RECURSIVE ALGORITHMS FOR IMAGE LOCAL STATISTICS IN NON-RECTANGULAR WINDOWS

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ABSTRACT

A general framework is presented for recursive computation of image local statistics in sliding window of almost arbitrary shape with "per-pixel" computational complexity substantially lower than the window size. As special cases, recursive algorithms are described for computing image local statistics such as local mean, local variance, local kurtosis, local order statistics (minimum, maximum, median), local ranks, local DFT, DCT, DcST spectra in diamond, octagon, triangle-shaped windows as well as in windows with nonuniform weights, such as cosine, sine, Hann, Hamming and Blackman windows.

1. INTRODUCTION

Computing image local statistics, such as local means, variance, general local moments, local order statistics, ranks, histograms, spectra, etc., in sliding window is frequently required in image processing. Generally, for arbitrarily shaped window of $WinSz$ pixels, the "per-pixel" computational complexity of this process is $\mathcal{O}(WinSz)$ or even, for spectra, $\mathcal{O}(WinSz \log WinSz)$. Even for moderate window sizes, this complexity might be formidably large, especially in real-time processing applications. Substantial reduction of the computational complexity is possible with the use of recursive computation methods which utilize information common to consecutive overlapping windows and compute local statistics for the current window position by means of an appropriate modification of the results obtained for the previous window position [1].

Recursive algorithms for computing local statistics in the window of a rectangular shape are well known. These algorithms include mean, histogram and median, order statistics, spectra, e.g. DFT, DCT, DcST ([1], [2], [3], [4]).

However, in many cases the rectangular window is far from being optimal and windows of other shape are required. Recently, a number of recursive algorithms for computing image local statistics in the window of non-rectangular shape were suggested:

- mean and variance in octagonal window [5], moments in diamond, hexagon, general polygon windows [6], mean in arbitrary window [7];

- histogram, median and order statistics in arbitrary window [8].

In this paper, we suggest unification and extension of these algorithms and present a general solution for recursive computing local statistics in windows of virtually arbitrary shape.

2. IMAGE LOCAL STATISTICS

We will define scanning window for local statistics measurements by a window weight coefficients $\{\mathbf{w}_n\}$ and define local statistics as following.

The local moment of order P of a signal $\mathbf{a}^{(k)}$ in a window of $\mathbf{N}_w$ pixels in its $\mathbf{k}$-th position is defined as follows:

$$\mathbf{M}_{[P]}^{(k)} \triangleq \sum_{n=0}^{N_w-1} w_n \left[a_n^{(k)}\right]^P . \tag{1}$$

Common local statistical parameters (mean, variance, skew, kurtosis) are linear combinations of the local moments. These parameters are used for data statistical analysis, image smoothing, enhancement and feature extraction.

The local weighted histogram counts, for each gray level q, the weighted number of of signal samples with this gray level. The value of local weighted histogram at gray level q is:

$$\mathbf{H}^{(k)}\{q\} \triangleq \sum_{n=0}^{N_w-1} w_n \delta \left\{q - a_n^{(k)}\right\} . \tag{2}$$

Conventionally, local histograms are computed for windows with uniform weights. Local histograms and closely related local variational rows and local order statistics are used in rank filtering for signal/image denoising, smoothing, enhancement, extraction of object details and their boundaries.

Yet another representatives of local statistics are local spectra in different bases. The local signal spectrum $\alpha_r^{(k)}$ with respect to the basis $\psi_r(n)$ is defined as:

$$\alpha_r^{(k)\psi} \triangleq \sum_{n=0}^{N_w-1} w_n a_n^{(k)} \psi_r(n). \tag{3}$$

A variety of special cases of local spectra exist. The most important cases are DFT, DCT and DcST spectra in a window with uniform weight. Other examples are Walsh and Haar spectra. The applications of spectral analysis in scanning window include local adaptive signal/image restoration (denoising, deblurring, resampling with discrete sinc-interpolation, blind restoration, image enhancement), differentiating, integrating, target location and optical flow [1].

3. RECURSIVELY COMPUTED WINDOWS

3.1. Scanning modes

Recursive computations assume a certain arrangement of image data in computer memory and a certain method of scanning image data. According to a common convention, we assume that images are defined on a rectangular sampling grid. On this grid, the foolwing scanning modes are possible: *progressive row-wise — column-wise* scanning mode; *zig-zag row-wise — column-wise* mode; *zig-zag row-wise — column-wise* mode and *diagonal*-$45°$ — *diagonal*-$135°$ mode.

As it is shown in Figure 1,

1. In the *progressive row-wise — column-wise* scanning mode, all rows are scanned one after another from left to right, and pixels are accessed in row-wise order. In order to get pixels in column-wise order, the progressive row-wise — column-wise mode has to be applied to a transposed version of the original image.

2. In the *zig-zag row-wise — column-wise* mode, all rows are scanned in a one continuous scan. Even rows are scanned from left to right and odd rows are scanned from right to left. The pixels are accessed in the row-wise order. In order to get pixels in column-wise order, the zig-zag row-wise — column-wise mode has to be applied to a transposed version of the original image.

3. In the *diagonal*-$45°$ — *diagonal*-$135°$ mode, all pixels are accessed in the diagonal-$45°$ order. In order to access pixels in the diagonal-$135°$ order, the diagonal-$45°$ — diagonal-$135°$ mode has to be applied to a horizontally-flipped version of the original image.

The first two scanning modes are suited for rectangular windows with uniform weights, while the third scanning mode is suited for non-rectangular windows with uniform weights (e.g. diamond window).

The most simple way of local image processing is to access pixels in the progressive row-wise — column-wise scanning mode and to apply a recursion to each row separately. This method has two drawbacks:

1. The computation of the first result in each row is non-recursive;

2. No use is made of the similarity of neighboring "update structrures" in consecutive rows.

As it was shown in [3], for computation of median in a uniform rectangular window, the first drawback can be eliminated by changing the method of access to zig-zag row-wise — column-wise scanning mode. In this way all pixels are accessed in one continuous scan and there are no non-recursive computations except one at the beginning. As it was shown in [4], the second drawback can be eliminated by recursive computation of "update structures" in the current row based on the "update structures" in the previous row. One can generalize these methods to formulate a recursivity principle that applies to different local statistics operations in various windows.

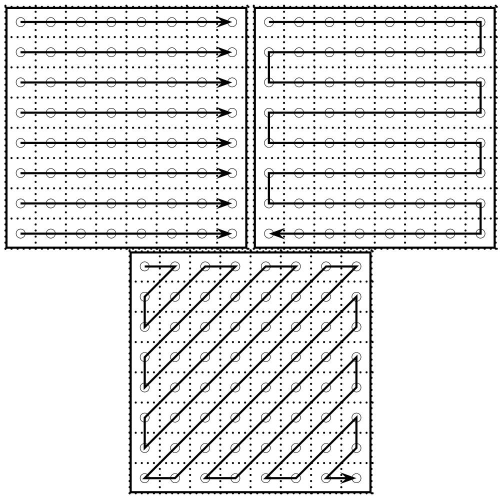

Figure 1. Different scanning modes in rectangular sampling grid. Top row, left figure: the progressive row-wise — column-wise scanning mode. Top row, right figure: the zig-zag row-wise — column-wise scanning mode. Bottom row: the diagonal-$45°$ — diagonal-$135°$ scanning mode.

For each row in image, the recursion can be performed in two steps. At the first step, local statistics in all "update structures" of the row are computed in parallel. This computation is column-wise recursive and it utilizes the "update structures" from the previous row. The results are stored in an LS_u buffers. At the second step, the local statistics in "update structures" for the current row are combined using row-wise recursion to obtain the local statistics within the window for this row. The results for the current row are stored in a LS_w buffer.

In Figure 2, the examples of "update structures" for different window shapes are presented. For instance, for the rectangular window and for the diamond window the "update structures" are pairs of window sides. The "update structures" lay on rows or columns of rectangular system of coordinates in case of rectangular window and on diagonals of the rectangular grid in case of diamond window. The "update structures" are mutually independent and therefore can be processed in parallel.

3.2. Recursivity in rectangular coordinates and interlaced scanning

The 2D windows can be divided into several classes. The class of "full" windows includes uniform rectangle, rectangle with cosine-masking, diamond, hexagon and octagon. The class of "sparse" windows includes sparse rectangle and sparse diamond. The representatives of "sector" class are sector diamond and sector octagon of different orientation (NW, NE, SW, SE) and their combinations. The class of "ring" windows consists of rectangular ring and octagonal ring.

The "leaving update structures" and the "arriving update structures" can be identified for window of an arbitrary shape [8]. As it is shown in Figure 2, the "update structures" of simple geometrical shapes lay on columns, rows and diagonals of the rectangular sampling grid.

In the rectangular sampling grid, the most convenient shape of the window is a rectangular shape. An important advantage of the rectangular window shape is its separability that enables decomposing computations to consecutive rows-wise and column-wise ones. The "update structures" of the rectangle lay on columns and the rows.

The "decimated" version of a rectangle that can be called *sparse rectangle* is also separable. The "update structures" of the sparse rectangle lay on "sparse" columns and rows. The "full" rectangle can be decomposed into two interlaced sparse rectangles that can be processed independently. This method of two-stage processing can be called *interlaced scanning*.

3.3. Recursivity in $45°$, $60°$-rotated coordinates and interlaced scanning

The diamond shape is not separable. For the diamond shape, the "update structures" lay on the diagonals of the rectangular grid. The diagonals can be easily accessed using diagonal-$45°$ — diagonal-$1355°$ scanning mode.

The "decimated" version of a diamond shape, called *sparse diamond*, is separable [6]. The "update structures" for sparse diamond lay on the "sparse" diagonals. The "full" diamond can be decomposed into two interlaced sparse diamonds that can be processed independently.

Other window shapes (e.g. hexagon, octagon) are non-separable. The "update structures" lay on columns, rows and diagonals of the rectangular grid.

4. PARALLELIZATION

4.1. Methods of computation parallelization

As a general way to recursive computing of local statistics in windows of arbitrary shape, including arbitrary weighted ones, we suggest parallelization of recursive computations by splitting a computational task into several independent sub-tasks that can be processed recursively. There are two different methods of parallelization:

- multiple windows method,
- expansion of the window function over recursive bases.

4.2. Multiple windows method

In multiple windows method, a given scanning window is decomposed onto several non-overlapping or overlapping

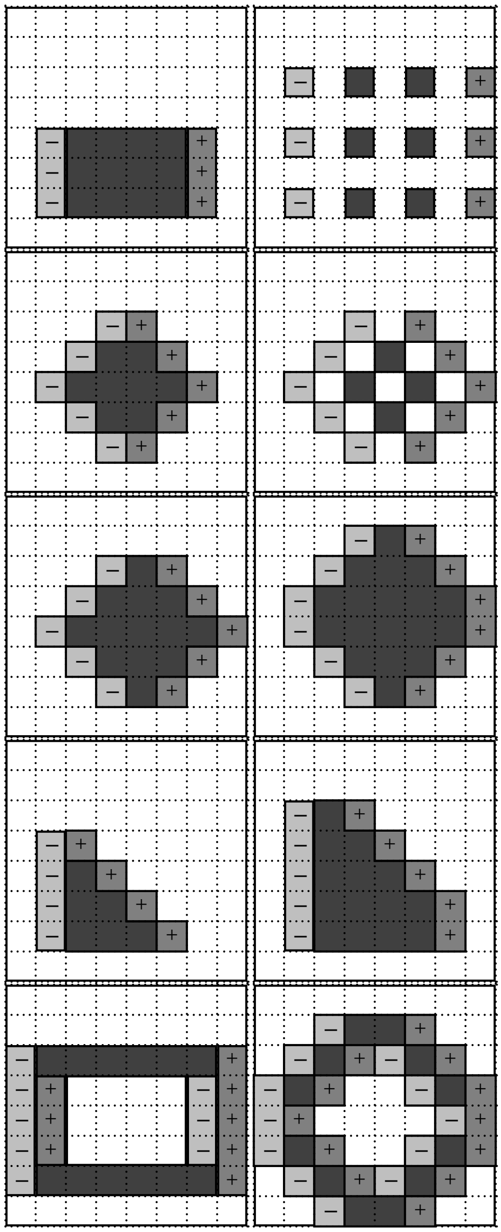

Figure 2. The "update structures" for windows of different geometrical shapes in the rectangular system of coordinates. First row: rectangle and sparse rectangle. Second row: diamond and sparse diamond. Third row: hexagon and octagon. Fourth row: diamond sector and octagon sector. Fifth row: rectangular ring and octogonal ring.

sub-windows, or "building-blocks", that allow recursive computations. The processing is performed in parallel on the sub-windows and then the results of sub-window computations are combined by addition with certain weight coefficients. If the sub-windows overlap, it will correspond to weighted windows with weight coefficient defined by the weight coefficients used in combining sub-window coefficient results. A simple example of composite window built from overlapping "building-blocks" is a combination of overlapping square and diamond. This window is an approximation of an octagon window, it is nearly isotropic and has "soft" edges. Other examples of composite windows are rings and combinations of sectors. The rings are obtained by subtraction of small window (usually of rectangular or octogonal shape) from the large window of the same type circumscribing the small one (usually two windows share a common center). The combinations of sectors are obtained by choosing the form of a sector window (usually of diamond sector or octagon sector shape) and combining several sectors of the same type and size but different orientation together to form parts of a diamond or an octagon.

4.3. Expansion of window function over recursive bases

The method of expansion of window over recursive bases allows to compute weighted local statistics in arbitrary shaped window with arbitrary weights assigned to each sample of the window. The most frequently used window functions are: Uniform (rectangle, Dirichlet), Sine lobe, Hann ("hanning", raised cosine, sine squared), Hamming and Blackman [9], [10]. The method parallelization of computations by expansion of the window window function over recursive bases will be elaborated in the next Section.

5. PARALLEL AND RECURSIVE COMPUTATION OF LOCAL MOMENTS

5.1. General formulation

Computing local moments in a weighted window defined by Eq. 1 can be regarded as a version of space invariant digital filtering of an input signal $\left[a_n^{(k)}\right]^P$ with a filter with point-spread function (PSF) w_n:

$$M_{[P]}^{(k)} = \sum_{n=0}^{N_w-1} w_n \left[a_n^{(k)}\right]^P . \tag{4}$$

An approach to parallel and recursive implementation of filtering was presented in [11]. According to this approach, filter with a given PSF is decomposed into a group of recursive sub-filters working in parallel. This decomposition is implemented through expansion of the filter PSF over recursive basis functions.

Assuming that the filter PSF can be expanded into a series over a system of basis functions $\psi_r(n)$, ($r = 0, \ldots, N_r - 1$, $N_r \leq N_w$):

$$w_n = \sum_{r=0}^{N_r-1} \lambda_r \psi_r(n), \tag{5}$$

the filter output can be found as a weighted sum of coefficients $\beta_r(k)$ of series expansion of the input signal:

$$M_{[P]}^{(k)} = \sum_{r=0}^{N_r-1} \lambda_r \alpha_r(k), \tag{6}$$

where

$$\alpha_r(k) = \sum_{n=0}^{N_w-1} \left[a_{k-n}\right]^P \psi_r(n). \tag{7}$$

If a given window is specified by its weight coefficients within a non-rectangular geomentrical shape, for the implementation of the above-described window function decomposition it must be re-defined as inscribed into a rectangular window with zero weight coefficients in those samples that are not occupied by the given window.

5.2. Recursive bases

It was found that the recursive basis functions belong to class of power functions [11]:

$$\psi_r(n) = \left[\psi_r(0)\right]^n . \tag{8}$$

Two most important special cases of 1-D recursive basis functions are the basis of exponential functions and the basis of rectangular functions. Sine and cosine functions od DCT and DcST transforms, being linear combinations of complex exponential functions, also belong to recursive basis functions. Walsh, Walsh-Paley, Walsh-Hadamard and Haar basis functions are additional examples of recursive basis functions. Two-dimensional recursive bases are obtained as separable combinations of one-dimensional ones.

6. PARALLEL AND RECURSIVE COMPUTATION OF LOCAL MOMENTS THROUGH COMBINATIONS OF MULTIPLE WINDOWS

Assume that a window $\bar{W}$ outlines L "building-block" windows W_l $(l = 0, \ldots, L-1)$:

$$\bar{W} \triangleq \bigcup_l W_l, \tag{9}$$

and a constant weight w_l is assigned to each "building-block" window W_l. Parallel and recursive computation of local moments by means of combining results of computations in multiple recursive windows is based on the following theorem.

Theorem 1. *The weighted sum of moments over any combination of windows is equal to the weighted moment over the outline (union) of the windows.*

Proof. The moment of order P over the window W_l is given by:

$$\mathbf{M}_{[P]}^{(W_l)} \triangleq \sum_{n \in W_l} [a_n]^P . \tag{10}$$

It can be computed over the outline window $\bar{W}$, using the indicator function of the window W_l:

$$\mathbf{M}_{[P]}^{(W_l)} = \sum_{n \in \bar{W}} \delta(n \in W_l) [a_n]^P . \tag{11}$$

The weighted sum of moments over a combination of windows is given by:

$$\bar{\mathbf{M}}_{[P]} \triangleq \sum_l w_l \mathbf{M}_{[P]}^{(W_l)}. \tag{12}$$

Then:

$$\begin{aligned} \bar{\mathbf{M}}_{[P]} &= \sum_l w_l \sum_{n \in \bar{W}} \delta(n \in W_l) [a_n]^P \\ &= \sum_{n \in \bar{W}} \left\{ \sum_l w_l \delta(n \in W_l) \right\} [a_n]^P = \sum_{n \in \bar{W}} \bar{w}_n [a_n]^P \\ &\triangleq \mathbf{M}_{[P]}^{\mathbf{w}}, \end{aligned} \tag{13}$$

where $\mathbf{M}_{[P]}^{\mathbf{w}}$ denotes the weighted moment of order P over the outline window $\bar{W}$ and $\bar{w}_n$ denotes the weight of the sample a_n. This weight is equal to the weighted sum of indicator functions over the windows:

$$\bar{w}_n = \sum_l w_l \delta(n \in W_l). \tag{14}$$

This proves the theorem. □

7. PARALLEL AND RECURSIVE COMPUTATION OF LOCAL HISTOGRAMS AND THEIR DERIVATIVES THROUGH COMBINATIONS OF MULTIPLE WINDOWS

7.1. Weighted histogram theorem

Assume that window $\bar{W}$ can be represnted as composed of L "building-block" windows W_l ($l = 0, \ldots, L-1$):

$$\bar{W} \triangleq \bigcup_l W_l, \tag{15}$$

and a constant weight w_l is assigned to each "building-block" window W_l. Assume also that the weight of sample in the outline window $\bar{W}$ is equal to the sum of weights of this sample in the "building-block" windows owning it. Then the weighted histogram over the outline window $\bar{W}$ can be found as a weighted sum of histograms over "building-block" windows W_l.

Theorem 2. *The weighted sum of histograms over any combination of windows is equal to the weighted histogram over the outline (union) of the windows.*

Proof. The histogram over the window W_l is given by:

$$\mathbf{H}^{(W_l)}\{q\} \triangleq \sum_{n \in W_l} \delta\{q - a_n\}. \tag{16}$$

It can be computed from the histogram over the outline window $\bar{W}$, using the indicator function of the window W_l:

$$\mathbf{H}^{(W_l)}\{q\} = \sum_{n \in \bar{W}} \delta(n \in W_l) \delta\{q - a_n\}. \tag{17}$$

The weighted sum of histograms over a combination of windows is given by:

$$\bar{\mathbf{H}}\{q\} \triangleq \sum_l w_l \mathbf{H}^{(W_l)}\{q\}. \tag{18}$$

Then:

$$\begin{aligned} \bar{\mathbf{H}}\{q\} &= \sum_l w_l \sum_{n \in \bar{W}} \delta(n \in W_l) \delta\{q - a_n\} \\ &= \sum_{n \in \bar{W}} \left\{ \sum_l w_l \delta(n \in W_l) \right\} \delta\{q - a_n\} \\ &= \sum_{n \in \bar{W}} \bar{w}_n \delta\{q - a_n\} \triangleq \mathbf{H}^{\mathbf{w}}\{q\}, \end{aligned} \tag{19}$$

where $\mathbf{H}^{\mathbf{w}}\{q\}$ denotes the weighted histogram over the outline window $\bar{W}$ and $\bar{w}_n$ denotes the weight of the gray level $q = a_n$ in the histogram. This weight is equal to the weighted sum of indicator functions over the windows:

$$\bar{w}_n = \sum_l w_l \delta(n \in W_l). \tag{20}$$

This equation proves the theorem. □

7.2. Computing local weighted variational rows and order statistics

Local weighted histograms can be used as bases for computing local weighted variational rows that are defined as cumulative sum of the weighted histogram and local weighted order statistics, such as weighted median, and their derivatives, such as inter-quantile distances and alike. Local minima and local maxima over composite windows can be found directly from recursively computed local minima and maxima over the window "building-blocks".

8. CONCLUSIONS

We briefly reviewed known methods of efficient recursive computation of image local statistics, such as local moments, histograms and order statistics, and local spectra in uniform windows of different geometrical shapes, and presented a general approach to recursive computation, in different ways of scanning image data, of local statistics in windows of virtually arbitrary shapes and weights. The approach exploits the idea of parallelization of computations by means of decomposition of given arbitrary window functions to a combination of either certain standard uniform windows, such as rectangular, diamond, octagon, diamond and octagon sectors, or of window functions that are basis functions of orthogonal transforms such as DFT, DCT, DcST, that allow recursive computation. Different implementations of the approach to computing local image moments and their derivatives and local histograms and their derivatives are outlined. This opens new opportunities for real-time implementation of many image and video processing algorithms that are based on the image local statistics.

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