COVERS OF KLEIN SURFACES*

by

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Abstract

We consider ramified (Galois) covers of the upper half plane in the category of Klein surfaces. We study the connection between the group theoretical ramification data of the cover and its geometrical properties, such as the number of the connected components of the boundary and orientability of the surface.

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Introduction

In this paper we study certain aspects of the *real forms* of a ramified covering $p\colon X \to \mathbb{P}^1$ of the complex projective line by a compact Riemann surface X. That is, we are dealing with coverings of Klein surfaces $\bar{p}\colon Y \to \mathbb{H}$, where $\mathbb{H}$ denotes the upper half plane as a Klein surface, $\eta\colon X \to Y$ is the canonical double covering of Y, and $\bar{p} \circ \eta = \xi \circ p$, where $\xi\colon \mathbb{P}^1 \to \mathbb{H}$ is the canonical double covering of the upper half plane by the Riemann sphere. We are especially interested in the case where $X/\mathbb{H}$ is Galois, that is, $|\mathrm{Aut}(X/\mathbb{H})| = \deg(X/\mathbb{H})$.

It is well known that the existence of both Y and $\bar{p}$ (*real form of the covering*) is equivalent to the existence of an antianalytic involution u of X such that $p \circ u = \xi \circ p$; we call it a "lifting of ξ". In particular, the set of ramification points $T := \{c_1, \ldots, c_{r+2s}\} \subset \mathbb{P}^1$ must be invariant under complex conjugation.

Assume that $X/\mathbb{H}$ is Galois and fix $x_0 \in \mathbb{P}^1 \smallsetminus T$. Put $\Pi = \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ and $G = \mathrm{Aut}(X/\mathbb{P}^1)$. There is an epimorphism $f\colon \Pi \to G$ called the canonical map. Let g_i be the image in G of a closed loop through x_0 containing in its interior only one ramification point c_i. Then, g_i generates the inertia group at a point lying above c_i and $(G, T, g_i \mid i = 1, \ldots, r+2s)$ is called *ramification data**. The aim of the paper is to characterize the existence, as well as the topology, of the possible real forms $(Y, \bar{p})$ in terms of the Galois group G, and the ramification data of p.

We recall that the topological type of a compact Klein surface Y is uniquely determined by the topological type (the genus) of its canonical double covering $\hat{Y}$ (which is a Riemann surface), the number of connected components of the preimage R in $\hat{Y}$ of its boundary $\partial(Y)$, (also called *the set of real points*) and whether Y is orientable. This last property is equivalent to $\hat{Y} \smallsetminus R$ being disconnected [BEGG, Prop. A.28].

By the Riemann Existence Theorem we know that given G and any ramification data, a compact Riemann surface X can be found that is a Galois unramified covering of $\mathbb{P}^1$ outside T with these data. One usually proves this by a cut-and-paste method, first constructing only a topological model of X and then showing that it carries a

* Usually one considers the conjugacy class of g_i instead, since changing x_0 changes the canonical map up to group conjugation.

unique analytic structure. In Section 1 we give an alternative construction for X, called the *explicit Galois cover of Riemann surfaces corresponding to the ramification data* $(G, T, g_i \mid i = 1, \dots, r + 2s)$, or shortly, the *explicit model for* X. Namely, we fit a suitable path going through the ramification points T and glue the half spheres along this path according to the ramification data. Unlike the classical construction, this one gives the Riemann surface together with its analytic structure. We shall show how this modification has the advantage that allows studying the topology of the set of points invariant under antianalytic involutions.

The construction is quite simple, though its description requires a lot of notation, probably since a sheet of paper is not the most appropriate media to convey intuitively simple geometrical constructions (in which pieces of the surfaces are allowed to pass through each other).

In Section 2 we introduce suitable systems of generators, with respect to the complex conjugation for both the fundamental group $\Pi = \Pi_1(\mathbb{P}^1 \setminus T, x_0)$ and the fundamental group $\Phi = \Phi(\mathbb{H} \setminus \xi(T), \xi(x_0))$ of the complement of $\xi(T)$ in $\mathbb{H}$ (see also [KN] and [HJ]).

In Sections 2 and 3 we answer the question of the existence of a real form of a covering, in terms of the group G itself, and in a more general context. In fact, assume we are given a Galois covering of Riemann surfaces p: $X \to \hat{Y}$ such that $\hat{Y}$ is the double covering of a Klein surface $Y = \hat{Y}/u_0$ where u_0 is an antianalytic involution. We show that $X \to Y$ is a Galois covering iff there is an antianalytic lifting u of u_0 to X, or, equivalently, there is such a lifting having some fixed point, or iff there is a lifting u: $X \to X$ which is an antianalytic involution having some fixed point (*real involution*, cf. Lemma 2.1). There is an algebraic counterpart of this Lemma in the frame of ramification theory of real valuations, saying that every lifting of a real involution is a real involution (cf. [AV]).

Consequently, each lifting u of an antianalytic involution downstairs induces an involutive automorphism of the group G of deck transformations, which we still call u. Namely, for $g = f(\alpha) \in G$, $f(\alpha)^u = f(\alpha^{u_0}) = u \circ f(\alpha) \circ u$. Therefore, one can describe the group H of deck transformations of $X \to Y$ as a semidirect product $< u > G$. In this way, in the case of the covering $X \to \mathbb{H}$, the canonical map $f\colon \Pi \to G$ lifts to an epimorphism $f\colon \Phi \to H$. Moreover, we prove that the existence of this $u \in \mathrm{Aut}(G)$ also suffices for the existence of a lifting of the complex conjugation. We heavily use here the *explicit model for* X: to get the antianalytic involution u on X we glue together a piece-wise definition, suggested by the automorphism of the group (cf. Proposition 3.2). We call this construction *explicit real involution*, since we have an description of it in terms of the group operation, and in particular a condition on a point on X to be *real* that is, fixed by u, in terms of G (cf. Corollary 3.5).

Next, we are able to find out adjacencies of real segments (between two ramification points) to recover the connected components, the *real ovals*. In fact, we obtain a formula in terms of the index of the centralizer of u in H and the ramification orders (cf. Theorem 4.10).

In Section 5 we conclude the study of the topology of the real form describing in terms of the action of u on G whether or not the Klein surface $X/<u>$ is orientable. To this end we consider the image $\mathcal{A}$ of a suitable system of generators of Φ under the canonical map $f\colon \Phi \to H$, and associate with H and $\mathcal{A}$ a graph (a subgraph of the Cayley graph of H with respect to $\mathcal{A}$) whose connectedness is equivalent to $X/<u>$ being not orientable (cf. Theorem 5.3 and Corollary 5.4).

In Section 6 we use this presentation to study whether one can always cover a given Klein surface Y, which is a covering of $\mathbb{H}$ unramified outside $\xi(T)$, by another Klein surface $\hat{Y}$, such that $\hat{Y}/\mathbb{H}$ is still unramified outside $\xi(T)$ and $\hat{Y}$ is orientable, respectively, not orientable. Because of the existence of the unramified orienting double [AG, §6], this is interesting only if we require the complex double $\hat{X}$ of $\hat{Y}$ to be Galois over $\mathbb{H}$. We reduce this question to a problem in combinatorial group theory and partially solve this. The full solution seems to be difficult – we hope that this paper will stimulate experts to try to solve it.

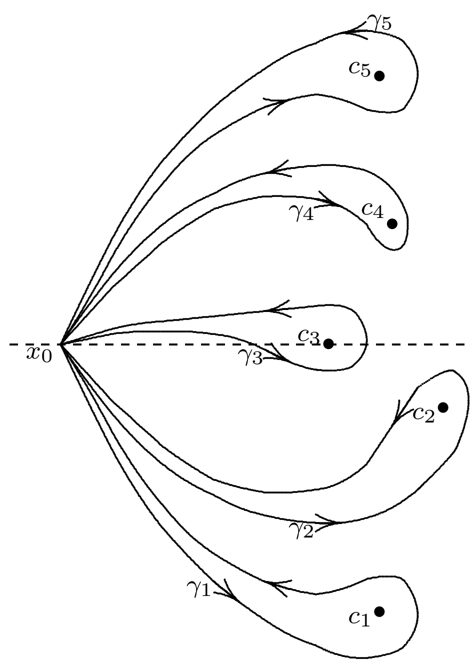
Finally we stress that our results and methods are heavily algorithmic and mostly inspired by classical ideas of Galois covers and combinatorial group theory.

1. An explicit model of a Riemann surface

Let X be a compact connected Riemann surface and let $p\colon X \to \mathbb{P}^1$ be a **Galois** (branched) cover of the Riemann sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$ with group $G = \mathrm{Aut}(X/\mathbb{P}^1)$. Let $T = \{c_1, \ldots, c_n\}$ be a set of $2 \le n < \infty$ points in $\mathbb{P}^1$ that contains the branch points of $X/\mathbb{P}^1$.

Fix a point $x_0 \in \mathbb{P}^1 \smallsetminus T$ and let $\Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ be the fundamental group of $\mathbb{P}^1 \smallsetminus T$. Then $\Pi_1(\mathbb{P}^1 \smallsetminus T, x_0) = \langle \gamma_1, \ldots, \gamma_n \mid \gamma_1 \cdots \gamma_n = 1 \rangle$, where $\gamma_1, \ldots, \gamma_n$ are certain closed loops from x_0 around $c_1, \ldots, c_n$, respectively [Voe, Theorem 4.27]. In particular, $\gamma_1 \cdots \gamma_n = 1$. (The joining of paths is in the order "from left to right": path $\alpha\beta$ starts at the origin of α and ends at the terminus of β.)

(1)



Furthermore, fix $x \in X$ such that $p(x) = x_0$. There is an epimorphism

$$f\colon \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0) \to G = \mathrm{Aut}(X/\mathbb{P}^1) \tag{2}$$

given as follows: $f(\gamma)$ is the unique element $g \in \mathrm{Aut}(X/\mathbb{P}^1)$ such that $g(x) \in p^{-1}(x_0)$ is the terminus of the path $\hat{\gamma}$ on X from x that lifts γ. Put $g_i = f(\gamma_i)$, for $i = 1, \ldots, n$.

Thus p yields a **branch data**

$$(G, c_1, g_1, \ldots, c_n, g_n), \tag{3}$$

namely, a finite group G, elements $g_1, \dots, g_n \in G$ such that $g_1 g_2 \cdots g_n = 1$, and n distinct points $c_1, \dots, c_n \in \mathbb{P}^1$.

As is well known [Voe, Theorem 4.32], this **branch data** completely determines the Galois cover $p\colon X \to \mathbb{P}^1$, up to an isomorphism.

Given branch data (3), we are going to describe a cut-and-paste procedure that *explicitly* produces such X and $p\colon X \to \mathbb{P}^1$, together with the appropriate analytic structure.

Construction 1.1: We first introduce some geometric description, together with terminology and notation.

PART A: *Use of* $\mathbb{P}^1$ *for charts.* We visualize $\mathbb{P}^1$ as the unit sphere $S = S^2$ in $\mathbb{R}^3$. Now, a Riemann surface is a topological space with an atlas of charts with maps into $\mathbb{C}$. Since any *proper* subset $U \subset \mathbb{P}^1$ is mapped by a suitable Möbius transformation onto a subset of $\mathbb{C}$, we may as well give an atlas of charts with maps into proper subsets of S. This seems at times more natural; for instance, $\mathbb{P}^1$ itself can be given by an open covering $\{U_j\}_j$, where the maps of the charts are just the identities $U_j \to U_j$.

PART B: *A closed path on the sphere.* An **arc** is a simple path on S lying on a circle, with distinct endpoints. A path on S is **piecewise linear**, if it is the join (concatenation) of finitely many arcs such that the angle between no two consecutive arcs is 0.

Write $T = \{c_i\}_{i \in I}$, where $I = \mathbb{Z}/n\mathbb{Z}$ (thus, in this notation, $c_0 = c_n$ and $c_{n+1} = c_1$). For each $i \in I$ we choose a simple piecewise linear path from c_i to c_{i+1}. We denote it (as well as its underlying set) $[c_i, c_{i+1}]$; put also $(c_i, c_{i+1}) = [c_i, c_{i+1}] \smallsetminus \{c_i, c_{i+1}\}$. It is easy to see that we may choose these paths so that the join of

$$[c_0, c_1], [c_1, c_2], \dots, [c_{n-1}, c_n]$$

is a simple closed piecewise linear path C on S.

Path C divides $S \smallsetminus C$ into two disjoint open regions: the 'upper part' and the 'lower part'. Let S^- be the lower part, and let S^+ be the closure of the upper part with $T = \{c_i\}_{i \in I}$ removed, so that S is the disjoint union $S = S^+ \cup S^- \cup T$.

Part C: *Sheets.* For each $g \in G$ consider a copy $S_g = \{g\} \times S$ of the Riemann sphere S. We use the following

Notation: The elements of S_g will be written as $\llparenthesis g, z \rrparenthesis$, where $z \in S$, rather than (g, z), to avoid confusion with the path notation (c_i, c_{i+1}) introduced above.

For an ambient subset A of S put

(i) $A^+ = A \cap S^+$ and $A^- = A \cap S^-$, and

(ii) $A_g = \{g\} \times A = \{\llparenthesis g, z \rrparenthesis \in S_g \mid z \in A\}$.

For each $i \geq 0$ put $h_i = g_1 g_2 \cdots g_i$; this, in fact, defines h_i for each $i \in I$, since $h_n = g_1 \cdots g_n = 1$. We have $g_i = h_{i-1}^{-1} h_i$ and $h_n = 1$.

Part D: *Cut and Paste.* Let $\dot\bigcup_{g \in G} S_g = \{\llparenthesis g, z \rrparenthesis \mid g \in G,\ z \in S\}$ be the disjoint union of the S_g; we visualize the S_g as concentric spheres, infinitesimally close to each other. Let $\dot\bigcup_{g \in G} S_g \to S$ be the projection $\llparenthesis g, z \rrparenthesis \mapsto z$. For each $i \in I$, cut each S_g along the arc $[c_i, c_{i+1}]_g$ and glue the lower part S_g^- of S_g with the upper part $(S^+ \mathbin{\dot\cup} T)_{gh_i}$ of S_{gh_i} along this cut. I.e., if $z_\nu \to z$, where $z \in [c_i, c_{i+1}] \subseteq S^+ \mathbin{\dot\cup} T$ and $\{z_\nu\}_{\nu=1}^\infty \subseteq S^-$, then $\llparenthesis g, z_\nu \rrparenthesis \to \llparenthesis gh_i, z \rrparenthesis$. This implies that $\llparenthesis g, c_i \rrparenthesis$ identifies with $\llparenthesis h, c_i \rrparenthesis$ if and only if $h\langle g_i \rangle = g\langle g_i \rangle$.

By abuse of notation, $\llparenthesis g, c_i \rrparenthesis$ denotes also the equivalence class of the point $\llparenthesis g, c_i \rrparenthesis$.

It is easy to see that the resulting set X has an obvious structure of a Riemann surface that covers $\mathbb{P}^1$ and the map $p\colon X \to \mathbb{P}^1$ induced from $\dot\bigcup_{g \in G} S_g \to S$ is a cover with the prescribed ramification data. However, for the sake of rigour and to have a future reference for the precise behaviour in the neighbourhoods of the critical points $\llparenthesis g, c_i \rrparenthesis$, we give a more formal proof in the next two parts.

Part E: *An atlas for the unramified cover.* Firstly, it is easier to see that $X' = X \smallsetminus \{\llparenthesis g, c_i \rrparenthesis \mid i \in I,\ g \in G\}$ is an unramified analytic cover of $\mathbb{P}^1 \smallsetminus T$. Indeed, for each $i \in I$ choose an open subset U_i of the Riemann sphere S containing the open path (c_i, c_{i+1}), so that $U_1, \ldots, U_n$ are disjoint. (In particular, $c_1, \ldots, c_n \notin \dot\bigcup_{i \in I} U_i$.) Let

$$
\begin{aligned}
\hat{S}_g^+ &= S_g^+ \mathbin{\dot\cup} \dot\bigcup_{i \in I} (U_i^-)_{gh_i^{-1}}, \qquad g \in G \\
\hat{S}_g^- &= S_g^- \mathbin{\dot\cup} \dot\bigcup_{i \in I} (U_i^+)_{gh_i}, \qquad g \in G.
\end{aligned}
$$

Then $X' = \hat{S}_g^+ \cup \hat{S}_g^-$ and the projections on the first coordinate

$$p\colon \hat{S}_g^+ \mapsto S^+ \mathbin{\dot\cup} \dot{\bigcup_{i\in I}} U_i^- = S^+ \cup \bigcup_{i\in I} U_i \subseteq \mathbb{P}^1 \text{ by } (h, z) \mapsto z;$$
$$p\colon \hat{S}_g^- \mapsto S^- \mathbin{\dot\cup} \dot{\bigcup_{i\in I}} U_i^+ = S^- \cup \bigcup_{i\in I} U_i \subseteq \mathbb{P}^1 \text{ by } (h, z) \mapsto z$$

obviously define an analytic atlas on X'.

PART F: *An atlas for the ramified cover.* To complete the above atlas to an analytic atlas on X, let D_i be an open disk on the sphere S around c_i of radius ρ_i so small that

(i) $D_1, \ldots, D_n$ are disjoint;

(ii) $D_i \cap U_j = \emptyset$ for $j \neq i-1, i$; and

(iii) $D_i \cap [c_{i-1}, c_i]$ and $D_i \cap [c_i, c_{i+1}]$ are arcs, say, $(a_i, c_i]$ and $[c_i, b_i)$, respectively.

For each $i \in I$ and each left coset $g\langle g_i\rangle$ of $\langle g_i\rangle$ in G put

$$D_{i,g\langle g_i\rangle} = \{(g, c_i)\} \mathbin{\dot\cup} \dot{\bigcup_{h\in g\langle g_i\rangle}} \big((D_i^+)_h \mathbin{\dot\cup} (D_i^-)_{hh_i^{-1}}\big)$$

Then, for each $i \in I$, we have

$$p^{-1}(D_i) = \bigcup_{g\in G} \{(g, c_i)\} \cup (D_i^+)_g \cup (D_i^-)_g = \dot{\bigcup_{g\langle g_i\rangle}} D_{i,g\langle g_i\rangle},$$

where $g\langle g_i\rangle$ runs through the left cosets of $\langle g_i\rangle$ in G. It now suffices to find, for each $i \in I$ and each coset $g\langle g_i\rangle$ of G, a bijection $\psi\colon D_{i,g\langle g_i\rangle} \to D_i \subset \mathbb{P}^1$ such that

$$(4) \qquad \begin{aligned} p\circ\psi^{-1}&\colon \psi(S_g^+ \cap D_{i,g\langle g_i\rangle}) \to p(S_g^+ \cap D_{i,g\langle g_i\rangle}) \\ p\circ\psi^{-1}&\colon \psi(S_g^- \cap D_{i,g\langle g_i\rangle}) \to p(S_g^- \cap D_{i,g\langle g_i\rangle}) \end{aligned}$$

are the restrictions of an analytic map $\phi_i\colon D_i \to D_i$.

Applying an appropriate Möbius transformation of $S = \mathbb{P}^1$ we may assume that $c_i = 0$ and D_i is the unit disk in $\mathbb{C}$. Let $0 < \alpha < 2\pi$ be that angle between the arcs $(a_i, c_i]$ and $[c_i, b_i)$ that contains S^+; the complementary angle of size $2\pi - \alpha$ contains S^-. Let e be the order of $\langle g_i\rangle$ and consider a representative gg_i^j of $g\langle g_i\rangle$. In this case we may write

$$(D_i^+)_{gg_i^j} = \{(gg_i^j, z^e) \mid 0 < |z| < 1,\ \frac{2j\pi}{e} \le \arg z \le \frac{2j\pi + \alpha}{e}\}$$

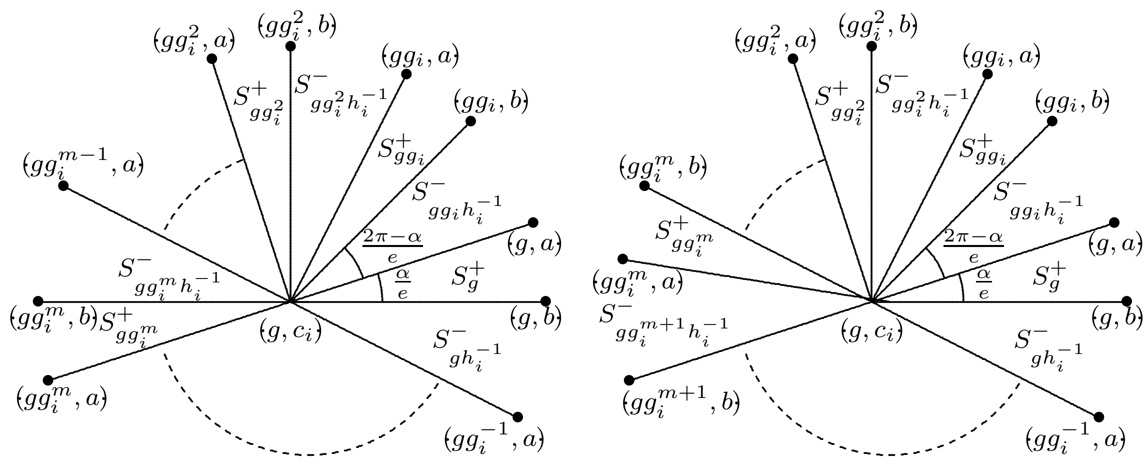
$$(D_i^-)_{gg_i^j h_i^{-1}} = \{(gg_i^j h_i^{-1}, z^e) \mid 0 < |z| < 1,\ \frac{2j\pi - 2\pi + \alpha}{e} < \arg z < \frac{2j\pi}{e}\}$$

$$j = 0, 1, \ldots, e-1,$$

$$(g, c_i) = (g, 0^e).$$

Using this presentation, define ψ by $\psi(h, z^e) = z$. Then $p \circ \psi^{-1}$ is the map $z \mapsto z^e$.

Thus the analytic neighbourhood $D_{i,g\langle g_i\rangle}$ of (g, c_i) (more precisely, its image under the chart ψ) looks as follows



$e = 2m$ even $\qquad\qquad$ $e = 2m + 1$ odd

Figure 5

(Here $a = a_i$ and $b = b_i$. Observe that $(gg_i^{-1}, a) = (gg_i^{e-1}, a)$ and $S^-_{gh_i^{-1}} = S^-_{gg_i^{e-1} h_{i-1}^{-1}}$.)

Part G: *Action of G and the map $f\colon \Pi_1(\mathbb{P}^1 \setminus T, x_0) \to G$.* It is clear that G acts on X by $h(g, z) = (hg, z)$, and, for each $h \in G$, the map $(g, z) \mapsto (hg, z)$ is analytic. As $p\colon X \to \mathbb{P}^1$ is of degree $|G|$, we have $G = \mathrm{Aut}(X/\mathbb{P}^1)$.

To verify that $f(\gamma_i) = g_i$, for each $i \in I$, let us assume that $x_0 \in S^+$ and γ_i starts in S^+, passes at (c_{i-1}, c_i) from S^+ to S^-, then passes at (c_i, c_{i+1}) from S^- back to S^+, and returns, through S^+, to x_0. Let $x = (1, x_0) \in S_1^+$; then $x \in X$ and $p(x) = x_0$. The lifting of γ_i to a path on X from x starts in S_1^+, passes at the cut $(c_{i-1}, c_i)_1$ from S_1^+ to $S^-_{h_{i-1}^{-1}}$, then passes at $(c_i, c_{i+1})_{g_i}$ from $S^-_{h_{i-1}^{-1}}$ to $S^+_{h_{i-1}^{-1} h_i} = S^+_{g_i}$, and continues, through $S^+_{g_i}$, to $(g_i, x_0) = g_i x$. Thus $f(\gamma_i) = g_i$.

Definition 1.2: The above constructed Riemann surface X and the cover $p\colon X \to \mathbb{P}^1$ will be called **the explicit Galois cover corresponding to branch data (3)**. ∎

Remark 1.3: Let $\Pi = \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$.

(a) $\Pi = \langle \gamma_1, \dots, \gamma_n \mid \gamma_1 \cdots \gamma_n = 1 \rangle$ is a free group on $n-1$ generators $\gamma_1, \dots, \gamma_{n-1}$ [Voe, Corollary 4.29].

(b) For each $1 \le i \le n$ put

$$\delta_i = \gamma_1 \gamma_2 \cdots \gamma_i \qquad \text{and} \qquad h_i = g_1 g_2 \cdots g_i. \tag{6}$$

Thus δ_i is homotopic to a closed simple loop that contains in its interior exactly the first i points of T and $f(\delta_i) = h_i = g_1 \cdots g_i$, for each $i \in I$.

(c) Π is a free group on $\delta_1, \dots, \delta_{n-1}$. Indeed, by (6), $\gamma_i = \delta_{i-1}^{-1}\delta_i$, for each i, so $\delta_1, \dots, \delta_{n-1}$ generate Π. Since Π is free on $n-1$ generators, $\delta_1, \dots, \delta_{n-1}$ are necessarily free generators.

(d) It follows from Part F in the above construction that the order of $g_i = f(\gamma_i)$ is the ramification index of c_i in $X/\mathbb{P}^1$. In particular, c_i is unramified in $X/\mathbb{P}^1$ if and only if $g_i = f(\gamma_i) = 1$.

(e) Let T' be a subset of T containing c_0, say, $T' = \{c_{i_1}, \dots, c_{i_{n'}} = c_0\}$. Let $\Pi' = \Pi_1(\mathbb{P}^1 \smallsetminus T', x_0)$. The inclusion $\iota\colon \mathbb{P}^1 \smallsetminus T \to \mathbb{P}^1 \smallsetminus T'$ induces a homomorphism $\iota_*\colon \Pi \to \Pi'$ [Ma, §II.4], by mapping the class in Π of a path γ onto its class in Π'. In particular, if $1 \le i \le n$ and $c_i \notin T'$, then $\iota_*(\gamma_i) = 1$; by (c), Π' is a free group on $\gamma_{i_1}, \dots, \gamma_{i_{n'-1}}$. Thus $\iota_*\colon \Pi \to \Pi'$ is surjective. It easily follows that $\mathrm{Ker}(\iota_*)$ is the normal subgroup of Π generated by $\{\gamma_i \mid c_i \notin T'\}$.

By (d), the canonical map $f\colon \Pi \to G$ of (2) factors through $\iota_*\colon \Pi \to \Pi'$ if and only if $T \smallsetminus T'$ is unramified in $X/\mathbb{P}^1$. If this happens, the induced homomorphism $f'\colon \Pi' \to G$ is the corresponding canonical map.

Remark 1.4: Let E be a subgroup of G. Then $p\colon X \to \mathbb{P}^1$ induces a cover $\bar{p}\colon E\backslash X \to \mathbb{P}^1$. Conversely, each (branched) cover of $\mathbb{P}^1$ arises this way, for a suitable Galois cover $p\colon X \to \mathbb{P}^1$. In this sense the above construction produces all covers of $\mathbb{P}^1$, not just Galois covers.

2. Real involutions and fundamental groups

Fix a complex conjugation $z \xrightarrow{c} \overline{z}$ of $\mathbb{P}^1$ and denote $\mathbb{H} = \mathbb{P}^1/\langle c\rangle$. Then $\xi\colon \mathbb{P}^1 \to \mathbb{H}$ is a cover of Klein surfaces of degree 2, the so-called complex double of $\mathbb{H}$ [BEGG, Construction 0.1.12]. For each path α on $\mathbb{P}^1$ let $\overline{\alpha}$ be its complex conjugate, i.e., if $\alpha\colon [0,1] \to \mathbb{P}^1$, then $\overline{\alpha}(t) = \overline{\alpha(t)}$ for every $t \in [0,1]$.

More generally, let $\xi\colon \hat{Y} \to Y$ be the complex double of a Klein surface Y. There is a unique dianalytic automorphism $z \xrightarrow{c} \overline{z}$ of $\hat{Y}/Y$. For each path α on $\hat{Y}$ let $\overline{\alpha}$ be its conjugate. I.e., if $\alpha\colon [0,1] \to \hat{Y}$, then $\overline{\alpha}(t) = \overline{\alpha(t)}$ for every $t \in [0,1]$.

Let T be a finite subset of $\hat{Y}$ closed under c. Choose $x_0 \in \hat{Y} \smallsetminus T$ such that $\overline{x_0} = x_0$. Then $\alpha \to \overline{\alpha}$ is an involution of $\Pi_1(\hat{Y} \smallsetminus T, x_0)$.

Let $p\colon X \to \hat{Y}$ be a cover of Riemann surfaces. Denote by $\mathrm{Aut}(X/\hat{Y})$ the group of analytic automorphisms of p. Further denote by $\mathrm{Aut}(X/Y)$ the group of dianalytic automorphisms of the composite cover $X \xrightarrow{p} \hat{Y} \xrightarrow{\xi} Y$. In particular, $\mathrm{Aut}(\hat{Y}/Y)$ is of order 2; let c be its generator. In general, $\mathrm{Aut}(X/\hat{Y}) \le \mathrm{Aut}(X/Y)$ and either $\mathrm{Aut}(X/Y) = \mathrm{Aut}(X/\hat{Y})$ or $(\mathrm{Aut}(X/Y) : \mathrm{Aut}(X/\hat{Y})) = 2$, in which case $\mathrm{Aut}(X/Y) \smallsetminus \mathrm{Aut}(X/\hat{Y})$ consists of those automorphisms u that restrict to c on $\hat{Y}$, that is, satisfy $p \circ u = c \circ p$.

The composite cover X/Y is **Galois**, if $|\mathrm{Aut}(X/Y)| = \deg(X/Y)$. By the preceding paragraph this happens if and only if

(a) $X/\hat{Y}$ is Galois, i.e., $|\mathrm{Aut}(X/\hat{Y})| = \deg(X/\hat{Y})$, and

(b) there is a dianalytic automorphism of X that restricts to c on $\hat{Y}$.

Let $p\colon X \to \hat{Y}$ be a Galois cover of Riemann surfaces unramified outside a finite subset T of $\hat{Y}$. Let u be a dianalytic automorphism of X. We say that u is a **real involution** of $\xi \circ p$ if it restricts to c on $\hat{Y}$ and there is a point $x \in X$ such that $u(x) = x$. We may assume that $x \in X \smallsetminus p^{-1}(T)$. Indeed, in the chart of a sufficiently small neighbourhood of x, the map p is given by $z \mapsto z^e$ in the unit disk, for a suitable integer e. As $p \circ u = c \circ p$, the map u is given in this chart by $z \mapsto \eta(z)\overline{z}$, where $\eta(z)$ is an e-the root of unity for each z. By continuity, $\eta(z)$ is constant, say, η, and hence $z \mapsto \eta\overline{z}$ is a reflection with respect to the line through 0 and $\sqrt{\eta}$; each point on this line corresponds to a fixed point of u.

LEMMA 2.1: *Let $\xi\colon \hat{Y} \to Y$ be the complex double of a Klein surface Y and let $p\colon X \to \hat{Y}$ be a Galois cover of Riemann surfaces unramified outside T.*

(a) *Let u be a real involution of $\xi \circ p$. Then u is of order 2.*

(b) *If c lifts to a dianalytic automorphism of X (i.e., if X/Y is Galois) then c lifts to a real involution.*

(c) *Let $x_0 \in \hat{Y} \smallsetminus T$ and $x \in X$ such that $p(x) = x_0$. Consider the canonical map $f\colon \Pi_1(\hat{Y} \smallsetminus T, x_0) \to \mathrm{Aut}(X/\hat{Y})$. Let u be a real involution of $\xi \circ p$ such that $u(x) = x$. Then $f(\overline{\alpha}) = f(\alpha)^u = uf(\alpha)u$ for every $\alpha \in \Pi_1(\hat{Y} \smallsetminus T, x_0)$.*

Proof: (a) Let $x \in X$ such that $u(x) = x$ and let $x_0 = p(x)$. If θ is a closed loop on X with origin at x, then $c(p(\theta)) = p(u(\theta))$. Thus [Ma, Theorem V.5.1 and its proof], u is necessarily obtained in the following way. For $z \in X' = p^{-1}(\hat{Y} \smallsetminus T)$, choose a path α in $\hat{Y} \smallsetminus T$ that lifts to a path from x to z; then $u(z)$ is the terminus of the unique lifting of $\overline{\alpha}$ to a path with origin at x. Extend u from X' to $u\colon X \to X$ by continuity. As $\overline{\overline{\alpha}} = \alpha$, we have $u^2 = 1$.

(c) Let $\alpha \in \Pi_1(\hat{Y} \smallsetminus T, x_0)$. By the definition of $f(\alpha)$, α lifts to a path on X from x to $f(\alpha)(x)$. The above description of u implies that $\overline{\alpha}$ lifts to a path on X from x to $u(f(\alpha)(x)) = uf(\alpha)u(x) = f(\alpha)^u(x)$. Thus, by the definition of $f(\overline{\alpha})$, we have $f(\overline{\alpha}) = f(\alpha)^u$.

(b) Let $x_0 \in \hat{Y} \smallsetminus T$ such that $c(x_0) = x_0$ and choose $x \in X$ such that $p(x) = x_0$. Suppose c lifts to u. As $X/\hat{Y}$ is Galois, we have $u(x) = g(x)$ for some $g \in G$. Multiply u by g^{-1} from the left to assume that $u(x) = x$. ∎

Notation 2.2: Let $Y = \mathbb{H}$ and $\hat{Y} = \mathbb{P}^1$. Assume that $T = \{c_1, \ldots, c_n\}$ consists of $2s$ complex points

$$c_1, c_2, \ldots, c_{2s-1}, c_{2s},$$

and r real points

$$c_{2s+1}, \ldots, c_{2s+r},$$

where $c_{2j} = \overline{c_{2j-1}}$ and $\mathrm{Im} c_{2j-1} < 0 < \mathrm{Im} c_{2j}$ for $j = 1, \ldots, s$, and $n = r + 2s \geq 2$. Let $\gamma_1, \ldots, \gamma_n = \gamma_0 \in \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ be loops as in Section 1, and for each $0 \leq i \leq n$ let $\delta_i = \gamma_1\gamma_2 \cdots \gamma_i$.

Denote the involution $\gamma \mapsto \overline{\gamma}$ of $\Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ by ε. We call the semidirect product

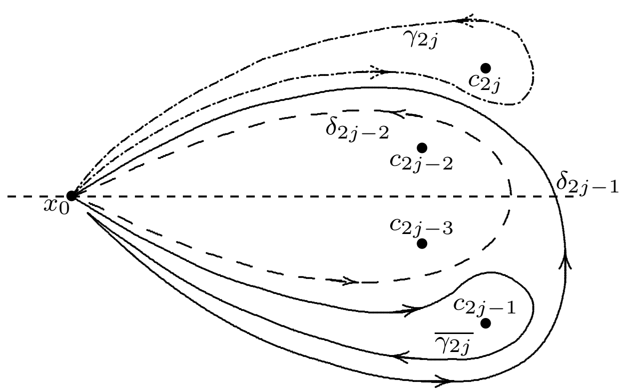
$$\Phi_1(\mathbb{H} \smallsetminus \xi(T), \xi(x_0)) = \langle \varepsilon \rangle \ltimes \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0) \tag{1}$$

the **fundamental group of** $\mathbb{H} \smallsetminus \xi(T)$ **based at** $\xi(x_0) \in \mathbb{H}$. For $\gamma \in \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ we may write $\overline{\gamma} = \gamma^\varepsilon$, where $\gamma \mapsto \gamma^\varepsilon$ is the conjugation by ε in $\Phi_1(\mathbb{H} \smallsetminus \xi(T), \xi(x_0))$.

If $r \geq 1$, let $\varepsilon_i = \varepsilon\delta_{2s+i} = \varepsilon\gamma_1\gamma_2\cdots\gamma_{2s+i} \in \Phi_1(\mathbb{H} \smallsetminus \xi(T), \xi(x_0))$, for $i = 1, \ldots, r$. Then $\varepsilon_r = \varepsilon$. ∎

Lemma 2.3: *The involution $\gamma \mapsto \overline{\gamma}$ of $\Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ satisfies:*
(a) $\overline{\delta_{2j}} = \delta_{2j}^{-1}$, *for* $j = 0, 1, \ldots, s$;
(b) $\overline{\delta_i} = \delta_i^{-1}$, *for* $i = 2s, 2s+1, \ldots, 2s+r$;
(c) $\overline{\gamma_{2j}} = \delta_{2j-2}\delta_{2j-1}^{-1}$, *for* $j = 1, \ldots, s$;

Proof: Conditions (a) and (b) follow as the subset of T contained in the interior of δ_{2j} or δ_{2s+i} is closed under complex conjugation by Remark 1.3(b). Condition (c) requires a straightforward verification: In the following picture the dashed path δ_{2j-2} is homotopic to the solid path $\overline{\gamma_{2j}}\delta_{2j-1}$:



∎

Lemma 2.4: *Put $\Pi = \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ and $\Phi = \Phi_1(\mathbb{H} \smallsetminus \xi(T), \xi(x_0))$.*
(a) $\Pi = \langle \gamma_2, \overline{\gamma_2}, \gamma_4, \overline{\gamma_4}, \ldots, \gamma_{2s}, \overline{\gamma_{2s}}, \delta_{2s+1}, \ldots, \delta_{2s+r}\rangle$. *Moreover,*

(a1) If $r \geq 1$, then Π is the free group on

$$\gamma_2, \overline{\gamma_2}, \gamma_4, \overline{\gamma_4}, \ldots, \gamma_{2s}, \overline{\gamma_{2s}}, \delta_{2s+1}, \ldots, \delta_{2s+r-1}. \tag{2}$$

(a2) If $r = 0$, then Π has presentation

$$\Pi = \langle \gamma_2, \overline{\gamma_2}, \gamma_4, \overline{\gamma_4}, \ldots, \gamma_{2s}, \overline{\gamma_{2s}}, \mid \ \omega = \overline{\omega}\rangle, \qquad \text{where } \omega = \gamma_2\gamma_4\cdots\gamma_{2s}.$$

(b) *$\langle\varepsilon\rangle$ acts on Π by*

$$\gamma_{2j}^{\varepsilon} = \overline{\gamma_{2j}},\ \overline{\gamma_{2j}}^{\varepsilon} = \gamma_{2j},\ j = 1, \ldots, s, \qquad \delta_{2s+i}^{\varepsilon} = \delta_{2s+i}^{-1},\ i = 1, \ldots, r-1.$$

(c) *If $r \geq 1$, then Φ has presentation*

$$\Phi = \langle \gamma_2, \gamma_4, \ldots, \gamma_{2s}, \varepsilon_1, \ldots, \varepsilon_r \mid \varepsilon_1^2 = \cdots = \varepsilon_r^2 = 1 \rangle; \tag{3}$$

hence Φ is the free product of r copies $\langle\varepsilon_1\rangle, \ldots, \langle\varepsilon_r\rangle$ of $\mathbb{Z}/2\mathbb{Z}$ and s copies of $\mathbb{Z}$.

If $r = 0$, then Φ has presentation

$$\Phi = \langle \gamma_2, \gamma_4, \ldots, \gamma_{2s}, \varepsilon \mid \varepsilon^2 = 1,\ [\varepsilon, \omega] = 1 \rangle, \qquad \text{where } \omega = \gamma_2\gamma_4\cdots\gamma_{2s}.$$

Proof: (a) Let $\Pi_0 = \langle \gamma_2, \overline{\gamma_2}, \gamma_4, \overline{\gamma_4}, \ldots, \gamma_{2s}, \overline{\gamma_{2s}}, \delta_{2s+1}, \ldots, \delta_{2s+r} \rangle$. We show that $\delta_k \in \Pi_0$, for $1 \leq k \leq 2s + r$.

We may assume that $k \leq 2s$. Trivially, $\delta_0 = 1 \in \Pi_0$. Assume, by induction, that $\delta_1, \ldots, \delta_{2j-2} \in \Pi_0$ for some $1 \leq j \leq s$. Then, by Lemma 2.3(c), $\delta_{2j-1} = \overline{\gamma_{2j}}^{-1}\delta_{2j-2} \in \Pi_0$, and hence, $\delta_{2j} = \delta_{2j-1}\gamma_{2j} \in \Pi_0$.

In fact, by induction on j, $\delta_{2j} = \overline{\gamma_{2j}}^{-1} \cdots \overline{\gamma_4}^{-1}\overline{\gamma_2}^{-1}\gamma_2\gamma_4\cdots\gamma_{2j}$. In particular,

(5) $\delta_{2s} = \overline{\omega}^{-1}\omega$, where $\omega = \gamma_2\gamma_4\cdots\gamma_{2s}$.

(a1) Assume $r \geq 1$. As $n = 2s + r$, by Remark 1.3(c), Π is the free group on $2s+r-1$ generators. As $\delta_{2s+r} = 1$, by the first paragraph of this proof (2) is a sequence of $2s + r - 1$ generators of Π. Hence Π is free on them.

(a2) Assume $r = 0$. As $n = 2s$, Π is the free group on $\delta_1, \ldots, \delta_{2s}$ modulo the relation $\delta_{2s} = 1$. Thus, by the first paragraph of this proof and (5), Π is the free group on $\gamma_2, \overline{\gamma_2}, \gamma_4, \overline{\gamma_4}, \ldots, \gamma_{2s}, \overline{\gamma_{2s}}$ modulo the relation $\overline{\omega}^{-1}\omega = 1$.

(b) See Lemma 2.3(b).

(c) First assume $r \geq 1$. Recall that $\delta_{2s+r} = 1$. By (a) and (b)

$$\begin{aligned}
\Phi =&\langle \varepsilon, \gamma_2, \gamma_4, \ldots, \gamma_{2s}, \overline{\gamma_2}, \overline{\gamma_4}, \ldots, \overline{\gamma_{2s}}, \delta_{2s+1}, \ldots, \delta_{2s+r-1} \mid \\
&\qquad \varepsilon^2 = 1, \gamma_2^\varepsilon = \overline{\gamma_2},\ \gamma_4^\varepsilon = \overline{\gamma_4}, \ldots, \gamma_{2s}^\varepsilon = \overline{\gamma_{2s}}, \delta_{2s+1}^\varepsilon = \delta_{2s+1}^{-1}, \ldots, \delta_{2s+r-1}^\varepsilon = \delta_{2s+r-1}^{-1} \rangle \\
=&\langle \varepsilon, \gamma_2, \gamma_4, \ldots, \gamma_{2s}, \delta_{2s+1}, \ldots, \delta_{2s+r-1} \mid \\
&\qquad \varepsilon^2 = 1, \delta_{2s+1}^\varepsilon = \delta_{2s+1}^{-1}, \ldots, \delta_{2s+r-1}^\varepsilon = \delta_{2s+r-1}^{-1} \rangle \\
=&\langle \varepsilon, \delta_{2s+1}, \ldots, \delta_{2s+r-1}, \gamma_2, \gamma_4, \ldots, \gamma_{2s} \mid \\
&\qquad (\varepsilon\delta_{2s+r})^2 = 1, (\varepsilon\delta_{2s+1})^2 = 1, \ldots, (\varepsilon\delta_{2s+r-1})^2 = 1 \rangle \\
=&\langle \varepsilon_1, \ldots, \varepsilon_r, \gamma_2, \gamma_4, \ldots, \gamma_{2s} \mid \ \varepsilon_1^2 = 1, \ldots, \varepsilon_r^2 = 1 \rangle.
\end{aligned}$$

Now assume $r = 0$. Then $s = \frac{n}{2} \geq 1$. By (a) and (b),

$$\begin{aligned}
\Phi =&\langle \varepsilon, \gamma_2, \gamma_4, \ldots, \gamma_{2s}, \overline{\gamma_2}, \overline{\gamma_4}, \ldots, \overline{\gamma_{2s}} \mid \\
&\qquad \overline{\omega} = \omega, \varepsilon^2 = 1, \gamma_2^\varepsilon = \overline{\gamma_2},\ \gamma_4^\varepsilon = \overline{\gamma_4}, \ldots, \gamma_{2s}^\varepsilon = \overline{\gamma_{2s}} \rangle \\
=&\langle \varepsilon, \gamma_2, \gamma_4, \ldots, \gamma_{2s} \mid \ \varepsilon^2 = 1, \varepsilon\omega\varepsilon = \omega \rangle \\
=&\langle \varepsilon, \gamma_2, \gamma_4, \ldots, \gamma_{2s} \mid \ \varepsilon^2 = 1, [\varepsilon, \omega] = 1 \rangle
\end{aligned}$$

So

$$\begin{aligned}
\Phi =&\langle \varepsilon, \gamma_2, \gamma_4, \ldots, \gamma_{2s-2}, \omega \mid \ \varepsilon^2 = 1, [\varepsilon, \omega] = 1 \rangle \\
\cong&(\langle \varepsilon \rangle \times \langle \omega \rangle) \star \langle \gamma_1 \rangle \star \cdots \star \langle \gamma_{2s-2} \rangle,
\end{aligned}$$

where $\langle \varepsilon \rangle \cong \mathbb{Z}/2\mathbb{Z}$ and $\langle \omega \rangle \cong \langle \gamma_1 \rangle \cong \cdots \cong \langle \gamma_{2s-3} \rangle \cong \mathbb{Z}$. ∎

Furthermore, we may rewrite the assertion of Lemma 2.1(c) as follows:

LEMMA 2.5: *Let $p\colon X \to \mathbb{P}^1$ be a Galois cover of Riemann surfaces unramified outside a finite set T closed under the complex conjugation and let u be a real involution. Let $x_0 \in \mathbb{P}^1 \smallsetminus T$ and $x \in X$ such that $p(x) = x_0$. Then the canonical map $f\colon \Pi = \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0) \to \mathrm{Aut}(X/\mathbb{P}^1)$ extends to an epimorphism $f\colon \Phi_1(\mathbb{H} \smallsetminus \xi(T), \xi(x_0)) \to \mathrm{Aut}(X/\mathbb{H})$ such that $f(\varepsilon) = u$. (We call this extension also the 'canonical map'.)*

Remark 2.6: Let $T' \subseteq T$ be a set closed under the complex conjugation, say, consisting of $2s'$ complex points $c_{j_1}, c_{j_2}, \ldots, c_{2j_{s'}-1}, c_{2j_{s'}}$, and r' real points $c_{2s+i_1} \ldots, c_{2s+i_{r'}}$ where $1 \leq j_1 < \cdots < j_{s'} \leq s$ and $1 \leq i_1 < \cdots < i_{r'} \leq r$. Let

$$\begin{aligned}
\Pi = \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0), &\qquad \Phi = \Phi_1(\mathbb{H} \smallsetminus \xi(T), \xi(x_0)) = \langle \varepsilon \rangle \ltimes \Pi, \\
\Pi' = \Pi_1(\mathbb{P}^1 \smallsetminus T', x_0), &\quad \Phi' = \Phi_1(\mathbb{H} \smallsetminus \xi(T'), \xi(x_0)) = \langle \varepsilon' \rangle \ltimes \Pi',
\end{aligned}$$

where ε, ε' are the complex conjugations of paths $\gamma \mapsto \overline{\gamma}$ in Φ, Φ', respectively.

(a) By Remark 1.3(e), there is an epimorphism $\iota_*\colon \Pi \to \Pi'$ induced by the identity map $\gamma \mapsto \gamma$ on paths. Since $\iota_*(\overline{\gamma}) = \overline{\iota_*(\gamma)}$, this epimorphism extends to an epimorphism $\iota_*\colon \Phi \to \Phi'$ by $\varepsilon \mapsto \varepsilon'$.

(b) Assume $r \geq 1$ and $r' \geq 1$. By Lemma 2.4, Φ' has presentation

$$\Phi' = \langle \varepsilon'_{i_1}, \ldots, \varepsilon'_{i_{r'}}, \gamma_{2j_1}, \gamma_{2j_2}, \ldots, \gamma_{2j_{s'}} \mid (\varepsilon'_{i_1})^2 = \cdots = (\varepsilon'_{i_{r'}})^2 = 1 \rangle, \tag{3'}$$

where $\varepsilon'_{i_k} = \varepsilon' \gamma_{j_1} \gamma_{j_2} \cdots \gamma_{j_{s'}} \gamma_{2s+i_1} \cdots \gamma_{2s+i_k}$, for $1 \leq k \leq r'$. (In particular, $\varepsilon'_{i_{r'}} = \varepsilon'$.) We observe that

$$\iota_*(\varepsilon_i) = \iota_*(\varepsilon \gamma_1 \gamma_2 \cdots \gamma_s \gamma_{2s+1} \cdots \gamma_{2s+i}) = \varepsilon' \gamma_{j_1} \gamma_{j_2} \cdots \gamma_{j_{s'}} \gamma_{2s+i_1} \cdots \gamma_{2s+i_k} = \varepsilon'_{i_k}, \qquad i = 1, \ldots, r,$$

where k is the largest integer such that $i_k \leq i$.

(c) By Remark 1.3(e), the canonical map $f\colon \Phi \to \mathrm{Aut}(X/\mathbb{H})$ factors through $\iota_*\colon \Phi \to \Phi'$ if and only if $T \smallsetminus T'$ is unramified in $X/\mathbb{P}^1$. If this happens, the induced homomorphism $f'\colon \Phi' \to \mathrm{Aut}(X/\mathbb{H})$ is the corresponding canonical map.

3. Complex conjugation

Let $\mathbb{H} = \{z \in \mathbb{P}^1 \mid \mathrm{Im} z \geq 0\}$ be the upper half plane, as a Klein surface. We want to study ramified covers $\bar{p}$: $Y \to \mathbb{H}$, where Y is a Klein surface with nonempty boundary. We are especially interested in the boundary and in the orientability of Y. This study can be essentially reduced to Section 1 in the following way. Let η: $\hat{Y} \to Y$ be the complex double [BEGG, Construction 0.1.12] of Y and let u be the generator of $\mathrm{Aut}(\hat{Y}/Y) \cong \mathbb{Z}/2\mathbb{Z}$. Then $\hat{Y}$ is a compact Riemann surface. The boundary of Y is the image of $\delta\hat{Y} = \{y \in \hat{Y} \mid u(y) = y\}$ in Y and its connected components are in bijection with the connected components of $\hat{Y}$. Moreover, Y is orientable if and only if $\hat{Y} \smallsetminus \delta\hat{Y}$ is disconnected. Thus, instead of Y, we consider the pair $(\hat{Y}, u)$.

The complex conjugation c on $\mathbb{P}^1$ ($z \mapsto \bar{z}, \infty \mapsto \infty$) induces a cover of Klein surfaces ξ: $\mathbb{P}^1 \to \mathbb{H} = \mathbb{P}^1/\langle c \rangle$. This is the complex double of $\mathbb{H}$. We have $\mathrm{Aut}(\mathbb{P}^1/\mathbb{H}) = \langle c \rangle \cong \mathbb{Z}/2\mathbb{Z}$.

By the universal property of complex doubles, $\bar{p}$: $Y \to \mathbb{H}$ is induced from a cover $\hat{p}$: $\hat{Y} \to \mathbb{P}^1$ of Riemann surfaces. There is a Klein surface X and a cover $X \to Y$ such that its composition $X \to \mathbb{H}$ with $\bar{p}$ is Galois and factors into a cover p: $X \to \mathbb{P}^1$ and ξ.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "p"'] & \hat{Y} \arrow[r, "\eta"] \arrow[d, "\hat{p}"] & Y \arrow[d, "\bar{p}"] \\ & \mathbb{P}^1 \arrow[r, "\xi"] & \mathbb{H} \end{tikzcd} \qquad \begin{tikzcd} F & F_0\mathbb{C} \arrow[l] & F_0 \arrow[l] \\ & \mathbb{C}(t) \arrow[ul] \arrow[u] & \mathbb{R}(t) \arrow[l] \arrow[u] \end{tikzcd}$$

(I.e., the corresponding extension of fields of meromorphic functions [BEGG, Appendix] $F/\mathbb{R}(t)$ is Galois and $\mathbb{C}(t) \subseteq F$. So X is a Riemann surface [BEGG, Remarks A.1(3)].) Let $G = \mathrm{Aut}(X/\mathbb{P}^1)$, $H = \mathrm{Aut}(X/\mathbb{H})$, and $H_0 = \mathrm{Aut}(X/Y)$. Then $Y = H_0\backslash X$ and $\bar{p}$ is induced from $\xi \circ p$.

We want to restrict ourselves to Klein surfaces $H_0\backslash X$ that have a nonempty boundary and are "maximal" in the sense that there is no proper subgroup H_1 of H_0 such that $H_1\backslash X$ has a nonempty boundary. In Lemma 2.1 we have seen that for such surfaces H_0 is of order 2, that is, $\hat{Y} = X$. Thus $\bar{p}$: $Y \to \mathbb{H}$ is a real form of p: $X \to \mathbb{P}^1$.

For each path γ on $\mathbb{P}^1$ let $\overline{\gamma}$ be its complex conjugate.

As in Section 1, let $T = \{c_1, \dots, c_n\}$ be a set of $2 \leq n < \infty$ points in $\mathbb{P}^1$ that contains the branch points of $X/\mathbb{P}^1$. It should be the lifting of its image in $\mathbb{H}$, and hence closed under the complex conjugation. Enlarging T, if necessary, we may assume that T contains at least one real point. (This allows us to avoid a separate treatment of the easier special case.) Thus T consists of $2s \geq 0$ complex points

$$c_1, c_2, \ldots, c_{2s-1}, c_{2s},$$

and $r \geq 1$ real points

$$c_{2s+1}, \ldots, c_{2s+r} = c_0,$$

where $c_{2j} = \overline{c_{2j-1}}$ and $\mathrm{Im} c_{2j-1} < 0 < \mathrm{Im} c_{2j}$ for $j = 1, \ldots, s$, and $n = r + 2s$.

As in Section 1, let C be a closed simple path on $S = \mathbb{P}^1$, the join of piecewise linear paths

$$[c_0, c_1], [c_1, c_2], \ldots, [c_{r+2s-1}, c_{r+2s} = c_0]. \tag{1a}$$

However, we now may require some additional properties for C:

(1b) The real points of T lie on the **equator** $\mathbb{R} \cup \{\infty\}$, which is a great circle on S. So after reordering them we may assume that C has been constructed so that its components $[c_{2s+1}, c_{2s+2}], [c_{2s+2}, c_{2s+3}], \ldots, [c_{2s+r-1}, c_0]$ are arcs lying on the equator. If $s = 0$, also $[c_0, c_{2s+1}]$ lies on the equator.

(1c) The rest of C may be chosen to look as in the following picture:

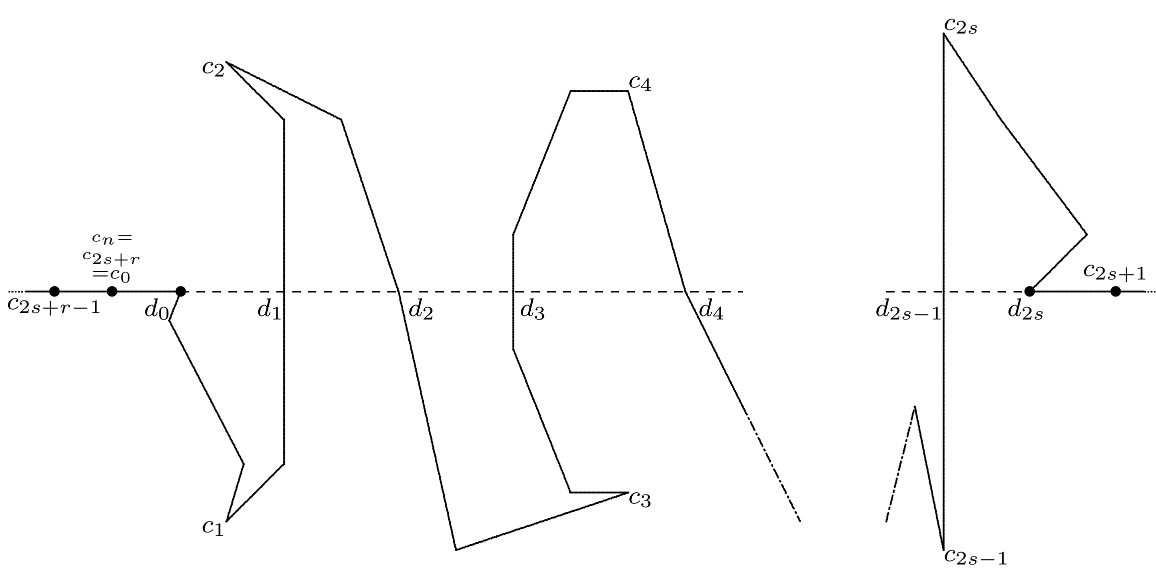


Figure 2

Here, if $s = 0$, then $d_0 = d_{2s}$ is an arbitrary point on $[c_0, c_{2s+1}]$. Otherwise $[c_0, d_0]$ is the intersection of $[c_0, c_1]$ with the equator, $[d_{2s}, c_{2s+1}]$ is the intersection of $[c_{2s}, c_{2s+1}]$ with the equator, and d_j is the unique point of intersection of $[c_j, c_{j+1}]$ with the equator, $j = 1, \ldots, 2s-1$. More specifically:

(1d) Path $[c_0, c_{2s+1}]$ from c_0 to c_{2s+1} on the equator (that avoids $c_{2s+2}, \ldots, c_{2s+r-1}$ and – unless $s = 0$ – is not a subpath of C) passes through the points $d_0, d_1, d_2, \ldots, d_{2s}$ in this order.

(1e) For $1 \le j \le s$ the complex conjugate of the piecewise linear path $[c_{2j-1}, c_{2j}]$ is its inverse (that is, the underlying set of $[c_{2j-1}, c_{2j}]$ is closed under complex conjugation). It crosses the equator in d_{2j-1}. Moreover, if D_{2j-1}, D_{2j} are open disks (as in Construction 1.1, Part F(iii)) around c_{2j-1}, c_{2j}, respectively and $D_{2j-1} \cap [c_{2j-1}, c_{2j}] = [c_{2j-1}, b_{2j-1})$, and $D_{2j} \cap [c_{2j-1}, c_{2j}] = (a_{2j}, c_{2j}]$, then $\overline{b_{2j-1}} = a_{2j}$.

(1f) We also introduce the following notation. Recall the definition of S^+, S^- from Construction 1.1, Part B. For each $1 \le j \le s$ let M_j be the region of S^+ surrounded by the paths $[d_{2j-2}, c_{2j-1}]$, its complex conjugate, and $[c_{2j-1}, c_{2j}]$, *together* with these three paths with their endpoints $c_{2j-1}, c_{2j}, d_{2j-2}$.

Also, let L_j be the open region of S^- surrounded by the paths $[c_{2j}, d_{2j}]$, its complex conjugate, and $[c_{2j-1}, c_{2j}]$. (In particular, points that lie on these three paths are *not* in L_j.)

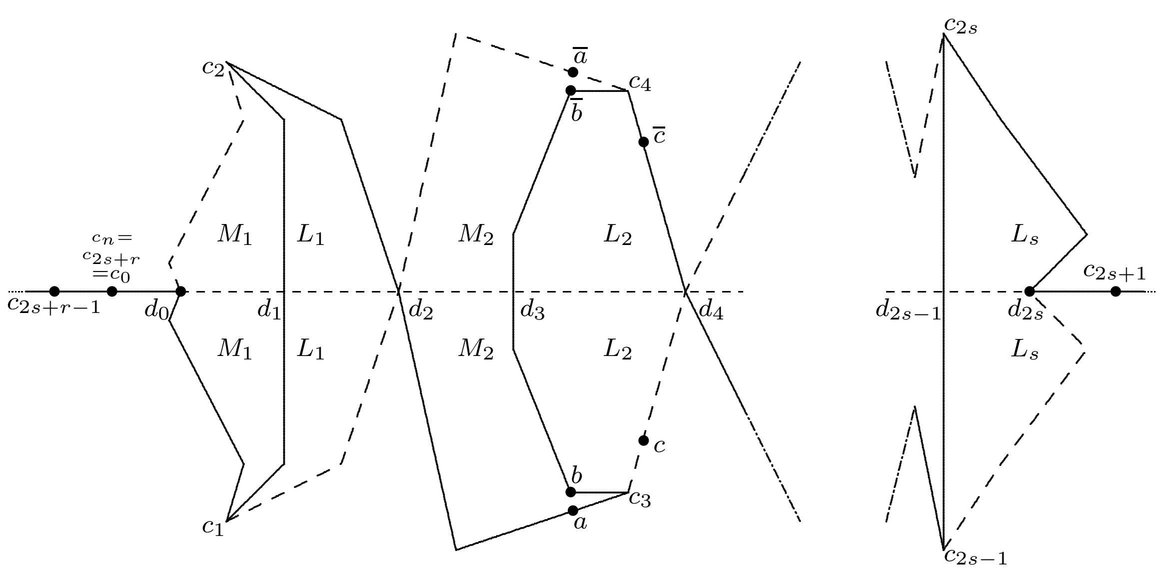


Figure 3

(Here $a = a_3, b = b_3 = \overline{a_4}, c = \overline{b_4}$ in the notation of Construction 1.1, Part F(iii).)

Choose $x_0 \in (c_0, d_0)$. Then $\overline{x_0} = x_0$ and $x_0 \in \mathbb{P}^1 \smallsetminus T$. Recall that $\Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ is generated by $\delta_1, \ldots, \delta_{r+2s}$ (Remark 1.3) and $\gamma \mapsto \overline{\gamma}$ is an involution of $\Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ (Section 2).

For the rest of this section let $p\colon X \to \mathbb{P}^1$ be the explicit Galois cover of Riemann surfaces corresponding to the branch data $(G, c_1, g_1, \ldots, c_n, g_n)$ constructed in Section 1 and let f be the canonical map from $\Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ to $\mathrm{Aut}(X/\mathbb{P}^1)$. Assume that there is a group automorphism $g \mapsto \overline{g}$ of G such that $\overline{f(\gamma)} = f(\overline{\gamma})$ for every $\gamma \in \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$. By Lemma 2.1(c), this is a necessary condition for the existence of a real involution. We will now show, by an explicit construction, that this condition is also sufficient. But first we need a better description of the automorphism $g \mapsto \overline{g}$. Put

$$h_i = g_1 \cdots g_i, \qquad \text{for } i = 0, 1, \ldots, n.$$

Lemma 3.1: *The automorphism $g \mapsto \overline{g}$ of G satisfies:*

(a) $\overline{h_i} = h_i^{-1}$, *for* $i = 2s, 2s+1, \ldots, 2s+r$,

(b) $\overline{h_{2j}} = h_{2j}^{-1}$, *for* $j = 0, 1, \ldots, s$,

(c) $\overline{g_{2j-1}} = h_{2j-2} g_{2j}^{-1} h_{2j-2}^{-1}$, *for* $j = 1, \ldots, s$,

(d) $\overline{g_{2j}} = h_{2j-2} h_{2j-1}^{-1} = h_{2j-2} g_{2j-1}^{-1} h_{2j-2}^{-1}$, *for* $j = 1, \ldots, s$,

(e) $\overline{g_i} = h_{i-1} h_i^{-1} = h_i g_i^{-1} h_i^{-1}$, *for* $i = 2s+1, \ldots, 2s+r$.

Proof: We have $g_i = f(\gamma_i)$, for $i = 1, \ldots, n$ and $\delta_k = \gamma_1 \cdots \gamma_k$, for $k = 0, \ldots, n$. Therefore $h_k = f(\delta_k)$, for $k = 0, \ldots, n$. So assertions (a), (b), and the left equality of (d) follow from Lemma 2.3. The second equality of (d) follows from $h_{2j-2} = h_{2j-1} g_{2j-1}$. Assertion (e) follows from (a) and from $g_i = h_{i-1}^{-1} h_i$. Finally, applying the automorphism $g \mapsto \overline{g}$ to (d) we get $g_{2j} = \overline{h_{2j-2}}\,\overline{g_{2j-1}}^{-1}\,\overline{h_{2j-2}}^{-1}$, from which (c) follows by (b). ∎

Here is the definition of the real involution:

Proposition 3.2: *For each point $(g, z) \in X$ put*

$$\overline{(g,z)} = \begin{cases} (\overline{g}h_i, \overline{z}) = (\overline{g}h_i, z) & \text{if } z \in [c_i, c_{i+1}], \text{ for } i = 2s+1, \ldots, 2s+r-1, \\ (\overline{g}h_{2s}, \overline{z}) = (\overline{g}h_{2s}, z) & \text{if } z \in [d_{2s}, c_{2s+1}], \\ (\overline{g}h_{2j}^{-1}, \overline{z}) & \text{if } z \in L_j, \text{ for } j = 1, \ldots, s, \\ (\overline{g}h_{2j-2}, \overline{z}) & \text{if } z \in M_j, \text{ for } j = 1, \ldots, s, \\ (\overline{g}, \overline{z}) & \text{otherwise.} \end{cases} \tag{5}$$

Then $(g,z) \mapsto \overline{(g,z)}$ is a real involution u of X which satisfies $ugu = \overline{g}$ for every $g \in G$.

Proof: Definition (5) is good: Firstly, the cases in (5) are disjoint, except for the definition of $\overline{(g,c_i)}$, for $2s+1 \le i \le 2s+r-1$. Namely, we have $c_i = \overline{c_i} \in [c_i, c_{i+1}]$, but also $c_i \in [c_{i-1}, c_i]$, for $i > 2s+1$, and $c_i \in [d_{2s}, c_i]$, for $i = 2s+1$, so $\overline{(g,c_i)}$ is defined to be $(\overline{g}h_i, c_i)$ and $(\overline{g}h_{i-1}, c_i)$ at the same time. But $\overline{g}h_i = \overline{g}h_{i-1}g_i$, so these points are equal.

Secondly, although for each i and k we have $(g,c_i) = (gg_i^k, c_i)$, the definition of $\overline{(g,c_i)}$ does not depend on k. Indeed, for $2s+1 \le i \le 2s+r$ we have, by Lemma 3.1(e),

$$\overline{(gg_i^k, c_i)} = (\overline{g}\,\overline{g_i}^k h_i, \overline{c_i}) = (\overline{g}h_i g_i^{-k}, \overline{c_i}) = (\overline{g}h_i, \overline{c_i}) = \overline{(g,c_i)}.$$

(Case $i = 2s+r$ is included here since $h_{2s+r} = 1$.) For $i = 2j-1$, where $1 \le j \le s$, we have $c_{2j-1}, \overline{c_{2j-1}} = c_{2j} \in M_j$, and hence by Lemma 3.1(c)

$$\overline{(gg_i^k, c_i)} = (\overline{g}\,\overline{g_{2j-1}}^k h_{2j-2}, \overline{c_{2j-1}}) = (\overline{g}h_{2j-2}g_{2j}^{-k}, c_{2j}) = (\overline{g}h_{2j-2}, c_{2j}) = \overline{(g,c_i)}.$$

Similarly for $i = 2j$, where $1 \le j \le s$, by Lemma 3.1(d),

$$\overline{(gg_i^k, c_i)} = (\overline{g}\,\overline{g_{2j}}^k h_{2j-2}, \overline{c_{2j}}) = (\overline{g}h_{2j-2}g_{2j-1}^{-k}, c_{2j-1}) = (\overline{g}h_{2j-2}, c_{2j-1}) = \overline{(g,c_i)}.$$

Notice that the map $(g,z) \mapsto \overline{(g,z)}$ is of order 2 by Lemma 3.1(a), (b).

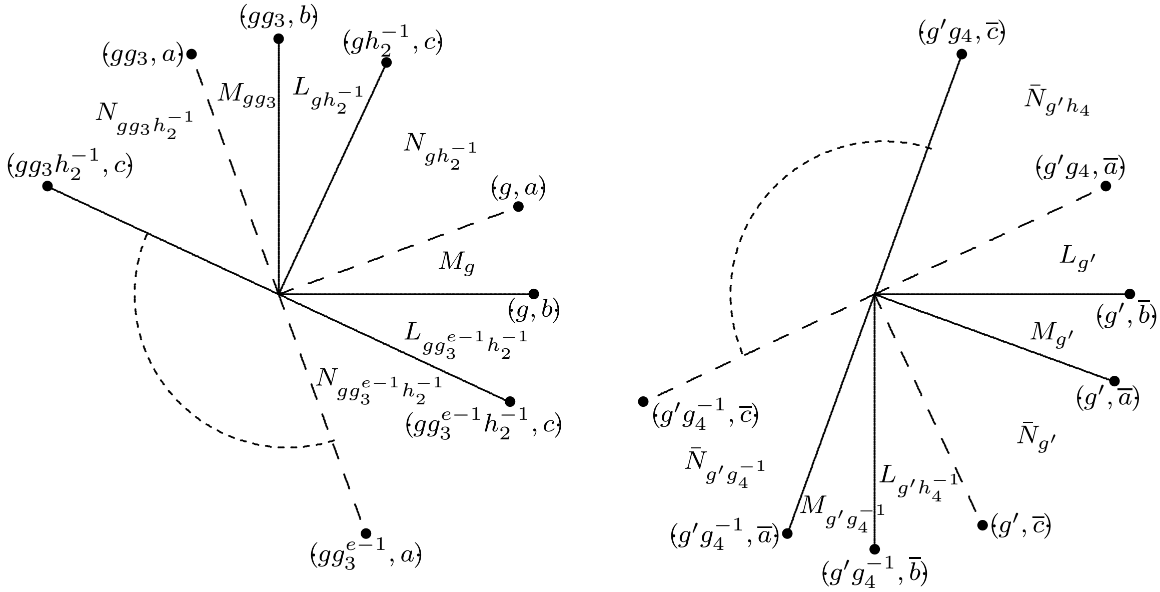
By (5), $p(\overline{(g,z)}) = \overline{z} = \overline{p(g,z)}$, so $(g,z) \mapsto \overline{(g,z)}$ restricts to $z \mapsto \overline{z}$ on $\mathbb{P}^1$. Let $x = (1, x_0)$; as $x_0 \in (c_0, d_0)$ and $\overline{x_0} = x_0$, we have $\overline{x} = x$.

Using Lemma 3.1 and the glueing instructions from Section 1, it is straightforward (although tedious) to verify that the map $(g,z) \mapsto \overline{(g,z)}$ is antianalytic.

For instance, if $s \ge 0$, then $c_3, c_4 \in M_2$, so $\overline{(g,c_3)} = (\overline{g}h_2, c_4)$. Put $g' = \overline{g}h_2$ and consider analytic neighbourhoods of (g,c_3) and (g', c_4) as described in Figure 5 of Section 1. Denote

$$L = L_2, M = M_2, N = S^- \smallsetminus L_2, \text{ and } \bar{N} = S^+ \smallsetminus M_2.$$

Let $h \in G$. As we go around (g,c_3) in the anti-clockwise direction, we pass from region M_h to $N_{hh_2^{-1}}$, from $N_{hh_2^{-1}}$ to $L_{hh_2^{-1}}$, and from there to $M_{hh_2^{-1}h_3} = M_{hg_3}$. As we go around (g', c_4) in the clockwise direction, we pass from region M_h to $\bar{N}_h$, from $\bar{N}_h$ to $L_{hh_4^{-1}}$, and from there to $M_{hh_4^{-1}h_3} = M_{hg_4^{-1}}$. So the neighbourhoods look as follows.



Neighbourhood of (g, c_3) Neighbourhood of (g', c_4)

Rule (5) says that in these neighbourhoods we have, by Lemma 3.1(b) and Lemma 3.1(c),

$$\begin{aligned}
\overline{M_g} &= M_{\overline{g}h_2} = M_{g'}, \\
\overline{N_{gh_2^{-1}}} &= \bar{N}_{\overline{gh_2^{-1}}} = \bar{N}_{\overline{g}h_2} = \bar{N}_{g'}, \\
\overline{L_{gh_2^{-1}}} &= L_{\overline{gh_2^{-1}}h_4^{-1}} = L_{\overline{g}h_2h_4^{-1}} = L_{g'h_4^{-1}}, \\
\overline{M_{gg_3}} &= M_{\overline{gg_3}h_2} = M_{\overline{g}h_2g_4^{-1}} = M_{g'g_4^{-1}}, \\
\overline{N_{gg_3h_2^{-1}}} &= \bar{N}_{\overline{gg_3h_2^{-1}}} = \bar{N}_{\overline{g}\,\overline{g_3}h_2} = \bar{N}_{\overline{g}h_2g_4^{-1}} = \bar{N}^{-1}_{g'g_4},
\end{aligned}$$

etc.

Thus if we identify these neighbourhoods with the unit disk around 0 via the appropriate charts, then $(g, z) \mapsto \overline{(g, z)}$ in this disk is the antianalytic map $z \mapsto \overline{z}$.

Finally, let $g \in G$. Then

$$ugu(1, x_0) = ug\overline{(1, x_0)} = ug(1, x_0) = u(g, x_0) = \overline{(g, x_0)} = (\overline{g}, x_0) = \overline{g}(1, x_0).$$

So both ugu and $\overline{g}$ coincide on $x = (1, x_0)$. Therefore $ugu = \overline{g}$. ∎

Definition 3.3: We call the above constructed involution u **the explicit real involution corresponding to the automorphism** $g \mapsto \overline{g}$. ∎

Remark 3.4: Let $p'\colon X' \to \mathbb{P}^1$ be a Galois cover of Riemann surfaces and let u' be a real involution of $X'/\mathbb{H}$. Then, up to an isomorphism, X' is an explicit cover $p\colon X \to \mathbb{P}^1$ and u' is an explicit real involution u.

Indeed, by definition, u' fixes a point $x' \in X'$ unramified over $\mathbb{P}^1$. As u' lifts the complex conjugation c of $\mathbb{P}^1$, the point $p'(x')$ lies on the equator of $\mathbb{P}^1$. Let T be a finite subset of $\mathbb{P}^1$ containing the branch points and avoiding $p'(x')$, with $r \geq 1$ real points and $s \geq 1$ pairs of complex conjugate points. We may choose the ordering of T and the path C so that $p'(x') \in (c_0, d_0)$. Let $x_0 = p'(x')$.

As we have already mentioned in Section 1, we may assume that X' is the explicit cover X corresponding to this choice.

Let $x = \langle 1, x_0 \rangle \in X$. Then $p'(x) = x_0 = p'(x')$, hence there is $g' \in G$ such that $x' = x^{g'}$. Replacing u' by $(u')^{g'}$, we may assume that $u'(x) = x$. So $u'u^{-1} \in \mathrm{Aut}(X/\mathbb{P}^1)$ and $u'u^{-1}(x) = x$. Therefore $u'u^{-1} = 1$. ∎

A point $\langle g, z \rangle \in X$ is **real**, if $\overline{\langle g, z \rangle} = \langle g, z \rangle$. The explicit definition (5) allows us to identify all real points:

COROLLARY 3.5: *Let $\langle g, z \rangle \in X$. Then $\langle g, z \rangle$ is real if and only if*

(a) *either $z = c_i$ and $\overline{g} h_i \langle g_i \rangle = g \langle g_i \rangle$, for $i = 2s+1, \ldots, 2s+r$,*

(b) *or $\langle g, z \rangle \in (c_i, c_{i+1})_g$ and $\overline{g} h_i = g$, for $i = 2s+1, \ldots, 2s+r-1$,*

(c) *or $\langle g, z \rangle \in (c_0, d_0)_g$ and $\overline{g} = g$,*

(d) *or $\langle g, z \rangle \in [d_{2j-2}, d_{2j-1}]_g$ and $\overline{g} h_{2j-2} = g$, for $j = 1, \ldots, s$,*

(e) *or $\langle g, z \rangle \in (d_{2j-1}, d_{2j})_g$ and $\overline{g} h_{2j}^{-1} = g$, for $j = 1, \ldots, s$,*

(f) *or $\langle g, z \rangle \in [d_{2s}, c_{2s+1})_g$ and $\overline{g} h_{2s} = g$.*

Proof: Only case $i = 2s+r = n$ in (a) needs clarification. As $c_n \in [c_{2s+r-1}, c_n]$, we have

$$\overline{\langle g, c_n \rangle} = \langle \overline{g} h_{2s+r-1}, c_n \rangle = \langle \overline{g} h_{2s+r-1} g_n, c_n \rangle = \langle \overline{g} h_n, c_n \rangle = \langle \overline{g}, c_n \rangle. \quad ∎$$

Remark 3.6: Let E be a subgroup of G. Consider the cover $\bar{p}\colon E\backslash X \to \mathbb{P}^1$ induced from $p\colon X \to \mathbb{P}^1$. The real involution u of X induces an automorphism of $E\backslash X$ if and only if $\overline{E} = E^u = E$. Assume this is the case. It is easy to see that Corollary 3.5 holds also for $E\backslash X$ instead of X, if we replace everywhere g by Eg. That is, $(Eg, z) \in E\backslash X$ is real (i.e. u-invariant) if and only if

(a) either $z = c_i$ and $E\overline{g}h_i\langle g_i\rangle = Eg\langle g_i\rangle$, for $i = 2s+1, \ldots, 2s+r$,

(b) or $(Eg, z) \in (c_i, c_{i+1})_{Eg}$ and $E\overline{g}h_i = Eg$, for $i = 2s+1, \ldots, 2s+r-1$,

etc.

4. Real points and segments

The main goal of this section is a formula (Theorem 4.10) for the number of the connected components of the boundary of a Klein surface which is a ramified Galois cover of $\mathbb{H}$. This formula looks similar to the one obtained by Gromadzki in [Gro] in terms of signature of NEC groups*. Since Gromadzki's formula relies on the representation of H as an NEC group, it is not clear how to relate both.

Our result is framed in group theoretic terms. To this end we introduce the notion of real segments which represent subsets of the boundary and show how to glue them together. Retain the notation from the preceding sections. In particular, let $p\colon X \to \mathbb{P}^1$ be the explicitly constructed Galois cover of Riemann surfaces corresponding to the branch data $(G, c_1, g_1, \dots, c_n, g_n)$ with respect to a path C satisfying condition (1) of Section 3. Let f be the canonical map from $\Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$ to $\mathrm{Aut}(X/\mathbb{P}^1)$. Assume that there is a group automorphism $g \mapsto \overline{g}$ of G such that $\overline{f(\gamma)} = f(\overline{\gamma})$ for every $\gamma \in \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$. Let $u\colon X \to X$ be the corresponding explicit real involution, given by Proposition 3.2.

We fix the analytic atlas for X introduced in Construction 1.1, Parts E,F.

A connected subset C' of X is **smooth at a point** $z \in C'$, if it is locally an arc at z, i.e., there is an open neighbourhood V of z contained in some chart of the atlas such that the restriction of the chart map $V \to \mathbb{P}^1$ maps $C' \cap V$ into a circle on $\mathbb{P}^1$. We say that C' is **smooth**, if it is smooth at every $z \in C'$.

EXAMPLE 4.1: *Let $2s + 1 \le i \le 2s + r - 1$. The paths $(c_i, c_{i+1})_g$, for $g \in G$, on X are the liftings of the path (c_i, c_{i+1}) on $\mathbb{P}^1$. They are smooth.*

Proof: Chart $\hat{S}_g^+ \to \mathbb{P}^1$ maps $(c_i, c_{i+1})_g$ onto the arc (c_i, c_{i+1}) on $\mathbb{P}^1$. ∎

LEMMA 4.2: *Let $g \in G$. Let $C(g)$ be the disjoint union of the following $2s + 2$ subsets*

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* Gromadzki's formula expresses the number of connected components of the boundary of a Klein surface S by means of its total group of automorphisms Λ_S, (more precisely in terms of its representation as an NEC group). It seems that the formula still holds for any group of automorphisms of S, $\Lambda' \subset \Lambda_S$.

of X

$$(c_0,d_0)_g, [d_0,d_1]_g, (d_1,d_2)_{gh_1^{-1}}, [d_2,d_3]_{gg_2}, (d_3,d_4)_{gg_2h_3^{-1}}, [d_4,d_5]_{gg_2g_4}, \dots \tag{1}$$
$$\dots, [d_{2s-2},d_{2s-1}]_{gg_2\cdots g_{2s-2}}, (d_{2s-1},d_{2s})_{gg_2\cdots g_{2s-2}h_{2s-1}^{-1}}, [d_{2s},c_{2s+1})_{gg_2\cdots g_{2s-2}g_{2s}}$$

Then $C(g)$ lifts the path (c_0,c_{2s+1}), which lies on the equator of $\mathbb{P}^1$. This lifting is smooth. As g ranges over all elements of G, the sets $C(g)$ are all the smooth liftings of (c_0,c_{2s+1}) to X.

Proof: The theory of lifting of paths to unramified coverings shows that (c_0,c_{2s+1}) has exactly $|G|$ disjoint liftings to a connected set on the unramified covering $X' = X \setminus \{(g,c_i) \mid i \in I,\ g \in G\}$ of $\mathbb{P}^1 \setminus \{c_i \mid i \in I\}$. Clearly $C(g)$ lifts (c_0,c_{2s+1}), for each $g \in G$. Each of the subsets in (1) is connected and smooth at each point, except, perhaps, at the points

$$(g,d_0), (g,d_1), (gg_2,d_2), \dots, (gg_2g_4\cdots g_{2s},d_{2s}).$$

We show that $C(g)$ is smooth also at these points and connected. E.g., for $\varepsilon > 0$ small enough,

$$(d_1-\varepsilon,d_1)_g \cup [d_1,d_2)_{gh_1^{-1}} \subseteq (U_1^+)_g \cup S^-_{gh_1^{-1}} \subseteq \hat{S}^-_{gh_1^{-1}}$$

(see Construction 1.1, Part E) and, using the chart $p\colon \hat{S}^-_{gh_1^{-1}} \to \mathbb{P}^1$, which maps the set on the left handed side onto the arc $(d_1-\varepsilon,d_1) \cup [d_1,d_2) = (d_1-\varepsilon,d_2)$, we see that $[d_1-\varepsilon,d_1)_g$ is in the connected component of $(d_1,d_2]_{gh_1^{-1}}$ and $C(g)$ is smooth at (g,d_1). ∎

To allow a uniform formulation of certain assertions, we introduce the following notation $C(2s,g), C(2s+1,g), \dots, C(2s+r,g)$. First denote

$$w = g_2g_4\cdots g_{2s}. \tag{2}$$

(3a) Put $C(i,g) = (c_i,c_{i+1})_g$ for $i = 2s+1,\dots,2s+r-1$; let $C(2s+r,g)$ be the union of the first $2s+1$ sets in (1), that is, all of them, except the last one; and let $C(2s,g) = (d_{2s},c_{2s+1})_g$. Call the sets $C(2s,g),\dots,C(2s+r,g)$ **segments**.

(3b) For $i = 2s+1, \ldots, 2s+r$ let (g, c_i) be the **left endpoint** of $C(i, g)$; let (g, d_{2s}) be the left endpoint of $C(2s, g)$.

(3c) For $i = 2s, \ldots, 2s+r-1$ let (g, c_{i+1}) be the **right endpoint** of $C(i, g)$; let (gw, d_{2s}) be the right endpoint of $C(2s+r, g)$.

Obviously, the endpoints of $C(i, g)$ belong to its closure in X, for each i. Distinct segments are disjoint, but they may have a common endpoint. By Example 4.1 and Lemma 4.2, segments are smooth. In fact, by Lemma 4.2, $C(2s+r, g) \cup \{(gw, d_{2s})\} \cup C(2s, gw)$ is smooth; here (gw, d_{2s}) is the common endpoint of the two segments.

With a_i, b_i defined as in Construction 1.1, Part F(iii) we have:

Remark 4.3: Let C' be a segment. Let $2s+1 \le i \le 2s+r$ and $g \in G$.

(a) $(c_i, b_i)_g \cap C' \neq \emptyset \Leftrightarrow (c_i, b_i)_g \subseteq C' \Leftrightarrow C' = C(i, g)$.

(b) $(a_i, c_i)_g \cap C' \neq \emptyset \Leftrightarrow (a_i, c_i)_g \subseteq C' \Leftrightarrow C' = C(i-1, g)$. ∎

A segment is **real** if all its points are real.

LEMMA 4.4: *The following assertions about a segment $C(i, g)$ are equivalent:*

(a) *$C(i, g)$ is real;*

(b) *$C(i, g)$ contains a real point;*

(c) $\bar{g} h_i = g$.

Moreover,

(d) *$C(2s+r, g)$ is real $\Leftrightarrow$ (gw, d_{r+2s}) is real $\Leftrightarrow$ $C(2s, gw)$ is real.*

Proof: Implication (a) $\Rightarrow$ (b) is trivial. For $2s < i < 2s+r$, (b) $\Rightarrow$ (c) $\Rightarrow$ (a) follows from Corollary 3.5(b). For $i = 2s$, (b) $\Rightarrow$ (c) $\Rightarrow$ (a) follows from Corollary 3.5(f).

(d) By Corollary 3.5(c)-(f) we have to show that the following conditions on $g \in G$ are equivalent:

$$\bar{g} = g,\ \bar{g} h_0 = g,\ \overline{g h_1^{-1}} h_2^{-1} = g h_1^{-1},\ \overline{g g_2} h_2 = g g_2,\ \overline{g g_2 h_3^{-1}} h_4^{-1} = g g_2 h_3^{-1},$$
$$\overline{g g_2 g_4} h_4 = g g_2 g_4,\ \ldots\ , \overline{g g_2 g_4 \cdots g_{2s}} h_{2s} = g g_2 g_4 \cdots g_{2s}.$$

Using Lemma 3.1, each of them is obtained from the preceding one by multiplication from the right with

$$h_0 = 1, h_1^{-1}, h_2, h_3^{-1}, h_4, \ldots, h_{2s-1}^{-1}, h_{2s},$$

respectively. (It follows from Lemma 3.1(b) that $h_{2j}^{-1} = \overline{h_{2j}} = \overline{h_{2j-1}}\ \overline{g_{2j}}$, hence $\overline{h_{2j-1}^{-1}} = \overline{g_{2j}} h_{2j}$.) Therefore they are equivalent. ∎

It follows from Corollary 3.5 that the set of real points on X consists of segments, points (g, d_{2s}), and points (g, c_i), for $2s+1 \le i \le 2s+r$, though not necessarily all of them. We therefore investigate the neighbourhoods $D_{i,g\langle g_i\rangle}$ (Construction 1.1, Part F) of the latter points.

LEMMA 4.5: *Let $2s+1 \le i \le 2s+r$ and let $g \in G$. Let R be the set of real points in $D_{i,g\langle g_i\rangle}$. Then $R \ne \emptyset$ if and only if $(g, c_i) \in R$. If $R \ne \emptyset$, then R is smooth: the chart map $\psi_i\colon D_{i,g\langle g_i\rangle} \to D_i$ maps R onto a diameter of D_i. More precisely, let e_i be the order of g_i in G. Then*

(a) *If $e_i = 2m+1$ is odd, then $R = (a_i, c_i)_{gg_i^{k+m}} \cup \{(g, c_i)\} \cup (c_i, b_i)_{gg_i^k}$ for some k, unique modulo e_i;*

(b) *If $e_i = 2m$ is even, then exactly one of the following holds: either*

  (b1) $R = (c_i, b_i)_{gg_i^{k+m}} \cup \{(g, c_i)\} \cup (c_i, b_i)_{gg_i^k}$, for some k, unique modulo e_i, or

  (b2) $R = (a_i, c_i)_{gg_i^{k+m}} \cup \{(g, c_i)\} \cup (a_i, c_i)_{gg_i^k}$, for some k, unique modulo e_i.

Proof: First observe that for every integer k we have

$$\begin{aligned} \overline{gg_i^k} h_i = gg_i^k &\Leftrightarrow \overline{g} h_i = gg_i^{2k}, \\ \overline{gg_i^k} h_{i-1} = gg_i^k &\Leftrightarrow \overline{g} h_i = gg_i^{2k+1}. \end{aligned} \tag{4}$$

Indeed, by Lemma 3.1(e), $\overline{g_i^k} h_i = \overline{g_i}^k h_i = h_i g_i^{-k}$ and we also have $h_i = h_{i-1} g_i$. This gives (4).

As $D_i \cap (\mathbb{R} \cup \{\infty\}) = (a_i, b_i) = (a_i, c_i) \cup \{c_i\} \cup (c_i, b_i)$, by Corollary 3.5 the real points in $D_{i,g\langle g_i\rangle}$ are contained in $\{(g, c_i)\} \cup \bigcup_{h\in G} (a_i, c_i)_h \cup \bigcup_{h \in G} (c_i, b_i)_h$. Since $(a_i, c_i), (c_i, b_i) \subseteq S^+ \cap D_i = D_i^+$, by the definition of $D_{i,g\langle g_i\rangle}$ we have

$$\begin{aligned} (c_i, b_i)_h \cap D_{i,g\langle g_i\rangle} \ne \emptyset \Leftrightarrow (c_i, b_i)_h \subseteq D_{i,g\langle g_i\rangle} &\Leftrightarrow h = gg_i^k \text{ for some } k \in \mathbb{Z}/e_i\mathbb{Z}, \\ (a_i, c_i)_h \cap D_{i,g\langle g_i\rangle} \ne \emptyset \Leftrightarrow (a_i, c_i)_h \subseteq D_{i,g\langle g_i\rangle} &\Leftrightarrow h = gg_i^{k'} \text{ for some } k' \in \mathbb{Z}/e_i\mathbb{Z}. \end{aligned} \tag{5}$$

By Corollary 3.5 there is a real point in $(c_i, b_i)_h$ if and only if $\overline{h} h_i = h$ and, if so, then every point of $(c_i, b_i)_h$ is real. Therefore by (5) and (4),

(i) $(c_i, b_i)_h \cap R \neq \emptyset \Leftrightarrow (c_i, b_i)_h \subseteq R \Leftrightarrow h = gg_i^k$, where $\overline{g}h_i = gg_i^{2k}$.

Similarly, since $(a_i, c_i) \subseteq (c_{i-1}, c_i)$ for $i > 2s+1$ and $(a_{2s+1}, c_{2s+1}) \subseteq (d_{2s}, c_{2s+1})$, by (4) and (4),

(ii) $(a_i, c_i)_h \cap R \neq \emptyset \Leftrightarrow (a_i, c_i)_h \subseteq R \Leftrightarrow h = gg_i^{k'}$, where $\overline{g}h_i = gg_i^{2k'+1}$.

Finally, by Corollary 3.5(a),

(iii) $(g, c_i) \in R \Leftrightarrow \overline{g}h_i = gg_i^\ell$ for some $\ell \in \mathbb{Z}/e_i\mathbb{Z}$.

It follows that if $(g, c_i) \notin R$ then certainly $R = \emptyset$.

Assume that $(g, c_i) \in R$ and let ℓ be such that $\overline{g}h_i = gg_i^\ell$. Then (i) is equivalent to

(i′) $h = gg_i^k$, where $2k \equiv \ell \pmod{e_i}$

and (ii) is equivalent to

(ii′) $h = gg_i^{k'}$, where $2k' + 1 \equiv \ell \pmod{e_i}$.

We divide the rest of the proof into two cases:

(a) Assume that e_i is odd, $e_i = 2m+1$. Then the congruences in (i′) and (ii′) have unique solutions k, k' modulo e_i; these solutions satisfy $k' \equiv k + m \pmod{e_i}$. This gives the asserted description of R.

(b) Assume that e_i is even, $e_i = 2m$. If ℓ is even, the congruence in (i′) has two solutions $k = \frac{\ell}{2}$ and $k = \frac{\ell}{2} + m$, and the congruence in (ii′) has no solution. This gives the asserted description (b1) of R. If ℓ is odd, the congruence in (i′) has no solution and the congruence in (ii′) has two solutions $k' = \frac{\ell-1}{2}$ and $k' = \frac{\ell-1}{2} + m$. This gives the asserted description (b2) of R.

Finally, descriptions (a) and (b) of R also show that R is smooth, since in Figure 5 of Section 1 these sets are in all cases diameters of the disk $D_{i,g\langle g_i\rangle}$. (In our setup, the angle α between (a_i, c_i) and (c_i, b_i) is π.) ∎

Lemma 4.5 and Lemma 4.4(d) say that a real segment has real endpoints and a real endpoint of a segment is the endpoint of a real segment. Moreover, by Corollary 3.5, real points other than endpoints lie on real segment. We may summarize this as follows:

Corollary 4.6: *The set of real points on X consists of (some) segments, each together with its endpoints.*

The next result specifies how the real segments are connected among themselves.

Notation 4.7: For each $2s+1 \le i \le 2s+r$ let e_i be the order of g_i in G and put $v_i = g_i^{\lfloor \frac{e_i}{2} \rfloor}$, where $\lfloor \frac{e_i}{2} \rfloor$ is the integer part of $\frac{e_i}{2}$. Also, let $e_{2s} = 1$ (so e_{2s} is odd) and put $v_{2s} = w^{-1}$. Extend the definition of $C(i,g), e_i, v_i$ from $i \in \{2s, \dots, 2s+r\}$ to all $i \in \mathbb{Z}$ by defining these entities modulo $r+1$. $\blacksquare$

PROPOSITION 4.8: *Let $C(i,g)$ be a real segment and let y be its endpoint. Then there exists a unique real segment $C' \ne C(i,g)$ which has y as one of its endpoints:*

(a) *If y is the left endpoint of $C(i,g)$ and e_i is even, then y is the left endpoint of C' and $C' = C(i, gv_i)$.*

(b) *If y is the left endpoint of $C(i,g)$ and e_i is odd, then y is the right endpoint of C' and $C' = C(i-1, gv_i)$.*

(c) *If y is the right endpoint of $C(i,g)$ and e_{i+1} is even, then y is the right endpoint of C' and $C' = C(i, gv_{i+1})$.*

(d) *If y is the right endpoint of $C(i,g)$ and e_{i+1} is odd, then y is the left endpoint of C' and $C' = C(i+1, gv_{i+1}^{-1})$.*

Moreover, the union $C' \cup \{y\} \cup C(i,g)$ is real and smooth.

Proof: We may assume that $2s \le i \le 2s+r$.

(a) We have $i \ne 2s$ and $y = (g, c_i)$. By assumption, $(c_i, b_i)_g \subseteq C(i,g)$ is real. Also, $(c_i, b_i)_g$ is contained in $D_{i,g\langle g_i \rangle}$ and in the set R of real points in it. By Lemma 4.5, $R = (c_i, b_i)_{gv_i} \cup \{y\} \cup (c_i, b_i)_g$. So a segment $C' \ne C(i,g)$ has y as endpoint and is real if and only if $(c_i, b_i)_{gv_i} \subseteq C'$. The existence, uniqueness and the explicit description of C' follow from Remark 4.3. The union $C' \cup \{y\} \cup C(i,g)$ is smooth at each point except y, since segments are smooth. It is smooth also at y, since its intersection with $D_{i,g\langle g_i \rangle}$ is the diameter R.

(b) If $i = 2s$, then $y = (g, d_{2s})$, and $C(2s, g), C(2s+r, gw^{-1}) = C(i-1, gv_i)$ are the only segments with endpoint y. So the assertion follows from Lemma 4.4(d). The smoothness follows from Lemma 4.2.

Assume $i \ne 2s$. Then, as in (a), $y = (g, c_i)$ and $(c_i, b_i)_g$ is contained in the set R of real points in $D_{i,g\langle g_i \rangle}$. By Lemma 4.5, $R = (a_i, c_i)_{gv_i} \cup \{y\} \cup (c_i, b_i)_g$. So a segment

$C' \neq C(i,g)$ has y as endpoint and is real if and only if $(a_i, c_i)_{gv_i} \subseteq C'$. The existence, uniqueness and the explicit description of C' follow from Remark 4.3. The smoothness follows as in (a).

(c) We have $i \neq 2s+r$ and $y = \langle g, c_j \rangle$, where $j = i+1$ with $2s+1 \leq j \leq 2s+r$. By Remark 4.3, $(a_j, c_j)_g \subseteq C(i,g)$. So, by assumption, $(a_j, c_j)_g$ is real. Also, $(a_j, c_j)_g$ is contained in $D_{j,g\langle g_j \rangle}$ and hence in the set R of real points in it. By Lemma 4.5, $R = (a_j, c_j)_{gv_j} \cup \{y\} \cup (a_j, c_j)_g$. So, again, the existence, uniqueness and the explicit description of C' follow from Remark 4.3. The smoothness follows as in (a).

(d) If $i = 2s+r$, then $y = \langle gw, d_{2s} \rangle$, and $C(2s+r, g), C(2s, gw) = C(2s+r+1, gv_{2s+r+1}^{-1})$ are the only segments with endpoint y. So the assertion follows from Lemma 4.4(d). The smoothness follows from Lemma 4.2.

Assume $i \neq 2s+r$. Then, as in (c), $y = \langle g, c_j \rangle$ and $(a_j, c_j)_g$ is contained in the set R of real points in $D_{j,g\langle g_j \rangle}$. By Lemma 4.5, $R = (a_j, c_j)_g \cup \{y\} \cup (c_j, b_j)_{gv_j^{-1}}$. So, again, the existence, uniqueness and the explicit description of C' follow from Remark 4.3. The smoothness follows as in (a). ∎

Proposition 4.9: *The set of real points on X is the disjoint union of finitely many smooth sets (ovals). Each of them consists of finitely many segments, including their endpoints. More precisely:*

(a) *Suppose $e_{2s+1}, \ldots, e_{2s+r}$ are odd. Let $v = v_{2s+r} v_{2s+r-1} \cdots v_{2s}$ and let $\ell = \mathrm{ord}(v)$ be its order in G. Then for each $g \in G$ with $\overline{g} = g$ there exists a real oval consisting of the following segments and their endpoints:*

$$C(2s+r, g), C(2s+r-1, gv_{2s+r}), C(2s+r-2, gv_{2s+r}v_{2s+r-1}), \ldots$$
$$\ldots, C(2s, gv_{2s+r} \cdots v_{2s+1}),$$
$$C(2s+r, gv), C(2s+r-1, gvv_{2s+r}), C(2s+r-2, gvv_{2s+r}v_{2s+r-1}), \ldots$$
$$\ldots, C(2s, gvv_{2s+r} \cdots v_{2s+1}),$$
$$\ldots$$
$$C(2s+r, gv^{\ell-1}), C(2s+r-1, gv^{\ell-1}v_{2s+r}), C(2s+r-2, gv^{\ell-1}v_{2s+r}v_{2s+r-1}), \ldots$$
$$\ldots, C(2s, gv^{\ell-1}v_{2s+r} \cdots v_{2s+1}),$$

$$C(2s+r, gv^\ell) = C(2s+r, g)$$

and every real oval is of this form.

(b) *Suppose not all e_i are odd. Let $i < k$ be integers such that e_i, e_k are even and $e_{i+1}, e_{i+2}, \ldots, e_{k-1}$ are odd. Let $v = v_{i+1}^{-1} v_{i+2}^{-1} \cdots v_{k-1}^{-1} v_k v_{k-1} \cdots v_{i+1} v_i$ and let $\ell = \operatorname{ord}(v)$ be its order in G. Then for each $g \in G$ with $\overline{g} h_i = g$ there exists a real oval consisting of the following segments and their endpoints:*

$$C(i,g), C(i+1, gv_{i+1}^{-1}), \ldots, C(k-1, gv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}),$$
$$C(k-1, gv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k), C(k-2, gv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k v_{k-1}), \ldots$$
$$\ldots, C(i, gv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k v_{k-1}\cdots v_{i+1}),$$
$$C(i,gv), C(i+1, gvv_{i+1}^{-1}), \ldots, C(k-1, gvv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}),$$
$$C(k-1, gvv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k), C(k-2, gvv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k v_{k-1}), \ldots$$
$$\ldots, C(i, gvv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k v_{k-1}\cdots v_{i+1}),$$
$$\ldots$$
$$C(i,gv^{\ell-1}), C(i+1, gv^{\ell-1}v_{i+1}^{-1}), \ldots, C(k-1, gv^{\ell-1}v_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}),$$
$$C(k-1, gv^{\ell-1}v_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k), C(k-2, gv^{\ell-1}v_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k v_{k-1}), \ldots$$
$$\ldots, C(i, gv^{\ell-1}v_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k v_{k-1}\cdots v_{i+1}),$$
$$C(i, gv^\ell) = C(i,g)$$

and every real oval is of this form.

Proof: Let R be a connected component of the set of real points on X. By Corollary 4.6 it contains a real segment $C(i,g)$; choose it. If e_i is odd, then, by Proposition 4.8(b), R contains also $C(i-1, gv_i)$. So, by induction, we may assume that either $i = 2s$ or e_i is even.

(a) Each segment in our list is the unique real segment connected, by Proposition 4.8(b), to its predecessor at its right endpoint (which is the left endpoint of the predecessor). The loop continues until $C(i,g)$ appears again.

(b) Starting with $C(i,g)$, using Proposition 4.8(d), the assertion successively lists the real segments whose left endpoint is the right endpoint of their predecessor, until the first segment $C(k-1, gv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1})$ with right endpoint with even ramification index e_k occurs. (The latter segment can be $C(i,g)$ itself; this happens if e_{i+1} is even. If e_i is the only even number among $e_{2s},\ldots,e_{2s+r}$, then $k = i+(r+1)$.) By Proposition 4.8(c), $C(k-1, gv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k)$ is the real segment with the same right endpoint as the preceding segment. Following it, using Proposition 4.8(b), the assertion then lists the real segments whose right endpoint is the left endpoint of their predecessor, until the first segment $C(i, gv_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_kv_{k-1}\cdots v_{i+1})$ with left endpoint with even ramification index e_i occurs. Using Proposition 4.8(a), this endpoint is also the left endpoint of $C(i,gv)$. From this point on we repeat the whole process, until $C(i,g)$ occurs again in the list. ∎

Let $H = \mathrm{Aut}(X/\mathbb{H})$ and let $u \in H$ be the real involution which satisfies $ugu = \overline{g}$ for every $g \in G$ (Proposition 3.2). Let $2s+1 \le i \le 2s+r$. By Lemma 3.1(a), $u_i = uh_i \in H$ is an involution. So condition $\overline{g}h_i = g$ (which, for $i = 2s+r$, reads $\overline{g} = g$) can be written as $uguh_i = g$, that is, $g^{-1}ug = u_i$. If $C_G(u) = \{g \in G \mid g^{-1}ug = g\}$, then the centralizer $C_H(u)$ of u in H is $C_H(u) = C_G(u) \cup uC_G(u)$. In particular, $|C_H(u)| = 2|C_G(u)|$.

THEOREM 4.10: *The number of real ovals (connected components of the set of real points on X) is*

(a) $\frac{|C_H(u)|}{2\mathrm{ord}(v_{2s+r}v_{2s+r-1}\cdots v_{2s})}$, *if $e_{2s+1},\ldots,e_{2s+r}$ are odd;*

(b) $\sum_i \frac{|C_H(u)|}{4\mathrm{ord}(v_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_kv_{k-1}\cdots v_{i+2}v_{i+1}v_i)}$, *otherwise. Here the sum runs through all $i \in \{2s,\ldots,2s+r\}$ with e_i even and u_i conjugate to u in H and, for each such i, k is the smallest integer such that $i < k$ and e_k is even.*

Proof: Let $2s \le i \le 2s+r$. By Lemma 4.4, the number of real segments of the form $C(i,g)$ is the cardinality of the set

$$N_i = \{g \in G \mid \overline{g}h_i = g\} = \{g \in G \mid u^{-1}guh_i = g\} = \{g \in G \mid g^{-1}ug = uh_i\}.$$

So if $u_i = uh_i$ is not conjugate to u in H, then $N_i = \emptyset$. If there is $h \in H$ such that $h^{-1}uh = u_i$, then without loss of generality $h \in G$ (otherwise replace h by uh) and $N_i = (G \cap C_H(u))h$. Thus $|N_i| = |G \cap C_H(u)| = \frac{1}{2}|C_H(u)|$ in this case.

(a) By Proposition 4.9(a), each connected component of the real part of X contains exactly $\ell = \mathrm{ord}(v_{2s+r}v_{2s+r-1}\cdots v_{2s})$ segments of the form $C(2s+r,g)$. So the number of components is $|N_{2s+r}|/\ell$.

(b) By Proposition 4.9(b), for each connected component R' of the real part of X there is a unique $2s \le i \le 2s+r$ with e_i even and $C(i,g) \in R'$. Moreover, R' contains exactly $2\ell_i$ such components, where $\ell_i = \mathrm{ord}(v_{i+1}^{-1}v_{i+2}^{-1}\cdots v_{k-1}^{-1}v_k v_{k-1}\cdots v_{i+2}v_{i+1}v_i)$. So the formula follows by the first paragraph of this proof. ∎

Corollary 4.11: *If $r = 0$, the number of real ovals is $\frac{|C_H(u)|}{2\mathrm{ord}(v_{2s})}$.*

Proof: Put $T' = T \cup \{c_{2s+1}\}$, where c_{2s+1} is real. The appropriate loop γ_{2s+1} around it is homotopic to 1, so $g_{2s+1} = f(\gamma_{2s+1}) = 1$. Hence $e_{2s+1} = 1$ and $v_{2s+1} = 1$. The formula follows from Theorem 4.10(a). ∎

Remark 4.12: For each $1 \le i \le r$ let $u_i = uh_i$. By Lemma 3.1(a), u_i is an element of order 2 in H. Moreover, u_i is a *real involution*, i.e., it has fixed points. In fact, $u_r = u$ is real by Proposition 3.2 and if $i < r$ then u_i fixes $\{1,z\}$ for every $z \in [c_j, c_{j+1}]$, where $j = i + 2s$. Indeed, $u_i(\{1,z\}) = uh_j(\{1,z\}) = u(\{h_j,z\}) = \overline{\{h_j,z\}}$. Now use the first case in the definition (5) in Proposition 3.2 and Lemma 3.1(a): $\overline{\{h_j,z\}} = \{\overline{h_j}h_j, z\} = \{h_j^{-1}h_j, z\} = \{1,z\}$. ∎

We can characterize geometrically the conjugacy classes of antianalytic involutions. Keeping the notation of Remark 4.12, we have

Corollary 4.13: *Let $\eta \in H$ be an antianalytic (i.e. $\eta \notin G$) involution. Then η has a fixed point lying over the arc $[c_0, c_{2s+1}]$, resp. $[c_{2s+i}, c_{2s+i+1}]$, if and only if η is conjugate to $u = u_r$, resp. to u_i, for $i = 1,\dots,r-1$. In particular, every real involution is conjugate to some u_i, for $1 \le i \le r$.*

Proof: Write $\eta = u\sigma$, where $\sigma \in G$. Then $\eta\{g,z\} = u\{\sigma g, z\} = \overline{\{\sigma g, z\}}$. By Proposition 3.2, $\overline{\{\sigma g, z\}} = \{\overline{\sigma g}h_k, \overline{z}\}$, where $k \in \{0,2,\dots,2s,2s+1,\dots,2s+r\}$ depends on z, as described there. Hence $\eta\{g,z\} = \{u\sigma g u h_k, \overline{z}\} = \{\eta g u h_k, \overline{z}\}$, whence $\eta\{g,z\} = \{g,z\} \Leftrightarrow \eta g u h_k = g,\ \overline{z} = z \Leftrightarrow \eta = g(uh_k)^{-1}g^{-1},\ \overline{z} = z$. Thus a fixed point of η must lie over one of the above mentioned arcs, and η has a fixed point over z in such an arc if and only if η is conjugate to $(uh_k)^{-1}$, for the appropriate k determined by z.

By Lemma 3.1, $uh_k u = \overline{h_k} = h_k^{-1}$. Hence uh_k is an involution.

If $z \in [c_{2s+i}, c_{2s+i+1}]$, then $k = i$, by Proposition 3.2, and $uh_k = u_i$.

If $z \in [c_0, c_{2s+1}]$, then $k = 2j$ for some $0 \leq j \leq s$. It therefore suffices to show that uh_{2j} is conjugate to u. This follows by induction on j. For $j = 0$ we have $uh_{2j} = u$. For $j \leq 1$ we have, by Lemma 3.1(d),

$$g_{2j}(uh_{2j})g_{2j}^{-1} = u(ug_{2j}u)(h_{2j}g_{2j}^{-1}) = u\overline{g_{2j}}h_{2j-1} = uh_{2j-2}h_{2j-1}^{-1}h_{2j-1} = uh_{2j-2}. \qquad \blacksquare$$

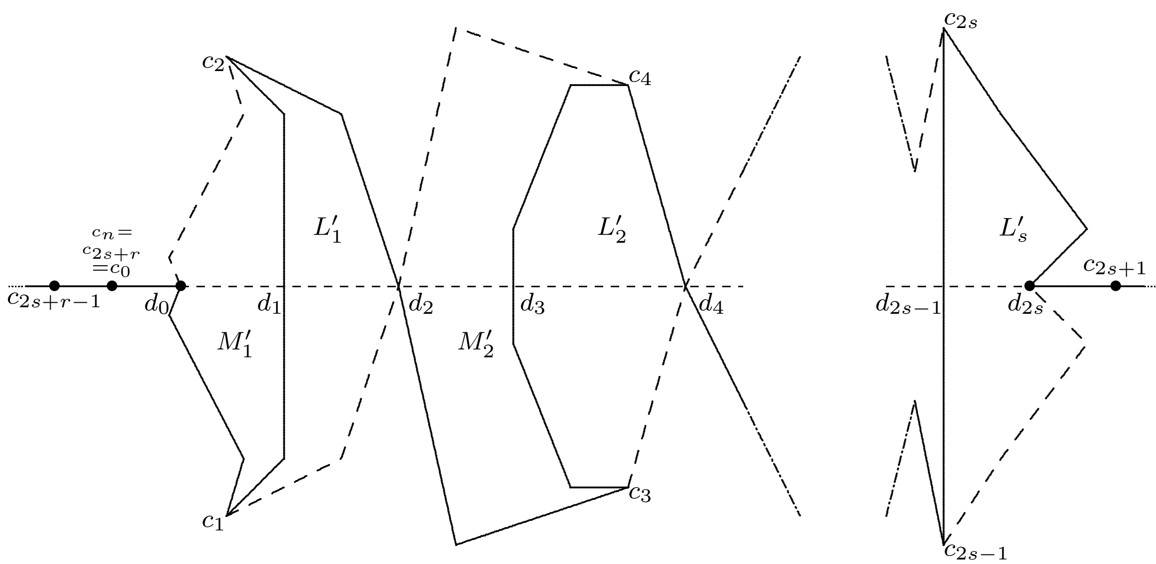
Remark 4.14: Let E be a subgroup of G such that $\overline{E} = E^u = E$. As in Remark 3.6 consider the cover $\bar{p}$: $E\backslash X \to \mathbb{P}^1$ instead of p: $X \to \mathbb{P}^1$. Then Proposition 4.9 still holds, if we replace g by Eg everywhere (including in the definition of $C(i, g)$) and replace ℓ by the smallest positive integer such that $Egv = Eg$, that is, the order of $\langle v \rangle / (\langle v \rangle \cap E^g)$. From this an analog of the formulae in Theorem 4.10 and Corollary 4.11 can be deduced, however, it is more complicated, since ℓ now depends on g.

5. Orientability of Klein surfaces

Retain the notation from the preceding section. So $p\colon X \to \mathbb{P}^1$ is a Galois cover of Riemann surfaces, unramified outside a finite set T closed under the complex conjugation, such that $X/\mathbb{H}$ is a Galois cover of Klein surfaces; furthermore, $u \in H = \mathrm{Aut}(X/\mathbb{H})$ is a real involution. Assume that T has $r \geq 0$ real points and $s \geq 0$ pairs of conjugate complex points, with $n = 2r + s$, and $p\colon X \to \mathbb{P}^1$ corresponds to the branch data $(G, c_1, g_1, \ldots, c_n, g_n)$. To start with, assume that $r \geq 1$.

Let R be the set of real points on X. Recall [BEGG, Proposition A.28] that the Klein surface $X/\langle u\rangle$ is orientable if and only if $X \smallsetminus R$ is not connected. This is equivalent to $X \smallsetminus (R \cup p^{-1}(T))$ being not connected, since $p^{-1}(T)$ consists of finitely many points with neighbourhoods homeomorphic to open disks.

In this section we give an algorithm to decide the orientability of $X/\langle u\rangle$ in terms of the group G and the action of the involution u on it (that is, in terms of H) by means of the connectedness of a graph associated to this data. The proof of the result (Theorem 5.3) is based in the following construction.



For each $1 \leq j \leq s$ let L'_j be that part of L_j which is above the equator, that is, above the arc (d_{2j-1}, d_{2j}) and including it, and let M'_j be that part of M_j which is under the equator, that is, under the arc (d_{2j-2}, d_{2j-1}), not including it. Put

$$S'^{-} = S^{-} \setminus \bigcup_{j=1}^{s} L'_j, \qquad S'^{+} = S^{+} \setminus \bigcup_{j=1}^{s} M'_j.$$

Then $X \setminus (R \cup p^{-1}(T))$ consists of the connected regions $S'^{+}_g, S'^{-}_g, (L'_j)_g, (M'_j)_g$, for $g \in G$ and $1 \le j \le s$. Moreover, these regions are connected as follows:

(1a) S'^{+}_g is always connected to $(L'_j)_{gh_{2j-1}^{-1}}$, through $(d_{2j-1}, c_{2j})_g$, for $1 \le j \le s$;

(1a$'$) S'^{-}_g is always connected to $(M'_j)_{gh_{2j-1}}$, through $(c_{2j-1}, d_{2j-1})_{gh_{2j-1}}$ for $1 \le j \le s$;

(1b) S'^{+}_g is connected to $S'^{-}_{gh_i^{-1}}$, through $(c_i, c_{i+1})_g$, if this arc is not real, that is, if $\overline{g}h_i \ne g$, for $i = 2s+1, \ldots, 2s+r-1$; also, S'^{+}_g is connected to $S'^{-}_g = S'^{-}_{gh_{r+2s}^{-1}}$, through $(c_0, d_0)_g$, if this arc is not real, that is, if $g \ne \overline{g} = \overline{g}h_{r+2s}$;

(1c) S'^{+}_g is connected to $S'^{-}_{gh_{2s}^{-1}}$, through $(d_{2s}, c_{2s+1})_g$, if this arc is not real, that is, if $\overline{g}h_{2s} \ne g$.

(1d) $(L'_j)_g$ is connected to S'^{-}_g, through $(d_{2j-1}, d_{2j})_g$, if this arc is not real, that is, if $\overline{g}h_{2j}^{-1} \ne g$, for $1 \le j \le s$;

(1e) S'^{+}_g is connected to $(M'_j)_g$, through $(d_{2j-2}, d_{2j-1})_g$, if this arc is not real, that is, if $\overline{g}h_{2j-2} \ne g$, for $1 \le j \le s$;

(1f) S'^{-}_g is connected to $(M'_j)_{gh_{2j-2}}$, through $(d_{2j-2}, c_{2j-1})_{gh_{2j-2}}$, for $1 \le j \le s$;

(1g) S'^{+}_g is connected to $(L'_j)_{gh_{2j}^{-1}}$, through $(c_{2j}, d_{2j})_g$, for $1 \le j \le s$;

Condition (1) determines the connected components of $X \setminus R$. We use (1f) to replace S'^{-}_g by the connected set $S'^{-}_g \cup \bigcup_{j=1}^{s} (M'_j)_{gh_{2j-2}}$. Similarly, we use (1g) to replace S'^{+}_g by the connected set $S'^{+}_g \cup \bigcup_{j=1}^{s} (L'_j)_{gh_{2j}^{-1}}$. Then the rest of the conditions may then be written as follows:

(2a) S'^{+}_g is connected to $S'^{+}_{gh_{2j-1}^{-1}h_{2j}} = S'^{+}_{gg_{2j}}$, for $1 \le j \le s$;

(2a$'$) S'^{-}_g is connected to $S'^{-}_{gh_{2j-1}h_{2j-2}^{-1}} = S'^{-}_{g(\overline{g_{2j}})^{-1}}$, for $1 \le j \le s$;

(2b) S'^{+}_g is connected to $S'^{-}_{gh_i^{-1}}$, if $\overline{g}h_i \ne g$, for $i = 2s+1, \ldots, 2s+r$;

(2c) S'^{+}_g is connected to $S'^{-}_{gh_{2s}^{-1}}$, if $\overline{g}h_{2s} \ne g$;

(2d) S'^{-}_g is connected to $S'^{+}_{gh_{2j}}$, if $\overline{g}h_{2j}^{-1} \ne g$, for $1 \le j \le s$;

(2e) S'^{+}_g is connected to $S'^{-}_{gh_{2j-2}^{-1}}$, if $\overline{g}h_{2j-2} \ne g$, for $1 \le j \le s$;

(In (2a$'$) we have used Lemma 3.1(d).) By change of variable, using Lemma 3.1(b), we see that (2d) is equivalent to:

(2d$'$) $S'^{-}_{gh_{2j}^{-1}}$ is connected to S'^{+}_{g}, if $\overline{gh_{2j}^{-1}}h_{2j}^{-1} \neq gh_{2j}^{-1}$, that is, if $\overline{g}h_{2j} \neq g$, for $1 \leq j \leq s$.

So the conjunction of (2d), (2c), and (2e) can be written as

(2e$'$) S'^{+}_{g} is connected to $S'^{-}_{gh_{2j}^{-1}}$, if $\overline{g}h_{2j} \neq g$, for $j = 0, 1, \ldots, s$.

We now show that condition (2e$'$) follows from (2a), (2a$'$), and (2b).

For $j = 0$, condition (2e$'$) is just (2b) with $i = 2s + r$, since $h_0 = h_{2s+r} = 1$. Let $1 \leq j \leq s$ and suppose, by induction, that (2e$'$) holds for $j - 1$, that is, (2e) holds for j. By (2a), S'^{+}_{g} is connected to $S'^{+}_{gg_{2j}}$. By (2a$'$), $S'^{-}_{gh_{2j-2}^{-1}}$ is connected to $S'^{-}_{gh_{2j-2}^{-1}\overline{g_{2j}}}$. By Lemma 3.1(d), $gh_{2j-2}^{-1}\overline{g_{2j}} = gh_{2j-1}^{-1} = gg_{2j}h_{2j}^{-1}$ and $\overline{g}h_{2j-2} = \overline{g}\,\overline{g_{2j}}h_{2j-1} = \overline{gg_{2j}}h_{2j}g_{2j}^{-1}$. Therefore $S'^{+}_{gg_{2j}}$ is connected to $S'^{-}_{gg_{2j}h_{2j}^{-1}}$, if $\overline{gg_{2j}}h_{2j}g_{2j}^{-1} \neq g$ that is, if $\overline{gg_{2j}}h_{2j} \neq gg_{2j}$. So if we change gg_{2j} to g, we get (2e$'$) for j.

We now make a change of notation:

NOTATION 5.1: *Denote S'^{+}_{g} simply by the symbol g and S'^{-}_{g} by gu. Then $\{S^{+}_{g}, S'^{-}_{g}\}_{g\in G}$ are replaced by the elements of $H = G \rtimes \langle u \rangle$. Furthermore, let $u_i = uh_{2s+i}$, for $i = 1, \ldots, r$. (By Lemma 3.1(a), these are elements of order 2 in H. Since $h_{2r+s} = h_n = 1$, we have $u_r = u$.)*

Then the remaining conditions (2a), (2a$'$), (2b) can be written as:

(3a) g is connected to gg_{2j}, for all $g \in G$ and all $1 \leq j \leq s$.

(3a$'$) gu is connected to $g(\overline{g_{2j}})^{-1}u = gug_{2j}^{-1}$, for all $g \in G$ and all $1 \leq j \leq s$.

(3b) g is connected to $gh_{2s+i}^{-1}u$, if $\overline{g}h_{2s+i} \neq g$, for all $g \in G$ and all $1 \leq i \leq r$. That is, g is connected to gu_i, if $ugu_i \neq g$, that is, $ug \neq gu_i$, for all $g \in G$ and all $1 \leq i \leq r$.

Conditions (3a) and (3a$'$) are equivalent to

(4a) h is connected to hg_{2j}, for all $h \in H$ and all $1 \leq j \leq s$.

As g runs through the elements of G, $h = gu_i$ runs through the elements of $H \setminus G$. Changing variable in (3b) we get the following equivalent condition to (3b):

(3b$'$) hu_i is connected to h, if $uh \neq hu_i$, for all $h \in H \setminus G$ and all $1 \leq i \leq r$.

So (3b) is equivalent to the conjunction of (3b) and (3b$'$), which can be concisely written as

(4b) hu_i is connected to h, for all $h \in H$ such that $uh \neq hu_i$ and all $1 \leq i \leq r$.

We may summarize this discussion as follows.

Definition 5.2: Let $r \geq 1, s \geq 0$. Let H be a finite group and let

$$(\sigma_1, \sigma_2, \ldots, \sigma_s; u_1, \ldots, u_r) \tag{5}$$

be a sequence of its generators such that $u_1, \ldots, u_r$ are of order 2. The associated **orientation graph** is the graph Γ whose set of vertices is H and whose edges are of two types:

(6a) an edge from h to $h\sigma_j$, for each $1 \leq j \leq s$ and every $h \in H$; and

(6b) an edge from h to hu_i, for each $1 \leq i \leq r$ and every $h \in H$ such that $u_r h \neq hu_i$.

THEOREM 5.3: *Assume $r \geq 1$. Let $u_i = uh_{2s+i} = ug_1g_2 \cdots g_{2s+i}$, for $i = 1, \ldots, r$. The Klein surface $X/\langle u \rangle$ is orientable if and only if the orientation graph Γ associated with $\mathcal{A} = (g_2, g_4, \ldots, g_{2s}; u_1, \ldots, u_r)$ is not connected.*

Proof: By Lemma 2.4(c), $\mathcal{A}$ generates H. Apply the discussion preceding Definition 5.2. ∎

COROLLARY 5.4: *Assume $r = 0$. The Klein surface $X/\langle u \rangle$ is orientable if and only if the orientation graph Γ associated with $\mathcal{A} = (g_2, g_4, \ldots, g_{2s}; u)$ is not connected.*

Proof: In this case $n = 2s \geq 2$. We put $T' = T \cup \{c_{2s+1}\}$, where c_{2s+1} is real. Consider $p\colon X \to \mathbb{P}^1$ as a cover ramified inside T'. By Theorem 5.3 (with $r = 1$), X is orientable if and only if the orientation graph associated with $(g_2, g_4, \ldots, g_{2s}; u_1 = u)$ is not connected. ∎

The following proposition gives some properties of the orientation graph.

PROPOSITION 5.5: *Let Γ be the orientation graph associated with a finite group H and a sequence (5). Put $u = u_r$. Let Γ' be the connected component of 1 and Γ'' the connected component of u in Γ. Then*

(a) *For every $h \in H$ either $h \in \Gamma'$ and $uh \in \Gamma''$ or $h \in \Gamma''$ and $uh \in \Gamma'$;*

(b) *$\Gamma = \Gamma' \cup \Gamma''$, that is, either Γ is connected or Γ' and Γ'' are its connected components.*

(c) *Γ is connected if and only if there is a path from 1 to u in Γ.*

Proof: (a) Since the group H is generated by (5), there are $a_1, \ldots, a_k \in \mathcal{A} \cup \mathcal{A}^{-1}$ such that $h = a_1 a_2 \cdots a_k$. We prove the assertion by induction on k. For $k = 0$ the assertion is clear: $1 \in \Gamma'$ and $u \in \Gamma''$ by definition. Let $h' = a_1 a_2 \cdots a_{k-1}$, then $h = h' a_k$. By induction hypothesis, say, $h' \in \Gamma'$ and $uh' \in \Gamma''$.

If there is no $1 \leq i \leq r$ such that $a_k = u_i$ and $uh' = h'u_i$, then there are edges in Γ either from h' to h and from uh' to uh (if $a_k \in \mathcal{A}$) or from h to h' and from uh to uh' (if $a_k^{-1} \in \mathcal{A}$). Therefore $h \in \Gamma'$ and $uh \in \Gamma''$.

If there is $1 \leq i \leq r$ such that $a_k = u_i$ and $uh' = h'u_i$, then u_i is of order 2 and

$$h = h'u_i = uh' \in \Gamma'' \qquad \text{and} \qquad uh = uh'u_i = h'u_i^2 = h' \in \Gamma'.$$

Conditions (b) and (c) follow from (a). ∎

We can express orientability in still another way:

Definition 5.6: Let $r \geq 1, s \geq 0$. Let H be a finite group and let (5) be a sequence of its generators such that $u_1, \dots, u_r$ are of order 2. Let H_0 be the subgroup of H generated by $\sigma_1, \sigma_2, \dots, \sigma_s$ and those u_i that are not conjugate to u_r in H, and let I_0 be the set of u_i conjugate to u_r in H. We say that H is **orientable** with respect to (5) if there are **no** $a_1, \dots, a_k \in H$ such that

(7a) $u_r = a_1 a_2 \cdots a_k$; and

(7b) for each $1 \leq \ell \leq k$ either $a_\ell = u_i \in I_0$ and $u_r a_1 a_2 \cdots a_{\ell-1} \neq a_1 a_2 \cdots a_{\ell-1} u_i$ or $a_\ell \in H_0$. ∎

COROLLARY 5.7: *The orientation graph Γ associated with (5) is connected if and only if H is not orientable with respect to (5). Hence the Klein surfaces $X/\langle u \rangle$ is orientable if and only if H is orientable with respect to $\mathcal{A}$ given in Theorem 5.3 and Corollary 5.4.*

Proof: We may replace condition (6) in Definition 5.2 by

(6′a) an edge from h to hg, for every $h \in H_0$; and

(6′b) an edge from h to hu_i, for each $u_i \in I_0$ and every $h \in H$ such that $u_r h \neq hu_i$.

Clearly, this graph contains Γ (has more edges) but is connected if and only if Γ is.

By Proposition 5.5(c), Γ is connected if and only if there is a path in Γ from 1 to u_r.

Consider an arbitrary sequence of vertices of Γ, written as

$$1, a_1, a_1 a_2, \dots a_1 a_2 \cdots a_{k-1}, a_1 a_2 \cdots a_{k-1} a_k.$$

Condition (7b) means that this is the sequence of vertices of a path in the graph. Condition (7a) says that its terminus is u. ∎

Example 5.8: Suppose $H = \tilde{G} \times \langle u \rangle$, where $\tilde{G} \leq H$. Then u is in the center of H, and $I_0 = \{u\}$. Therefore (7b) reduces to: $a_1, a_2, \ldots, a_k \in H_0$. Thus H is orientable if and only if $u \notin H_0$.

In particular, if also $\sigma_1, \sigma_2, \ldots, \sigma_s \in \tilde{G}$ and $u_1 = \cdots = u_r = u$, then $H_0 \subseteq \tilde{G}$, and hence $u \notin H_0$. Therefore H is orientable. ∎

When the number $n = r + 2s$ of the branch points is 2, the genus of X is 0, by the Riemann-Hurwitz formula. In this case $X/\langle u \rangle$ is orientable. Let us see how this also follows from the above characterization:

Corollary 5.9: *If $n = r + 2s = 2$, then $X/\langle u \rangle$ is orientable.*

Proof: We have either $r = 2, s = 0$ or $r = 0, s = 1$.

(a) Assume $r = 2, s = 0$. Then $H = \langle u_1, u_2 \rangle$, where $u_1, u_2 = u$ are of order 2. If $u_1 = u_2$, then, by Example 5.8, $X/\langle u \rangle$ is orientable. So assume $u_1 \neq u_2$. Let $h \in H$.

(i) If $u^h \neq u_1, u_2$, there are exactly two edges with endpoint at h, namely the edges from h to hu_1 and to hu_2.

(ii) If $u^h = u_i$, then $u^h \neq u_{3-i}$, and hence there is a unique edge with endpoint at h, namely the edge from h to hu_{3-i}.

So every vertex of Γ is of degree ≤ 2, and hence Γ consists of disjoint cycles and paths. But there are

$$|\{h \in H \mid \ u^h = u_1\}| + |\{h \in H \mid \ u^h = u_2 = u\}| = 2|C_H(u)| \geq 2|\langle u \rangle| \geq 4$$

vertices of degree 1, and hence at least 2 disjoint paths. Therefore Γ is not connected. By Theorem 5.3, $X/\langle u \rangle$ is orientable.

(b) Assume $r = 0, s = 1$. Recall (Lemma 2.4(c)) that $g_2 = f(\gamma_2), u = f(\varepsilon)$, where $\varepsilon, \gamma_2 \in \Phi(\mathbb{H}, \smallsetminus \xi(T), \xi(x_0))$ commute. Therefore $H = \langle g_2, u \rangle$ is commutative. So, $H = G \times \langle u \rangle$, where $G = \mathrm{Aut}(X/\mathbb{P}^1)$. By Example 5.8, $X/\langle u \rangle$ is orientable. ∎

Remark 5.10: Let E be a subgroup of $G = \mathrm{Aut}(X/\mathbb{P}^1)$ such that $\overline{E} = E^u = E$. As in Remark 4.12 consider the cover $\bar{p}\colon E\backslash X \to \mathbb{P}^1$ instead of $p\colon X \to \mathbb{P}^1$. Then Theorem 5.3 goes through, if we replace in the definition of the orientation graph in Definition 5.2 the elements $h \in H$ by the classes $Eh \in E\backslash H$.

I.e., its set of vertices is $E\backslash H = \{Eh \mid h \in H\}$ and its edges are of two types:

(6a′) an edge from Eh to Ehg_{2j}, for each $1 \le j \le s$ and every $Eh \in E\backslash H$; and

(6b′) an edge from Eh to Ehu_i, for each $0 \le i \le r$ and every $Eh \in E\backslash H$ such that $u_r Eh = Eu_r h \ne Ehu_i$, that is, $Eu_r \ne E(hu_i h^{-1})$.

We may also modify Definition 5.6 to this case. Let H_0 be the subgroup of H generated by $g_2, g_4, \ldots, g_{2s}$ and those u_i whose conjugacy class in H does not meet Eu_r, and let I_0 be the set of u_i whose conjugacy class in H meets Eu_r. We say that H is **orientable** with respect to $\mathcal{A}$ if there are no $a_1, \ldots, a_k \in H$ such that

(7a′) $Eu_0 = a_1 a_2 \cdots Ea_k$; and

(7b′) for each $1 \le \ell \le k$ either $a_\ell = u_i \in I_0$ and $Eu_r a_1 a_2 \cdots a_{\ell-1} \ne Ea_1 a_2 \cdots a_{\ell-1} u_i$ or $a_\ell \in H_0$.

Then Proposition 5.5 and Corollary 5.7 go through, if we replace the $1, h, u, uh \in H$ by their cosets E, Eh, Eu, Euh in $E\backslash H$.

6. Large non-orientable Klein covers

Fix a finite set $T \subseteq \mathbb{P}^1$ consisting of r real points and s pairs of complex conjugate points.

Let $Y/\mathbb{H}$ be a cover of Klein surfaces, unramified outside the image $\xi(T)$ of T in $\mathbb{H}$, such that Y has a nonempty boundary. Let X be the complex double of Y. We say that $Y/\mathbb{H}$ is **T-Galois** if $X/\mathbb{H}$ is a Galois cover of Klein surfaces (in particular, $X/\mathbb{P}^1$ is a Galois cover of Riemann surfaces unramified outside T and $Y/\mathbb{H}$ is its real form). Suppose $Y/\mathbb{H}$ is T-Galois. Let $H = \mathrm{Aut}(X/\mathbb{H})$ and let $u \in H$ be the generator of $\mathrm{Aut}(X/Y)$. Then u is a real involution and $Y = X/\langle u \rangle$.

We would like to address in this section the question of orientability of "sufficiently large" T-Galois covers. That is, we want to ask the following *two* questions:

(Q) Is there for every T-Galois cover $Y/\mathbb{H}$ a T-Galois cover $\hat{Y}/\mathbb{H}$ which factors through $Y/\mathbb{H}$ and $\hat{Y}$ is orientable, resp. $\hat{Y}$ is non-orientable?

If $r+2s = 2$, a complete answer is provided by Corollary 5.9: every T-Galois cover $\hat{Y}$ of $\mathbb{H}$ is orientable. So assume $r + 2s \geq 3$. We show below (Proposition 6.4)that the answer to (Q) with "is not orientable" is positive. The "is orientable" variant of (Q) remains an open question.

If $\hat{Y}/\mathbb{H}$ is a T-Galois cover then $\hat{Y} = \hat{X}/\langle \hat{u} \rangle$, where $\hat{X}$ is the complex double of $\hat{Y}$ and $\hat{u} \in \hat{H} = \mathrm{Aut}(\hat{X}/\mathbb{H})$ is a real involution. If $\hat{Y}/\mathbb{H}$ factors through $Y/\mathbb{H}$, say, via a cover $q\colon \hat{Y} \to Y$, then q lifts to a cover $q\colon \hat{X} \to X$ of Riemann surfaces. This cover induces a group epimorphism $q\colon \hat{H} \to H$ such that $q(\hat{u}) = u$. By Remark 3.4 we may assume that $\hat{X}, \hat{u}$ are given by our explicit construction in Sections 1 and 3. Thus, by Corollary 5.7, question (Q) can be translated into group theory:

Fix $x_0 \in \mathbb{P}^1 \smallsetminus T$ such that $\overline{x_0} = x_0$. Let $\Phi = \Phi_1(\mathbb{H} \smallsetminus \xi(T), \xi(x_0))$ and let $f\colon \Phi \to H$ be the canonical map. Recall (Lemma 2.4(c)) that $\Phi = \langle \gamma_2, \gamma_4, \ldots, \gamma_{2s}, \varepsilon, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r \rangle$, where $\varepsilon = \varepsilon_r$, if $r \geq 1$. To allow for a uniform treatment, let

$$(1) \qquad \bar{r} = \begin{cases} r & \text{if } r \geq 1; \\ 1 & \text{if } r = 0; \end{cases} \qquad \text{if } r = 0, \text{ put } \varepsilon_1 = \varepsilon.$$

Then $\Phi = \langle \gamma_2, \gamma_4, \ldots, \gamma_{2s}, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{\bar{r}} \rangle$.

A cover $q\colon \hat{X} \to X$ as above induces a group epimorphism $q\colon \hat{H} \to H$ and the canonical map $\hat{f}\colon \Phi \to \hat{H}$ satisfies $f = q \circ \hat{f}$.

Thus (Q) is equivalent to:

(Q$'$) Let H be a finite group and let $f\colon \Phi \to H$ be an epimorphism. Are there a finite group $\hat{H}$ and epimorphisms $\hat{f}\colon \Phi \to \hat{H}$ and $q\colon \hat{H} \to H$ such that $f = q \circ \hat{f}$ and $\hat{H}$ is orientable, resp., not orientable, with respect to

$$\hat{f}(\gamma_2), \hat{f}(\gamma_4), \ldots, \hat{f}(\gamma_{2s}); \hat{f}(\varepsilon_1), \hat{f}(\varepsilon_2), \ldots, \hat{f}(\varepsilon_{\bar{r}})? \tag{2}$$

Remark 6.1: Not every group epimorphism $f\colon \Phi \to H$ onto a finite group H corresponds to a Klein cover of $\mathbb{H}$. The necessary and sufficient condition is, as we have seen in Sections 2 and 3, that $f(\varepsilon_{\bar{r}}) \notin f(\Pi)$, where $\Pi = \Pi_1(\mathbb{P}^1 \smallsetminus T, x_0)$. That is, H is the semidirect product $f(\Pi) \rtimes \langle f(\varepsilon_{\bar{r}})\rangle$ and $f(\gamma)^{f(\varepsilon_{\bar{r}})} = f(\gamma^{\varepsilon}_{\bar{r}})$ for every $\gamma \in \Pi$. However, we may replace $f\colon \Phi \to H$ in (Q$'$) by $f_1\colon \Phi \to H_1$ with $f_1(\varepsilon_{\bar{r}}) \notin f_1(\Pi)$, if there is an epimorphism $q_1\colon H_1 \to H$ such that $q_1 \circ f_1 = f$. Such f_1 always exists. For instance, let $\bar{f}\colon \Phi \to \bar{H}$ correspond to a Klein cover and hence satisfy $\bar{f}(\varepsilon_{\bar{r}}) \notin \bar{f}(\Pi)$. Let H_1 be the image of Φ in $H \times \bar{H}$ under the product map $f_1 = (f, \bar{f})$ and let $q_1\colon H_1 \to H$ and $\bar{q}\colon H_1 \to \bar{H}$ be the coordinate projections. Then f_1 has the required property, since $\bar{f} = \bar{q} \circ f_1$.

Let $D_{r,s}$ be the free product of s infinite cyclic groups $\langle\sigma_1\rangle, \langle\sigma_2\rangle, \ldots, \langle\sigma_s\rangle$ and r groups $\langle\varepsilon_1\rangle, \langle\varepsilon_2\rangle, \ldots, \langle\varepsilon_r\rangle$ of order 2. Thus

$$D_{r,s} = \langle \varepsilon_1, \varepsilon_2, \ldots \varepsilon_r, \sigma_1, \sigma_2, \ldots, \sigma_s \mid \varepsilon_1^2 = \varepsilon_2^2 = \cdots = \varepsilon_r^2 = 1\rangle. \tag{3}$$

If $r \geq 1$, then $\Phi \cong D_{\bar{r},s} = D_{r,s}$; if $r = 0$, then $\bar{r} = 1$ and $\Phi \cong D_{1,s}/[\sigma_1\sigma_2\cdots\sigma_s, \varepsilon_1]$. (Lemma 2.4(c).) So we may replace Φ in (Q$'$) by $D_{\bar{r},s}$, but, for $r = 0$ (and hence $\bar{r} = 1$), we also have to require that $[\hat{f}(\sigma_1)\hat{f}(\sigma_2)\cdots\hat{f}(\sigma_s), \hat{f}(\varepsilon_1)] = 1$.

So the question can be reformulated as follows:

(Q$''$) Let H be a finite group with a set of generators

$$b_1, b_2, \ldots, b_s; u_1, u_2, \ldots, u_{\bar{r}} \tag{4}$$

with $u_1, u_2, \ldots, u_{\bar{r}}$ of order 2, such that $[b_1 b_2 \cdots b_s, u_1] = 1$ if $r = 0$. Is there a finite group $\hat{H}$ with a set of generators

$$\hat{b}_1, \hat{b}_2, \ldots, \hat{b}_s; \hat{u}_1, \hat{u}_2, \ldots, \hat{u}_{\bar{r}} \tag{5}$$

with $\hat{u}_1, \hat{u}_2, \ldots, \hat{u}_{\bar{r}}$ of order 2, such that $[\hat{b}_1 \hat{b}_2 \cdots \hat{b}_s, \hat{u}_1] = 1$ if $r = 0$, and an epimorphism $q\colon \hat{H} \to H$ mapping (4) onto (5) such that $\hat{H}$ is orientable, resp. not orientable, with respect to (5)?

Lemma 6.2: *Assume either $s \geq 1$ or $r \geq 2$. Then there exists $\gamma \in D_{r,s}$ with the following property. For every integer $m \geq 2$ there exists a finite group C and an epimorphism $f\colon D_{r,s} \to C$ such that*

(i) *$f(\varepsilon_1), f(\varepsilon_2), \ldots, f(\varepsilon_r)$ belong to distinct conjugacy classes in C;*

(ii) *if $s \geq 2$, then $f(\sigma_1) f(\sigma_2) \cdots f(\sigma_s) = 1$, and*

(iii) *$f(\gamma)$ is of order m.*

Proof: If $s \geq 1$, let $\gamma = \sigma_1$ and let C be the direct product of r cyclic groups $\langle u_1 \rangle, \langle u_2 \rangle, \ldots, \langle u_r \rangle$ of order 2 and a cyclic group $\langle c \rangle$ of order m. Define $f\colon D_{r,s} \to C$ by $f(\varepsilon_1) = u_1, \ldots, f(\varepsilon_r) = u_r$, $f(\sigma_1) = c$, $f(\sigma_2) = c^{-1}$, and $f(\sigma_3) = \cdots = f(\sigma_s) = 1$.

If $s = 0$ and $r \geq 2$, consider the dihedral group C_0 of order $2m$: this is the semidirect product of a group $\langle u_1 \rangle$ of order 2 acting on a cyclic group $\langle c \rangle$ of order m by $c^{u_1} = c^{-1}$. Notice that $u_2 = u_1 c$ is of order 2 and not conjugate to u_1. Let C be the direct product of C_0 and $r - 2$ cyclic groups $\langle u_3 \rangle, \ldots, \langle u_r \rangle$ of order 2. Define $f\colon D_{r,s} \to C$ by $f(\varepsilon_i) = u_i$, for $i = 1, 2, \ldots, r$. Let $\gamma = \varepsilon_1 \varepsilon_2$. Then $f(\gamma) = u_1 u_2 = c$. ∎

Lemma 6.3: *Let $D = D_{r,s}$, with either $r \geq 3$ or $r \geq 1, s \geq 1$, and let $m \geq 2$ be an integer. Let $D_0 = \langle \varepsilon_1, \varepsilon_2, \ldots \varepsilon_{r-1}, \sigma_1, \sigma_2, \ldots, \sigma_s \rangle$. Then there exist $\gamma \in D_0$, a finite group H, and an epimorphism $f\colon D \to H$ such that, denoting $u = f(\varepsilon_r)$ and $a = f(\gamma)$,*

(i) *$f(\varepsilon_1), f(\varepsilon_2), \ldots, f(\varepsilon_r)$ belong to distinct conjugacy classes in H;*

(ii) *if $s \geq 2$, then $f(\sigma_1) f(\sigma_2) \cdots f(\sigma_s) = 1$.*

(iii) *$(a^m u)(au)^{2m}(a^m u) = 1$,*

(iv) $a^m(au)^k \notin C_H(u)$ *for every integer* k.

Proof: By assumption, D is the free product of the group $\langle\varepsilon_r\rangle$ of order 2 and D_0, which is isomorphic to $D_{r-1,s}$. By Lemma 6.2, there is $\gamma \in D_0$, an epimorphism $f_0\colon D_0 \to C$ onto a finite group C such that

(i′) $f_0(\varepsilon_1), f_0(\varepsilon_2), \ldots, f_0(\varepsilon_{r-1})$ belong to distinct conjugacy classes in C,

(ii′) if $s \geq 2$, then $f_0(\sigma_1)f_0(\sigma_2)\cdots f_0(\sigma_s) = 1$, and

(iii′) $c = f_0(\gamma)$ is of order $2m$.

Let $\langle u\rangle$ be a group of order 2, and let H be the wreath product of $\langle u\rangle$ with C. That is, H is the semidirect product

$$H = \langle u\rangle \ltimes (C \times C),$$

where u acts on $C \times C$ by $(c_1, c_2)^u = (c_2, c_1)$. There is an epimorphism $f\colon D \to H$ given by $f(\sigma) = (1, f_0(\sigma)) \in C \times C$, for $\sigma \in D_0$, and $f(\varepsilon_r) = u$. Then (i) easily follows from (i′) and (ii) from (ii′).

Let $a = (1, c) = f(\gamma)$ and $b = (c, 1) \in C \times C \subseteq H$. Then $\operatorname{ord} a = \operatorname{ord} b = 2m$, $[a, b] = 1$, and $b = a^u = uau$. Therefore

$$(a^m u)(au)^{2m}(a^m u) = a^m u\big((ab)^m a^m\big)u = a^m(a^u b^u)^m(a^u)^m = a^m(ba)^m b^m = 1.$$

Let k be an integer. If k is even, say, $k = 2i$, then

$$a^m(au)^k = a^m(auau)^i = a^m(ab)^i = a^{m+i}b^i;$$

if k is odd, say, $k = 2i + 1$, then

$$a^m(au)^k = a^m(auau)^i au = a^m(ab)^i au = a^{m+i+1}b^i u.$$

As $a^m, a^{m+1} \neq 1$, none of these elements is fixed by the conjugation by u on H, and hence is not in $C_H(u)$. ∎

PROPOSITION 6.4: *Assume* $n = 2r + s \geq 3$. *Let* $f\colon D_{\bar{r},s} \to H$ *be an epimorphism onto a finite group* H *such that* $[f(\sigma_1)f(\sigma_2)\cdots f(\sigma_s), f(\varepsilon_1)] = 1$, *if* $r = 0$. *Then there exists a* *finite group $\hat{H}$ and epimorphisms $\hat{f}\colon D_{\bar{r},s} \to \hat{H}$ and $q\colon \hat{H} \to H$ such that $q \circ \hat{f} = f$, $\hat{H}$ is not orientable with respect to (2), and, if $r = 0$, then $[\hat{f}(\sigma_1)\hat{f}(\sigma_2)\cdots\hat{f}(\sigma_s), \hat{f}(\varepsilon_1)] = 1$.*

Proof: As $n \geq 3$, either $r \geq 3$ or $s \geq 1$; moreover, $\bar{r} \geq 1$ and, if $r = 0$, then $s \geq 2$.

Fix a multiple $m \geq 2$ of the order of H. Let $D = \langle \varepsilon_1, \varepsilon_2, \ldots \varepsilon_{\bar{r}-1}, \sigma_1, \sigma_2, \ldots, \sigma_s \rangle$. This subgroup of $D_{\bar{r},s}$ is isomorphic to $D_{\bar{r}-1,s}$.

By Lemma 6.3, there is $\gamma \in D$, a finite group $\tilde{H}$, and an epimorphism $\tilde{f}\colon D_{\bar{r},s} \to \tilde{H}$ such that $\tilde{u} = \tilde{f}(\varepsilon_{\bar{r}}), \tilde{a} = \tilde{f}(\gamma) \in \tilde{H}$ satisfy $(\tilde{a}^m\tilde{u})(\tilde{a}\tilde{u})^{2m}(\tilde{a}^m\tilde{u}) = 1$, and $\tilde{a}^m(\tilde{a}\tilde{u})^k \notin C_{\tilde{H}}(\tilde{u})$ for every k. Moreover, $\tilde{f}(\varepsilon_1), \tilde{f}(\varepsilon_2), \ldots, \tilde{f}(\varepsilon_{\bar{r}})$ belong to distinct conjugacy classes in $\tilde{H}$, and, if $r = 0$ (and hence $s \geq 2$), then $\tilde{f}(\sigma_1)\tilde{f}(\sigma_2)\cdots\tilde{f}(\sigma_s) = 1$.

Put $a = f(\gamma)$ and $u = f(\varepsilon_r)$. By our choice of m we have $(a^m u)(au)^{2m}(a^m u) = u^2 = 1$.

Define $\hat{f}\colon D_{\bar{r},s} \to \tilde{H} \times H$ by $\hat{f}(\sigma) = (\tilde{f}(\sigma), f(\sigma))$. Let $\hat{H}$ be the image of $\hat{f}$ and let $\tilde{p}\colon \hat{H} \to \tilde{H}$ and $q\colon \hat{H} \to H$ be the restrictions to $\hat{H}$ of the coordinate projections $\tilde{H} \times H \to \tilde{H}$, $\tilde{H} \times H \to H$. Then $\hat{f}\colon D_{\bar{r},s} \to \hat{H}$ satisfies $q \circ \hat{f} = f$ and $\tilde{p} \circ \hat{f} = \tilde{f}$.

Since $\tilde{f}(\varepsilon_1), \tilde{f}(\varepsilon_2), \ldots, \tilde{f}(\varepsilon_{\bar{r}})$ belong to distinct conjugacy classes in $\tilde{H}$, their preimages $\hat{f}(\varepsilon_1), \hat{f}(\varepsilon_2), \ldots \hat{f}(\varepsilon_{\bar{r}})$ under $\tilde{p}$ belong to distinct conjugacy classes in $\hat{H}$. If $r = 0$, then $\hat{f}(\sigma_1)\hat{f}(\sigma_2)\cdots\hat{f}(\sigma_s)$ commutes with $\hat{f}(\varepsilon_1)$, since $\tilde{f}(\sigma_1)\tilde{f}(\sigma_2)\cdots\tilde{f}(\sigma_s) = 1$ commutes with $\tilde{f}(\varepsilon_1)$, and $f(\sigma_1)f(\sigma_2)\cdots f(\sigma_s)$ commutes with $f(\varepsilon_1)$.

Put $\hat{a} = \hat{f}(\gamma) = (\tilde{a}, a)$ and $\hat{u} = \hat{f}(\varepsilon) = (\tilde{u}, u)$. Then

$$\hat{a} \in \hat{f}(D) = \langle \hat{f}(\varepsilon_1), \hat{f}(\varepsilon_2), \ldots \hat{f}(\varepsilon_{\bar{r}-1}), \hat{f}(\sigma_1), \hat{f}(\sigma_2), \ldots, \hat{f}(\sigma_s) \rangle = \hat{H}_0$$

and $(\hat{a}^m\hat{u})(\hat{a}\hat{u})^{2m}(\hat{a}^m\hat{u}) = 1$, that is,

$$\hat{a}^m \cdot \hat{u} \cdot \hat{a} \cdot \hat{u} \cdot \hat{a} \cdot \hat{u} \cdot \hat{a} \cdot \hat{u} \ \cdots \ \hat{a} \cdot \hat{u} \cdot \hat{a}^m = \hat{u}.$$

But none of the heads of the string on the left handed side (except the whole string) centralizes $\hat{u}$ in $\hat{H}$. Indeed, these heads are of the form either $\hat{a}^m(\hat{a}\hat{u})^k$ or $\hat{a}^m(\hat{a}\hat{u})^k\hat{u}$ for some k, and, by assumption, their images $\tilde{a}^m(\tilde{a}\tilde{u})^k$, $\tilde{a}^m(\tilde{a}\tilde{u})^k\tilde{u}$ are not in $C_{\tilde{H}}(\tilde{u})$. ∎

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