\varepsilon)$. This shows that $\mathbf{H}_0\in\mathrm{Sub}[F/E,\mathbf{P}]$.

Since $(M,\mathbf{P})$ is Frobenius, there exists an M-homomorphism ψ: $S\to\widetilde{M}$ such that $\psi(R)=M$ and $\mathbf{Ar}\,\psi=\mathbf{H}_0$. Let $\mathbf{a}=\psi\circ\varphi(\mathbf{x},y)$. Then $\mathbf{a}\in A(M)$ and $\pi(\mathbf{a})=\mathbf{b}$. Furthermore, by Remark 4.1(e),

$$\mathbf{Ar}(\psi\circ\varphi)=\varphi^*(\mathbf{Ar}\,\psi)=\varphi^*(\mathbf{H}_0)=\mathbf{H}\in\mathrm{Con}(A).$$

Therefore $\mathbf{Ar}(A,\mathbf{a})\subseteq\mathrm{Con}(A)$.

Conversely, let $\mathbf{a}\in A(M)$ such that $\pi(\mathbf{a})=\mathbf{b}$ and $\mathbf{Ar}(A,\mathbf{a})\subseteq\mathrm{Con}(A)$. By (*), $\mathbf{Ar}(A,\mathbf{a})\subseteq\mathcal{S}$. Let ρ: $C\to\widetilde{M}$ be an extension of $(\mathbf{x},y)\to\mathbf{a}$, and let φ_0 be the restriction of ρ to D. By Remark 4.1(e), $\mathbf{Ar}\,\varphi_0 = \mathrm{res}_{K(D)}\,\mathbf{Ar}\,\rho$, whence

$$\mathbf{Ar}(B,\mathbf{b}) = \mathrm{res}_{G(D/B)}\,\mathbf{Ar}(A,\mathbf{a}) \subseteq \mathrm{res}_{G(D/B)}\big(\mathcal{S}\cap \mathrm{Con}(A)\big) = \mathrm{Con}(B). \qquad \blacksquare$$

LEMMA 7.4: *Assume that* $\dim A = \dim B$ *and that* $(C/A, D/B)$ *is specialization compatible. Let* $\mathrm{Con}(A)$ *be be a conjugacy domain in* $\mathrm{Sub}[C/A,\mathbf{P}_0]$. *Define*

$$\mathrm{Con}(B) = \{\mathbf{G}^\sigma \mid \mathbf{G}\in \mathrm{Con}(A),\ \sigma \in G(C/B)\}$$

Let $(M,\mathbf{P})$ *be an extension of* $(K,\mathbf{P}_0)$. *Then each* $\mathbf{b}\in B(M)$ *satisfies (7.3).*

Proof: Assume that $\mathbf{Ar}(B,\mathbf{b}) \subseteq \mathrm{Con}(B)$. Extend $\mathbf{x}\to\mathbf{b}$ to a K-homomorphism $\varphi\colon C\to \widetilde{M}$ and put $c=\varphi(y)$. Then $\mathbf{Ar}\,\varphi \in \mathrm{Con}(B)$, so there are $\sigma\in G(C/B)$ and $\mathbf{G}\in\mathrm{Con}(A)$ such that $\mathbf{Ar}\,\varphi = \mathbf{G}^\sigma$. Replacing φ by $\varphi\circ\sigma^{-1}$ (Remark 4.1(f)) we may assume that $\sigma = 1$. In particular, $G \le G(C/A)$, and hence φ maps $K[A]$ into M. Thus $\mathbf{a} = (\mathbf{b}, c) \in A(M)$, and, by the above, $G = \mathbf{Ar}\,\varphi \in \mathbf{Ar}(A,\mathbf{a})$, whence $\mathbf{Ar}(A,\mathbf{a}) \subseteq \mathrm{Con}(A)$.

Conversely, let $\mathbf{a}\in A(M)$ such that $\pi(\mathbf{a}) = \mathbf{b}$ and $\mathbf{Ar}(A,\mathbf{a}) \subseteq \mathrm{Con}(A)$. Extend $\mathbf{x}\to\mathbf{a}$ to a K-homomorphism $\varphi\colon C\to\widetilde{M}$. It maps $K[B]$ into M. As $\mathbf{Ar}\,\varphi \in \mathrm{Con}(A) \subseteq \mathrm{Con}(B)$, we have $\mathbf{Ar}(A,\mathbf{b}) \subseteq \mathrm{Con}(B)$. ∎

Let us show how to make $(C/A, D/B)$ specialization compatible.

LEMMA 7.5 (cf. [FJ, Lemma 25.1]): *Let* K_1 *be a finite extension of* $K(D)$. *There are Zariski open subsets* $A' \subseteq A$, $B' \subseteq B$ *and a specialization compatible pair* $(C'/A', D'/B')$ *such that* $K(C) \subseteq K(C')$ *and* $K_1 \subseteq K(D')$.

Proof: Assume first that $\dim A = \dim B + 1$. Let K_1' be a finite Galois extension of $K(B)$ that contains both K_1 and the algebraic closure of $K(B)$ in $K(C)$. Let $h\in K[\mathbf{X},Y]$ be a polynomial that does not vanish on A. Put $A' = A\smallsetminus V(h)$, and let C' be the integral closure of $K[A']$ in $K_1'\cdot K(C)$. We may choose h so that for each intermediate field L of $K(C')/K(A')$ there is a generator $\zeta_L\in C'$ of L over $K(A')$ such that $\mathrm{discr}_{L/K(A')}\zeta_L \in (K[A'])^\times$. By [FJ, Lemma 5.3], $K[A'][\sqrt{-1},\zeta_L]$ is the integral closure of $K[A'][\sqrt{-1}]$ in $K(C')$, that is, $K[A'][\sqrt{-1},\zeta_L] = C'$. Furthermore, C'/A' is a cover.

Let $g\in K[\mathbf{X}]$ be a polynomial that does not vanish on B, but $g(\mathbf{b}) = 0$ for all $\mathbf{b}\in B$ with $h(\mathbf{b},y) = 0$. Put $B' = B\smallsetminus V(g)$, and let D' be the integral closure of $K[B']$ in K_1'. We may choose g so that D'/B' is a cover. Replacing h by gh we may assume that $\pi(A') = B'$. Use [FJ, Lemma 25.1] to achieve conditions (i) and (ii) for $(C'/A', D'/B')$. By Lemma 4.4 we may choose the field $K_1' = K(D')$ so that (iii) holds.

Let now φ be a specialization of $(C'/A', D'/B')$. Using the notation of Definition 7.1, let $\varepsilon \in G(F/E)$ be an involution such that $\delta = \varphi^*(\varepsilon)$ is real, and let P be an ordering on $N(\varepsilon)$. To show (iv), we have to verify that P extends to $F(\varepsilon)$. We need some preparations. Let $L = K(C')(\delta)$ and $L_0 = K(D')(\delta)$. As $\delta(\zeta_L) = \zeta_L$ and $\delta = \varphi^*(\varepsilon)$, we have $E(\varphi(\zeta_L)) \subseteq F(\varepsilon)$, by Remark 4.1(a). But since $[F\colon F(\varepsilon)] = 2$ and $C' = K[A'][\sqrt{-1}, \zeta_L]$, we get $F = E[\varphi(C')] \subseteq E(\sqrt{-1}, \varphi(\zeta_L))$, and hence $F(\varepsilon) = E(\varphi(\zeta_L))$. In particular, $F(\varepsilon) = N(\varepsilon)(y', \varphi(\zeta_L))$. Furthermore, $L = L_0(y, \zeta_L)$. By (ii), $[L : L_0(y)] = [F(\varepsilon) : N(\varepsilon)(y')]$, and hence φ maps $\mathrm{irr}(\zeta_L, L_0(y))$ onto $\mathrm{irr}(\varphi(\zeta_L), N(\varepsilon)(y'))$.

Let R_0 be the integral closure of $K[B']$ in L_0, and let $f(Y,Z) \in R_0[Y,Z]$ such that $f(y,Z) = \mathrm{irr}(\zeta_L, L_0(y))$. Then φ maps R_0 into $N(\varepsilon)$, the field L is the function field of $V(f)$ over L_0, and $F(\varepsilon)$ is the function field of $V(\varphi(f))$ over $N(\varepsilon)$.

Lemma 1.6 gives $0 \neq p \in R_0$ and a finite subset $\{q_{ij} \mid i \in I,\ j \in J(i)\}$ of R_0 such that $\mathrm{res}_{L_0} X_L = \bigcup_{i\in I} \bigcap_{j\in J(i)} H_{L_0}(q_{ij})$ and, if $\varphi(p) \neq 0$, then $\mathrm{res}_{N(\varepsilon)} X_{F(\varepsilon)} = \bigcup_{i\in I} \bigcap_{j\in J(i)} H_{N(\varepsilon)}(\varphi(q_{ij}))$. We may assume that g has been chosen so that $p \in R_0^\times$, and hence $\varphi(p) \neq 0$.

By Knebusch' Proposition 1.2 (applied to the ring R_0) there is an ordering Q on L_0 that is φ-compatible with P. By (iii) it extends to L, that is, $Q \in \bigcap_{j\in J(i)} H_{L_0}(q_{ij})$ for some $i \in I$. Hence $P \in \bigcap_{j\in J(i)} H_{N(\varepsilon)}(\varphi(q_{ij})) \subseteq \mathrm{res}_{N(\varepsilon)} X_{F(\varepsilon)}$. This shows (iv).

Now assume that $\dim A = \dim B$. Let K_1' be a finite Galois extension of $K(B)$ that contains both K_1 and $K(C)$. Let $g \in K[\mathbf{X}]$ be a polynomial that does not vanish on B, Put $A' = A \smallsetminus V(g)$, $B' = B \smallsetminus V(g)$, and let D' be the integral closure of $K[B']$ in K_1'. We may choose g so that $K[A']/K[B']$ is integral and D'/B' is a cover. Then D'/A' is also a Galois cover. ∎

8. Real Galois stratification

Definition 8.1: Let $(K, \mathbf{P}_0)$ be an e-fold ordered field. A normal stratification $\mathcal{A}_0 = \langle \mathbb{A}^n, C_i/A_i \mid i \in I\rangle$ of $\mathbb{A}^n$ over K [FJ, p. 410] is **real** if the covers C_i/A_i are real. It can be **augmented to a real Galois stratification**

$$\mathcal{A} = \langle \mathbb{A}^n, C_i/A_i, \mathrm{Con}(A_i) \mid i \in I\rangle, \tag{8.2}$$

where each $\mathrm{Con}(A_i)$ is a conjugacy domain in $\mathrm{Sub}[C_i/A_i, \mathbf{P}_0]$.

Put $\mathrm{Sub}\ \mathcal{A} = \mathrm{Sub}\ \mathcal{A}_0 = \bigcup_{i\in I} \mathrm{Sub}[C_i/A_i, \mathbf{P}_0]$.

Let $(M, \mathbf{P})$ be an extension of $(K, \mathbf{P}_0)$ and let $\mathbf{a} \in M^n$. Write $\mathbf{Ar}(\mathcal{A}, \mathbf{a}) \subseteq \mathrm{Con}(\mathcal{A})$ if $\mathbf{Ar}(A_i, \mathbf{a}) \subseteq \mathrm{Con}(A_i)$ for the unique i such that $\mathbf{a} \in A_i$. ∎

We have the following analogue of [FJ, Lemma 25.5].

LEMMA 8.3: *Let $n \geq 0$. For each real normal stratification $\mathcal{A}_0$ of $\mathbb{A}^{n+1}$ over K we can find a real normal stratification $\mathcal{B}_0$ of $\mathbb{A}^n$ over K and a finite family $\mathcal{H} \supseteq \mathrm{Sub}\ \mathcal{B}_0$ of (isomorphism types of) finite e-structures with the following property. Let $\mathcal{A} = \langle \mathbb{A}^{n+1}, C_i/A_i, \mathrm{Con}(A_i) |\ i \in I\rangle$ be an augmentation of $\mathcal{A}_0$ to a real Galois stratification, and let $\mathcal{S} \subseteq \mathcal{H}$. Then we can find an augmentation $\mathcal{B} = \langle \mathbb{A}^n, D_j/B_j, \mathrm{Con}(B_j) |\ j \in J\rangle$ of $\mathcal{B}_0$ with $\bigcup_j \mathrm{Con}(B_j) \subseteq \mathcal{S}$, that depends on $\mathcal{S}$, such that for each Frobenius field $(M, \mathbf{P})$ that contains $(K, \mathbf{P}_0)$ and satisfies $\mathrm{Im}\,\mathbf{G}(M, \mathbf{P}) \cap \mathcal{H} = \mathcal{S}$, and for each $\mathbf{b} \in \mathbb{A}^n(M)$ we have: $\mathbf{Ar}(\mathcal{B}, \mathbf{b}) \subseteq \mathrm{Con}(\mathcal{B})$ if and only if*

(∗) *there exists $\mathbf{a} \in \mathbb{A}^{n+1}(M)$ such that $\pi(\mathbf{a}) = \mathbf{b}$ and $\mathbf{Ar}(\mathcal{A}, \mathbf{a}) \subseteq \mathrm{Con}(\mathcal{A})$.*

Proof: Use Remark 1.3, the stratification lemma [FJ, Lemma 17.26], and Lemma 7.5 to construct real normal stratifications

$$\mathcal{A}'_0 = \langle \mathbb{A}^{n+1}, C_{jk}/A_{jk}, |\ j \in J,\ k \in K(j)\rangle, \quad \text{and} \quad \mathcal{B}_0 = \langle \mathbb{A}^n, D_j/B_j, |\ j \in J\rangle$$

over K with the following properties (see [FJ, Lemma 25.5]).

(a) For each $j \in J$ and $k \in K(j)$ there is a unique $i \in I$, denoted $i(j,k)$, such that $A_{jk} \subseteq A_i$.

(b) Let $i = i(j,k)$. The Galois cover C'_{jk}/A_{jk} induced from C_i/A_i satisfies $K(C'_{jk}) \subseteq K(C_{jk})$,

(c) $\pi^{-1}(B_j) = \bigcup_{j\in K(j)} A_{jk}$ and $\pi(A_{jk}) = B_j$,

(d) $(C_{jk}/A_{jk}, D_j/B_j)$ is specialization compatible.

Choose $\mathcal{H}$ so that $\mathcal{H} \supseteq \mathrm{Sub}\ \mathcal{A}_0 \cup \mathrm{Sub}\ \mathcal{A}'_0 \cup \mathrm{Sub}\ \mathcal{B}_0$, and let $\mathcal{S} \subseteq \mathcal{H}$.

Let $i = i(j,k)$. Then $\mathrm{Con}(A_i)$ induces a conjugacy domain $\mathrm{Con}(C'_{jk}/A_{jk}, \mathcal{S})$ in $\mathrm{Sub}[C'_{jk}/A_{jk}, \mathbf{P}_0]$ (Property 6.4). Use Property 6.1 to define a conjugacy domain $\mathrm{Con}(A_{jk}, \mathcal{S})$ in $\mathrm{Sub}[C_{jk}/A_{jk}, \mathbf{P}_0]$ that belongs to $\mathcal{S}$. The two properties ensure that the real Galois stratification

$$\mathcal{A}' = \langle \mathbb{A}^{n+1}, C_{jk}/A_{jk}, \mathrm{Con}(A_{jk}, \mathcal{S}) |\ j \in J, k \in K(j)\rangle,$$

satisfies for every extension $(M,\mathbf{P})$ of $(K,\mathbf{P}_0)$ and each $\mathbf{a}\in M^{n+1}$

$$\mathbf{Ar}(\mathcal{A},\mathbf{a})\subseteq \mathrm{Con}(\mathcal{A}) \quad \text{if and only if} \quad \mathbf{Ar}(\mathcal{A}',\mathbf{a})\subseteq \mathrm{Con}(\mathcal{A}').$$

Thus we may assume that $\mathcal{A}'=\mathcal{A}$. Apply Lemmas 7.2 and 7.4 to augment $\mathcal{B}_0$ to the desired real Galois stratification. ∎

Remark 8.4: For $\mathcal{B}_0$, $\mathcal{A}$ and $\mathcal{S}$ as in Lemma 8.3 we can also find another augmentation $\mathcal{B}$ of $\mathcal{B}_0$, with $\bigcup_j \mathrm{Con}(B_j)\subseteq \mathcal{S}$, such that for each Frobenius field $(M,\mathbf{P})$ that contains $(K,\mathbf{P}_0)$ and satisfies $\mathrm{Im}\,\mathbf{G}(M,\mathbf{P})\cap\mathcal{H}=\mathcal{S}$, and for each $\mathbf{b}\in\mathbb{A}^n(M)$ we have: $\mathbf{Ar}(\mathcal{B},\mathbf{b})\subseteq\mathrm{Con}(\mathcal{B})$ if and only if

($*'$) $\mathbf{Ar}(\mathcal{A},\mathbf{a})\subseteq\mathrm{Con}(\mathcal{A})$ for each $\mathbf{a}\in\mathbb{A}^{n+1}(M)$ such that $\pi(\mathbf{a})=\mathbf{b}$.

This can be deduced using the complementary real Galois stratification, analogously to [FJ, Lemma 25.7]. ∎

9. Applications

Let $m,n\geq 0$. Put $\mathbf{X}=(X_1,\ldots,X_m)$, $\mathbf{Y}=(Y_1,\ldots,Y_n)$, and let $Q_1,\ldots,Q_m$ be quantifiers. The following expression $\vartheta(\mathbf{Y})$

$$(Q_1X_1)\cdots(Q_mX_m)[\mathbf{Ar}(\mathbf{X},\mathbf{Y})\subseteq \mathrm{Con}\,\mathcal{A}],$$

where $\mathcal{A}$ is a real Galois stratification of $\mathbb{A}^{n+m}$, is called a **real Galois formula** in the free variables $\mathbf{Y}$. Its interpretation is clear from Definition 8.1.

Let $\mathcal{L}_e(K)$ be the first order predicate calculus language of e-fold ordered fields augmented by constant symbols for the elements of the field K.

LEMMA 9.1: *Every formula $\vartheta(\mathbf{Y})=\vartheta(Y_1,\ldots,Y_n)$ in the language $\mathcal{L}_e(K)$ is equivalent to a Galois formula over $(K,\mathbf{P}_0)$.*

Proof: Write $\vartheta(\mathbf{Y})$ in the prenex normal form. Without loss of generality it is quantifier free, i.e. of the form

$$\bigvee_{i\in I}\left[\vartheta_i(\mathbf{Y})\wedge\bigwedge_{j=1}^{e}\bigwedge_{k=1}^{r_j} h_{ijk}(\mathbf{Y})\geq_j 0\right],$$

where $\vartheta_i(\mathbf{Y})$ defines a K-constructible set A_i in $\mathbb{A}^n$, and $h_{ijk}(\mathbf{Y})\in K[\mathbf{Y}]$. We may assume that $\bigcup_i A_i=\mathbb{A}^n$, otherwise add ϑ_0 that defines the complement of $\bigcup_i A_i$, and put $h_{0jk}=-1$. Replacing the A_i by appropriate constructible subsets we may assume that they are disjoint. Finally, we may stratify each of the A_i into smaller sets, and thus assume that each A_i is a nonsingular basic sets over K, say, with generic point $\mathbf{y}_i$ over K, and $C_i = K[A_i][\sqrt{-1}, \sqrt{h_{ijk}(\mathbf{y}_i)} \mid j,k]$ is a real Galois cover of $K[A_i]$.

Augment the normal stratification $\langle \mathbb{A}^n, C_i/A_i \mid i \in I\rangle$ to a real Galois stratification $\mathcal{A}$ by letting $\mathrm{Con}(A_i)$ be the collection of all $\mathbf{H} \in \mathrm{Sub}[C_i/A_i, \mathbf{P}_0]$ with $\varepsilon(\sqrt{h_{ijk}(\mathbf{y}_i)}) = \sqrt{h_{ijk}(\mathbf{y}_i)}$ for all $\varepsilon \in \mathcal{E}_j(\mathbf{H})$, $j = 1, \ldots, e$, $k = 1, \ldots, r_j$.

Let $(M, \mathbf{P})$ be an e-ordered field that extends $(K, \mathbf{P}_0)$, and let $\mathbf{a} \in A_i(M)$. Let φ: $K[C_i] \to \widetilde{M}$ be an extension of $\mathbf{y}_i \to \mathbf{a}$, let $\delta \in \mathcal{E}_j(\widetilde{M}/M, \mathbf{P})$ and $\varepsilon = \varphi^*(\delta)$. Then $\sqrt{h_{ijk}(\mathbf{a})} = \varphi(\sqrt{h_{ijk}(\mathbf{y}_i)}) \in \widetilde{M}$, and, from the equation of Remark 4.1(a), ε fixes $\sqrt{h_{ijk}(\mathbf{y}_i)}$ if and only if δ fixes $\sqrt{h_{ijk}(\mathbf{a})}$. Therefore

$$h_{ijk}(\mathbf{a}) \in P_j \Leftrightarrow \sqrt{h_{ijk}(\mathbf{a})}) \in \widetilde{M}(\delta) \Leftrightarrow \varepsilon(\sqrt{h_{ijk}(\mathbf{y}_i)}) = \sqrt{h_{ijk}(\mathbf{y}_i)}.$$

Thus $\mathbf{Ar}(A_i, \mathbf{a}) \subseteq \mathrm{Con}_i(A_i)$ if and only if $\bigwedge_j \bigwedge_k h_{ijk}(\mathbf{a}) \geq_j 0$. Therefore the Galois formula $\mathbf{Ar}(\mathbf{Y}) \subseteq \mathrm{Con}\,\mathcal{A}$ is equivalent to $\vartheta(\mathbf{Y})$ over $(K, \mathbf{P}_0)$. ∎

Let $(K, \mathbf{P}_0)$ be an e-fold ordered field, and let Π be a class of (isomorphism types of) superprojective e-structures. Denote by $\mathrm{Frob}(K, \mathbf{P}_0; \Pi)$ the class of e-fold ordered Frobenius fields $(M, \mathbf{P})$ with $\mathbf{G}(M, \mathbf{P}) \in \Pi$ that contain $(K, \mathbf{P}_0)$.

THEOREM 9.2: *Let $(K, \mathbf{P}_0)$ be a presented e-fold ordered field with elimination theory, and let ϑ be a sentence in $\mathcal{L}_e(K)$.*

(a) *We can effectively find a finite Galois extension L/K with $\sqrt{-1} \in L$, a finite family $\mathcal{H} \supseteq \mathrm{Sub}[L/K, \mathbf{P}_0]$ of (isomorphism types of) finite e-structures, and for each $\mathcal{S} \subseteq \mathcal{H}$ a conjugacy domain $\mathrm{Con}(\mathcal{S})$ in $\mathrm{Sub}[L/K, \mathbf{P}_0]$ contained in $\mathcal{S}$ such that for every Frobenius field $(M, \mathbf{P})$ that contains $(K, \mathbf{P}_0)$ and satisfies $\mathrm{Im}\,\mathbf{G}(M, \mathbf{P}) \cap \mathcal{H} = \mathcal{S}$ we have: $(M, \mathbf{P}) \models \vartheta$ if and only if*

(9.3) $\mathbf{G}(L/L \cap M, \mathrm{res}_{L\cap M}\mathbf{P}) \in \mathrm{Con}(\mathcal{S})$.

(b) *We have $(M, \mathbf{P}) \models \vartheta$ for all $(M, \mathbf{P}) \in \mathrm{Frob}(K, \mathbf{P}_0; \Pi)$ if and only if $\mathrm{Sub}[L/K, \mathbf{P}_0] \cap \mathcal{S} = \mathrm{Con}(\mathcal{S})$ for all $\mathcal{S} \subseteq \mathcal{H}$ that satisfy*

(9.4) *there is $\mathbf{G} \in \Pi$ such that $\mathrm{Im}\,\mathbf{G} \cap \mathcal{H} = \mathcal{S}$.*

Proof: (a) By Lemma 9.1, ϑ is equivalent to a real Galois sentence ϑ'. Inductively apply Lemma 8.3 and Remark 8.4 to assume that ϑ' is quantifier free. Then ϑ' is associated with a real Galois stratification $\mathcal{A} = \langle \mathbb{A}^0, L/\mathbb{A}^0, \mathrm{Con}(\mathcal{S})\rangle$ of $\mathbb{A}^0$, with $\mathrm{Con}(\mathcal{S})$ depending on $\mathcal{S} \subseteq \mathcal{H}$. By Property 6.1 we may assume that $\sqrt{-1} \in L$. Let 0 be the unique point of $\mathbb{A}^0$. Then $\mathbf{Ar}(L/\mathbb{A}^0, M, \mathbf{P}, 0) = \mathbf{G}(L/L\cap M, \mathrm{res}_{L\cap M}\mathbf{P})$. Hence (9.3) is the interpretation of $(M,\mathbf{P}) \models \vartheta'$.

(b) Fix $\mathcal{S} \subseteq \mathcal{H}$. Observe that

(9.5)
$$\{\mathbf{G}(L/L\cap M, \mathrm{res}_{L\cap M}\mathbf{P}) \mid (M,\mathbf{P}) \in \mathrm{Frob}(K,\mathbf{P}_0;\Pi),\ \mathrm{Im}\,\mathbf{G}(M,\mathbf{P})\cap\mathcal{H} = \mathcal{S}\} = \begin{cases} \mathrm{Sub}[L/K,\mathbf{P}_0]\cap\mathcal{S}, & \text{if } \mathcal{S} \text{ satisfies (9.4);} \\ \emptyset, & \text{otherwise.} \end{cases}$$

Indeed, if $(M,\mathbf{P}) \in \mathrm{Frob}(K,\mathbf{P}_0;\Pi)$, then

$$\mathbf{G}(L/L\cap M, \mathrm{res}_{L\cap M}\mathbf{P}) = \mathbf{Ar}(L/\mathbb{A}^0, M, \mathbf{P}, 0) \subseteq \mathrm{Sub}[L/K,\mathbf{P}_0] \cap \mathrm{Im}\,\mathbf{G}(M,\mathbf{P})$$

and $\mathbf{G}(M,\mathbf{P}) \in \Pi$. This gives the inclusion "$\subseteq$" in (9.5).

Conversely, let $\mathbf{G} \in \Pi$ such that $\mathrm{Im}\,\mathbf{G}\cap\mathcal{H} = \mathcal{S}$, and let $\mathbf{H} \in \mathrm{Sub}[L/K,\mathbf{P}_0]\cap\mathcal{S}$. Then $\mathbf{H} \in \mathrm{Im}\,\mathbf{G}$, so there is an epimorphism $\pi\colon \mathbf{G} \to \mathbf{H}$. Put $H' = H \cap G(L/K(\sqrt{-1}))$, let $G' = \pi^{-1}(H')$, and $\mathfrak{H} = \langle H, H', \mathcal{E}(\mathbf{H})\rangle$. By Lemma 3.6, $\mathfrak{G} = \langle G, G', \mathcal{E}(\mathbf{G})\rangle$ is a projective Artin-Schreier structure. Let $\pi\colon \mathfrak{G} \to \mathfrak{H}$ be the epimorphism of weak structures induced by $\pi\colon \mathbf{G} \to \mathbf{H}$.

There exists a PRC field M containing K and an isomorphism $\theta\colon \mathfrak{G} \to \mathfrak{G}(M)$ such that $\mathrm{res}_L \circ \theta = \pi\colon \mathfrak{G} \to \mathfrak{H} \subseteq \mathfrak{G}(L/K)$ [HJ1, Theorem 10.2]. Then θ induces an isomorphism $\theta\colon \mathbf{G} \to \mathbf{G}(M,\mathbf{P})$, where the ordering P_j is induced by the real closure $\widetilde{M}(\varepsilon)$ for $\varepsilon \in \theta(\mathcal{E}_j(\mathbf{G}))$. Moreover, $\mathrm{res}_L \circ \theta = \pi\colon \mathbf{G} \to \mathbf{H}$. Thus $\mathbf{H} = \pi(\mathbf{G}) = \mathrm{res}_L\mathbf{G}(M,\mathbf{P}) = \mathbf{G}(L/L\cap M, \mathrm{res}_{L\cap M}\mathbf{P})$. By Lemma 3.5(a) the orderings $P_1,\ldots,P_e$ are distinct and they are all the orderings on M. So $(M,\mathbf{P})$ is PRCe. As $\mathbf{G}(M,\mathbf{P}) \cong \mathbf{G} \in \Pi$ is superprojective, $(M,\mathbf{P})$ is Frobenius (Proposition 5.6), whence $(M,\mathbf{P}) \in \mathrm{Frob}(K,\mathbf{P}_0;\Pi)$.

Assertion (b) follows immediately from (a) and (9.5). ∎

Condition $\mathrm{Sub}[L/K,\mathbf{P}_0]\cap\mathcal{S} \subseteq \mathrm{Con}(\mathcal{S})$ can be effectively checked for each subfamily $\mathcal{S}$ of $\mathcal{H}$. The only difficulty is to decide which $\mathcal{S}$ satisfy (9.4). We list a few interesting cases in which this is possible:

COROLLARY 9.6: *The theory of* $\mathrm{Frob}(K,\mathbf{P}_0;\Pi)$ *in* $\mathcal{L}_e(K)$ *is primitive recursive*

(a) *for* $\Pi = \{\mathbf{H}\}$, *where* $\mathbf{H}$ *is superprojective and* $\mathrm{Im}\,\mathbf{H}$ *is a primitive recursive family of finite e-structures,*

(b) *for* $\Pi = \{\hat{\mathbf{D}}_{e,n}\}$,

(c) *for* $\Pi = \{\hat{\mathbf{D}}(2)_{e,n}\}$,

(d) *for* Π = *the class of all superprojective e-structures.*

Proof: (a) Condition (9.4) is $\mathrm{Im}\,\mathbf{H} \cap \mathcal{H} = \mathcal{S}$. It can be effectively checked.

(b) and (c) are special cases of (a).

(d) Let $\mathcal{H} = \{\mathbf{A}_1, \ldots, \mathbf{A}_m, \mathbf{B}_1, \ldots, \mathbf{B}_n\}$ and $\mathcal{S} = \{\mathbf{B}_1, \ldots, \mathbf{B}_n\} \subseteq \mathcal{H}$ be given families of finite e-structures. By (9.4) it suffices to decide, whether there exists a superprojective e-structure $\mathbf{G}$ such that the $\mathbf{A}_i$ are quotients of $\mathbf{G}$ and the $\mathbf{B}_j$ are not. This is done by a straightforward translation of the notion of embedding covers to the category of e-structures [FJ, §23]. We refer the reader to [La, §2] for the details. ∎

It follows that the elementary theory of Geyer and v.d. Dries fields is primitive recursive.

LEMMA 9.7: *The expression "the e-structure* $\mathbf{B}$ *is realizable over* $(M, \mathbf{P})$*" is an elementary statement.*

Proof: This expression is equivalent to "there exists a Galois extension N of M such that $\mathbf{G}(N/M, \mathbf{P}) \cong \mathbf{B}$".

Let B be given as a subgroup of S_{2n} by its action on $\{1, 2, \ldots, 2n\}$. For a polynomial $f(Z)$ of degree $2n$, the statement "f is irreducible, normal and there exists an isomorphism of permutation groups β: $G(f, M) \to B$" is elementary [FJ, p. 256]. In particular, it asserts that there are polynomials $p_1 = Z, p_2, \ldots, p_{2n}$ of degree $< 2n$ such that $p_i(z)$ is the ith root of f, and hence $p_{\sigma(i)}(z) = p_i(z_{\sigma(i)})$, for each i.

Let z be a root of f, and let $N = M(z)$ be the splitting field of f over M. Condition $\sqrt{-1} \in N$ is equivalent to the statement "there exists a polynomial q of degree $< 2n$ such that $q(z)^2 + 1 = 0$". Finally, fix $1 \le j \le e$ and $\varepsilon \in \mathcal{E}_j(\mathbf{B})$, and let $\delta = \beta^{-1}(\varepsilon)$. The condition "$P_j$ extends to $N(\delta)$" can be expressed as follows. There is an irreducible polynomial h of degree n and a polynomial g of degree $< 2n$, such that $h(g(z)) = 0$ and $g(p_{\varepsilon(1)}(z)) = g(z)$, and h changes sign in the real closure of (M, P_j). Use Tarski's Principle 1.4 to express this in $\mathcal{L}_e(K)$. ∎

PROPOSITION 9.8: *Let* $(M_1, \mathbf{P}_1)$ *and* $(M_2, \mathbf{P}_2)$ *be two e-fold ordered Frobenius extensions of* $(K, \mathbf{P}_0)$. *Then* $(M_1, \mathbf{P}_1) \equiv (M_2, \mathbf{P}_2)$ *in* $\mathcal{L}_e(K)$ *if and only if*

$$\tilde{K} \cap (M_1, \mathbf{P}_1) \cong \tilde{K} \cap (M_2, \mathbf{P}_2) \text{ and } \mathrm{Im}\,\mathbf{G}(M_1, \mathbf{P}_1) = \mathrm{Im}\,\mathbf{G}(M_2, \mathbf{P}_2). \tag{9.9}$$

Proof: Assume (9.9). Theorem 9.2(a) implies that $(M_1, \mathbf{P}_1)$ and $(M_2, \mathbf{P}_2)$ satisfy the same sentences in $\mathcal{L}_e(K)$. Conversely, let $(M_1, \mathbf{P}_1) \equiv (M_2, \mathbf{P}_2)$. The first part of (9.9) follows from [J1, Lemma 5.1] and the second part from Lemma 9.7. ∎

Let $(K, \mathbf{P}_0)$ be a fixed e-fold ordered field with K a countable Hilbertian field. For a sentence $\vartheta \in \mathcal{L}_e(K)$ denote $A(\vartheta) = \{\sigma \in G(K)^e | \ \mathcal{K}_\sigma \models \vartheta\}$, where $\mathcal{K}_\sigma$ is the field defined in Example 5.2. Then $A(\vartheta)$ is a measurable set and the measure $\mu(A(\vartheta))$ is a rational number [J1, Theorem 8.1].

THEOREM 9.10: *The function that assigns to a sentence $\vartheta \in \mathcal{L}_e(K)$ the rational number $\mu(A(\vartheta))$ is primitive recursive.*

Proof: Put $\mathcal{H} = \operatorname{Im} \hat{\mathbf{D}}_e$, and let L and Con be as in Theorem 9.2. For each $1 \leq j \leq e$ let ε_j be the generator of $G(L/L \cap \overline{K}_j)$. Then $\mu(A(\vartheta))$ is equal to the number of e-tuples $\sigma \in G(L/K)^e$ such that $(\varepsilon_1^{\sigma_1}, \ldots, \varepsilon_e^{\sigma_j}) \in \mathbf{G}(L/K, \mathbf{P}_0)$ generates an e-structure in Con divided by $[L : K]^e$. ∎

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