Lecture 8 - Convex Optimization

- A convex optimization problem (or just a convex problem) is a problem consisting of minimizing a convex function over a convex set:

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & \mathbf{x} \in C, \end{aligned} \tag{1}$$

  - C - convex set.
  - f - convex function over C.
- A functional form of a convex problem can written as

$$\begin{aligned} \min \quad & f(\mathbf{x}) \\ \text{s.t.} \quad & g_i(\mathbf{x}) \leq 0, \quad i = 1, 2, \ldots, m \\ & h_j(\mathbf{x}) = 0, \quad j = 1, 2, \ldots, p, \end{aligned}$$

  $f, g_1, \ldots, g_m : \mathbb{R}^n \to \mathbb{R}$ are convex functions and $h_1, h_2, \ldots, h_p : \mathbb{R}^m \to \mathbb{R}$ are affine functions.
- Note that the functional form does fit into the general formulation (1).

"Convex Problems are Easy" - Local Minima are Global Minima

Theorem. Let $f : C \to \mathbb{R}$ be a convex function defined on the convex set $C \subseteq \mathbb{R}^n$. Let $\mathbf{x}^* \in C$ be a local minimum of f over C. Then $\mathbf{x}^*$ is a global minimum of f over C.

Proof.

- $\mathbf{x}^*$ is a local minimum of f over $C \Rightarrow \exists r > 0$ such that $f(\mathbf{x}) \geq f(\mathbf{x}^*)$ for any $\mathbf{x} \in C \cap B[\mathbf{x}^*, r]$.
- Let $\mathbf{x}^* \neq \mathbf{y} \in C$. We will show that $f(\mathbf{y}) \geq f(\mathbf{x}^*)$.
- Let $\lambda \in (0, 1)$ be such that $\mathbf{x}^* + \lambda(\mathbf{y} - \mathbf{x}^*) \in B[\mathbf{x}^*, r]$.
- Since $\mathbf{x}^* + \lambda(\mathbf{y} - \mathbf{x}^*) \in B[\mathbf{x}^*, r]$, it follows that $f(\mathbf{x}^*) \leq f(\mathbf{x}^* + \lambda(\mathbf{y} - \mathbf{x}^*))$ and hence by Jensen's inequality:

$$f(\mathbf{x}^*) \leq f(\mathbf{x}^* + \lambda(\mathbf{y} - \mathbf{x}^*)) \leq (1 - \lambda)f(\mathbf{x}^*) + \lambda f(\mathbf{y}).$$

- Thus, the desired inequality $f(\mathbf{x}^*) \leq f(\mathbf{y})$ follows.

More Results

A small variation of the proof of the last theorem yields the following.

Theorem. Let $f : C \to \mathbb{R}$ be a strictly convex function defined on the convex set C. Let $\mathbf{x}^* \in C$ be a local minimum of f over C. Then $\mathbf{x}^*$ is a strict global minimum of f over C.

Another important and easily deduced property of convex problems is that set of optimal solutions is also convex.

Theorem. Let $f : C \to \mathbb{R}$ be a convex function defined over the convex set $C \subseteq \mathbb{R}^n$. Then the set of optimal solutions of the problem

$$\min\{f(\mathbf{x}) : \mathbf{x} \in C\}$$

is convex. If, in addition, f is *strictly* convex over C, then there exists at most one optimal solution of the problem.

Proof. In class

Example

- A Convex Problem:

$$\begin{array}{ll} \min & -2x_1 + x_2 \\ \text{s.t.} & x_1^2 + x_2^2 \leq 3, \end{array}$$

- A Nonconvex Problem:

$$\begin{array}{ll} \min & x_1^2 - x_2 \\ \text{s.t.} & x_1^2 + x_2^2 = 3 \end{array}$$

Linear Programming

$$\text{(LP):} \quad \begin{array}{ll} \min & \mathbf{c}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{b}, \\ & \mathbf{B}\mathbf{x} = \mathbf{g}. \end{array}$$

- A convex optimization problem (constraints and objective function are linear/affine and hence convex).
- It is *also* equivalent to a problem of maximizing a convex (linear) function subject to a convex constraints set. Hence, if the feasible set is compact ans nonempty, then there exists at least one optimal solution which is an extreme point=basic feasible solution.
- A more general result drops the compactness assumption and is often called the fundamental theorem of linear programming.

Convex Quadratic Problems

- Convex quadratic problems are problems consisting of minimizing a convex quadratic function subject to affine constraints.
- The general form is

$$\begin{array}{ll} \min & \mathbf{x}^T\mathbf{Q}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} \\ \text{s.t.} & \mathbf{A}\mathbf{x} \leq \mathbf{c}, \end{array}$$

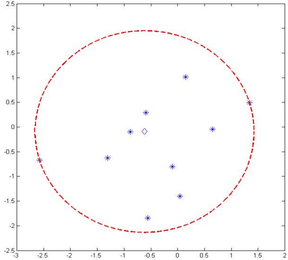
  $\mathbf{Q} \in \mathbb{R}^{n\times n}$ is positive semidefinite, $\mathbf{b} \in \mathbb{R}^n, \mathbf{A} \in \mathbb{R}^{m\times n}, \mathbf{c} \in \mathbb{R}^m$.

Chebyshev Center of a Set of Points

Chebyshev Center Problem. Given m points $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ in $\mathbb{R}^n$. The objective is to find the center of the minimum radius closed ball containing all the points.

- This ball is called the Chebyshev ball and the corresponding center is the Chebyshev center.
- In mathematical terms, the problem can be written as (r is the radius and $\mathbf{x}$ is the center):

$$\begin{array}{ll} \min_{\mathbf{x},r} & r \\ \text{s.t.} & \mathbf{a}_i \in B[\mathbf{x}, r], \quad i = 1, 2, \ldots, m \end{array}$$



- or:

$$\begin{array}{ll} \min_{\mathbf{x},r} & r \\ \text{s.t.} & \|\mathbf{x} - \mathbf{a}_i\| \le r, \quad i = 1, 2, \ldots, m. \end{array}$$

The Portfolio Selection Problem

- We are given n assets numbered as $1, 2, \ldots, n$. Let $Y_j (j = 1, 2, \ldots, n)$ be the RV representing the return from asset j.
- We assume that the expected returns are known:

$$\mu_j = E(Y_j), j = 1, 2, \ldots, n,$$

and that the covariances of all the pairs of variables are also known:

$$\sigma_{i,j} = COV(Y_i, Y_j), \quad i, j = 1, 2, \ldots, n.$$

- $x_j (j = 1, 2, \ldots, n)$ - the proportion of budget invested in asset j. The decision variables are constrained to satisfy $\mathbf{x} \in \Delta_n$.
- The overall return is the random variable:

$$R = \sum_{j=1}^{n} x_j Y_j,$$

whose expectation and variance are given by:

$$\mathbb{E}(R) = \boldsymbol{\mu}^T \mathbf{x}, \mathbb{V}(R) = \mathbf{x}^T \mathbf{C} \mathbf{x},$$

$\boldsymbol{\mu} = (\mu_1, \mu_2, \ldots, \mu_n)^T$ and $\mathbf{C}$ is the covariance matrix: $C_{i,j} = \sigma_{i,j}$

The Markowitz Model

- There are several formulations of the portfolio optimization problem, which are all referred to as the "Markowitz model" after Harry Markowitz (1952).
- Minimizing the risk under the constraint that a minimal return level is guaranteed:

$$\begin{array}{ll} \min & \mathbf{x}^T\mathbf{C}\mathbf{x} \\ \text{s.t} & \boldsymbol{\mu}^T\mathbf{x} \geq \alpha, \\ & \mathbf{e}^T\mathbf{x} = 1, \\ & \mathbf{x} \geq 0, \end{array}$$

- Maximize the expected return subject to a bounded risk constraint:

$$\begin{array}{ll} \max & \boldsymbol{\mu}^T\mathbf{x} \\ \text{s.t} & \mathbf{x}^T\mathbf{C}\mathbf{x} \leq \beta, \\ & \mathbf{e}^T\mathbf{x} = 1, \\ & \mathbf{x} \geq 0, \end{array}$$

- A penalty approach:

$$\begin{array}{ll} \min & -\boldsymbol{\mu}^T\mathbf{x} + \gamma(\mathbf{x}^T\mathbf{C}\mathbf{x}) \\ \text{s.t} & \mathbf{e}^T\mathbf{x} = 1, \\ & \mathbf{x} \geq 0, \end{array}$$

QCQP Problems

Quadratically Constrained Quadratic Problems:

$$
\text{(QCQP)} \quad \begin{array}{ll} \min & \mathbf{x}^T\mathbf{A}_0\mathbf{x} + 2\mathbf{b}_0^T\mathbf{x} + c_0 \\ \text{s.t.} & \mathbf{x}^T\mathbf{A}_i\mathbf{x} + 2\mathbf{b}_i^T\mathbf{x} + c_i \leq 0, \quad i = 1, 2, \ldots, m, \\ & \mathbf{x}^T\mathbf{A}_j\mathbf{x} + 2\mathbf{b}_j^T\mathbf{x} + c_j = 0, \quad j = m+1, m+2, \ldots, m+p. \end{array}
$$

$\mathbf{A}_0, \ldots, \mathbf{A}_{m+p}$- $n \times n$ symmetric, $\mathbf{b}_0, \ldots, \mathbf{b}_{m+p} \in \mathbb{R}^n$, $c_0, \ldots, c_{m+p} \in \mathbb{R}$.

- QCQPs are not necessarily convex problems.
- When there are no equality constrainers ($p = 0$) and all the matrices are positive semidefinite: $\mathbf{A}_i \succeq \mathbf{0}, i = 0, 1, \ldots, m$, the problem is convex, and is therefore called a convex QCQP.

The Orthogonal Projection Operator

- Definition. Given a nonempty closed convex set C, the orthogonal projection operator $P_C : \mathbb{R}^n \to C$ is defined by

$$P_C(\mathbf{x}) = \text{argmin}\{\|\mathbf{y} - \mathbf{x}\|^2 : \mathbf{y} \in C\}.$$

The first important result is that the orthogonal projection exists and is unique.

The First Projection Theorem. Let $C \subseteq \mathbb{R}^n$ be a nonempty closed and convex set. Then for any $\mathbf{x} \in \mathbb{R}^n$, the orthogonal projection $P_C(\mathbf{x})$ exists and is unique.

Proof. In class

Examples

- $C = \mathbb{R}^n_+$.

$$P_{\mathbb{R}^n_+}(\mathbf{x}) = [\mathbf{x}]_+,$$

  where $[\mathbf{v}]_+ = (\max\{v_1, 0\}, \max\{v_2, 0\}, \ldots, \max\{v_n, 0\})^T$.

- A box is a subset of $\mathbb{R}^n$ of the form

$$B = [\ell_1, u_1] \times [\ell_2, u_2] \times \cdots \times [\ell_n, u_n] = \{\mathbf{x} \in \mathbb{R}^n : \ell_i \leq x_i \leq u_i\},$$

  where $\ell_i \leq u_i$ for all $i = 1, 2, \ldots, n$.

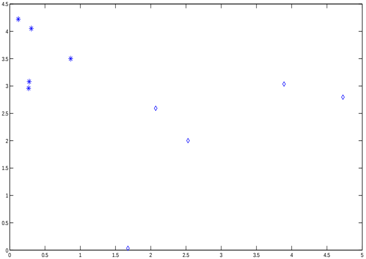
$$[P_B(\mathbf{x})]_i = \begin{cases} u_i & x_i \geq u_i \\ x_i & \ell_i < x_i < u_i, \\ \ell_i & x_i \leq \ell_i. \end{cases}$$

- $C = B[0, r]$.

$$P_{B[\mathbf{0},r]} = \begin{cases} \mathbf{x} & \|\mathbf{x}\| \leq r, \\ r\frac{\mathbf{x}}{\|\mathbf{x}\|} & \|\mathbf{x}\| > r. \end{cases}$$

Linear Classification

- Suppose that we are given two types of points in $\mathbb{R}^n$: type A and type B points.
- $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ - type A.
- $\mathbf{x}_{m+1}, \mathbf{x}_{m+2}, \ldots, \mathbf{x}_{m+p} \in \mathbb{R}^n$ - type B.



The objective is to find a linear separator, which is a hyperplane of the form

$$H(\mathbf{w}, \beta) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{w}^T\mathbf{x} + \beta = 0\}$$

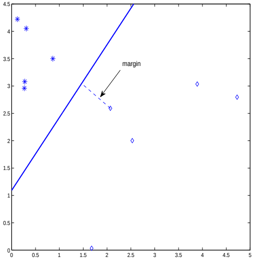
for which the type A and type B points are in its opposite sides:

$$\begin{aligned} \mathbf{w}^T\mathbf{x}_i + \beta &< 0, \quad i = 1, 2, \ldots, m, \\ \mathbf{w}^T\mathbf{x}_i + \beta &> 0, \quad i = m+1, m+2, \ldots, m+p. \end{aligned}$$

Underlying Assumption: the two sets of points are linearly separable, meaning that the set of inequalities has a solution.

Maximizing the Margin

The margin of the separator is the distance of the hyperplane to the closest point.



The separation problem will thus consist of finding the separator with the largest margin.

Lemma. Let $H(\mathbf{a}, b) = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\}$, where $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Let $\mathbf{y} \in \mathbb{R}^n$. Then the distance between $\mathbf{y}$ and the set H is given by

$$d(\mathbf{y}, H(\mathbf{a}, b)) = \frac{|\mathbf{a}^T\mathbf{y} - b|}{\|\mathbf{a}\|}.$$

Proof. Later on in lecture 10.

Mathematical Formulation

- $$\begin{array}{ll} \max & \left\{\min_{i=1,2,\ldots,m+p} \frac{|\mathbf{w}^T\mathbf{x}_i+\beta|}{\|\mathbf{w}\|}\right\} \\ \text{s.t.} & \mathbf{w}^T\mathbf{x}_i + \beta < 0, \quad i = 1,2,\ldots,m, \\ & \mathbf{w}^T\mathbf{x}_i + \beta > 0, \quad i = m+1, m+2, \ldots, m+p. \end{array}$$

  Nonconvex formulation $\Rightarrow$ difficult to handle.
- the problem has a degree of freedom in the sense that if $(\mathbf{w}, \beta)$ is an optimal solution, then so is any nonzero multiplier of it, that is, $(\alpha\mathbf{w}, \alpha\beta)$ for $\alpha \neq 0$. We can therefore decide that

$$\min_{i=1,2,\ldots,m+p} |\mathbf{w}^T\mathbf{x}_i + \beta| = 1,$$

- Thus, the problem can be written as

$$\begin{array}{ll} \max & \left\{\frac{1}{\|\mathbf{w}\|}\right\} \\ \text{s.t.} & \min_{i=1,2,\ldots,m+p} |\mathbf{w}^T\mathbf{x}_i + \beta| = 1, \\ & \mathbf{w}^T\mathbf{x}_i + \beta < 0, \quad i = 1,2,\ldots,m, \\ & \mathbf{w}^T\mathbf{x}_i + \beta > 0, \quad i = m+1, 2, \ldots, m+p. \end{array}$$

Mathematical Formulation Contd.

- $$\begin{array}{ll} \min & \frac{1}{2}\|\mathbf{w}\|^2 \\ \text{s.t.} & \min_{i=1,2,\ldots,m+p} |\mathbf{w}^T\mathbf{x}_i + \beta| = 1, \\ & \mathbf{w}^T\mathbf{x}_i + \beta \le -1, \quad i = 1,2,\ldots,m, \\ & \mathbf{w}^T\mathbf{x}_i + \beta \ge 1, \quad i = m+1,2,\ldots,m+p, \end{array}$$

- The first constraint can be dropped (why?)

$$\begin{array}{ll} \min & \frac{1}{2}\|\mathbf{w}\|^2 \\ \text{s.t.} & \mathbf{w}^T\mathbf{x}_i + \beta \le -1, \quad i = 1,2,\ldots,m, \\ & \mathbf{w}^T\mathbf{x}_i + \beta \ge 1, \quad i = m+1,m+2,\ldots,m+p. \end{array}$$

Convex Formulation.

Hidden Convexity in Trust Region Subproblems

- $$\text{(TRS):} \quad \min\{\mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c : \|\mathbf{x}\|^2 \leq 1\}.$$

  where $\mathbf{b} \in \mathbb{R}^n, c \in \mathbb{R}$ and $\mathbf{A}$ is an $n \times n$ symmetric matrix. **In general, this is a nonconvex problem**
- By the spectral decomposition theorem, there exist an orthogonal matrix $\mathbf{U}$ and a diagonal matrix $\mathbf{D} = \text{diag}(d_1, d_2, \ldots, d_n)$ such that $\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^T$, and hence (TRS) can be rewritten as

  $$\min\{\mathbf{x}^T\mathbf{U}\mathbf{D}\mathbf{U}^T\mathbf{x} + 2\mathbf{b}^T\mathbf{U}\mathbf{U}^T\mathbf{x} + c : \|\mathbf{U}^T\mathbf{x}\|^2 \leq 1\}.$$
- Making the linear change of variables $\mathbf{y} = \mathbf{U}^T\mathbf{x}$, the problem reduces to

  $$\min\{\mathbf{y}^T\mathbf{D}\mathbf{y} + 2\mathbf{b}^T\mathbf{U}\mathbf{y} + c : \|\mathbf{y}\|^2 \leq 1\}.$$
- Denoting $\mathbf{f} = \mathbf{U}^T\mathbf{b}$, we obtain

  $$\begin{array}{ll} \min & \sum_{i=1}^n d_i y_i^2 + 2\sum_{i=1}^n f_i y_i + c \\ \text{s.t.} & \sum_{i=1}^n y_i^2 \leq 1. \end{array} \tag{2}$$

Hidden Convexity in Trust Region Subproblems Contd.

Lemma. Let $\mathbf{y}^*$ be an optimal solution of (2). Then $f_i y_i^* \leq 0$ for all $i = 1, 2, \ldots, n$.

Proof.

- Denote the objective function of (2) by $g(\mathbf{y}) \equiv \sum_{i=1}^n d_i y_i^2 + 2\sum_{i=1}^n f_i y_i + c$.
- Let $i \in \{1, 2, \ldots, n\}$. Define $\tilde{\mathbf{y}}$ as

$$\tilde{y}_j = \begin{cases} y_j^* & j \neq i, \\ -y_i^* & j = i. \end{cases}$$

- $\tilde{\mathbf{y}}$ is feasible and $g(\mathbf{y}^*) \leq g(\tilde{\mathbf{y}})$.
- $\sum_{i=1}^n d_i (y_i^*)^2 + 2\sum_{i=1}^n f_i y_i^* + c \leq \sum_{i=1}^n d_i (\tilde{y}_i)^2 + 2\sum_{i=1}^n f_i \tilde{y}_i + c$.
- After cancelleation of terms, $2f_i y_i^* \leq 2f_i(-y_i^*)$,
- implying the desired inequality $f_i y_i^* \leq 0$.

Hidden Convexity in Trust Region Subproblems Contd.

Back to the TRS problem –

- Make the change of variable $y_i = -\mathrm{sgn}(f_i)\sqrt{z_i}(z_i \geq 0)$.
- problem (2) becomes

$$\begin{array}{ll} \min & \sum_{i=1}^n d_i z_i - 2\sum_{i=1}^n |f_i|\sqrt{z_i} + c \\ \text{s.t.} & \sum_{i=1}^n z_i \leq 1, \\ & z_1, z_2, \ldots, z_n \geq 0. \end{array}$$

- convex optimization problem.