The Macdonald Identities

From simle Lie algebras to the Dedekind eta function

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ABSTRACT. The irreducible finite dimensional representations of a semi-simple complex Lie Algebra are determined by their highest weight. In finding the dimension of the representation space V_λ with the highest weight λ, Hermann Weyl proved the identity

$$\prod_{\alpha>0}(e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) = \sum_{S\in W}(\det(S))e^{S\rho},$$

where α runs through the positive roots, S ranges over the elements of the Weyl group of $\mathfrak{g}$, and ρ is the Weyl vector. By passing from roots to "affine roots" and from the (finite) Weyl-group to the infinite "affine Weyl-group", Macdonald obtained a similar looking identity between an infinite product and an infinite series. Using appropriate substitutions, this leads to function theoretic identities between infinite sums and infinite products, perhaps the most spectacular of which represents $\eta(z)^{\dim(\mathfrak{g})}$ as a theta series related to the root system of $\mathfrak{g}$.

The talk will explain all necessary vocabular. The proof of Macdonald is very long and covers slightly more general root lattices than those belonging to simple Lie algebras. We propose a much shorter proof for the latter.