# Lecture 2 - Unconstrained Optimization

Definition **[Global Minimum and Maximum]** Let $f : S \to \mathbb{R}$ be defined on a set $S \subseteq \mathbb{R}^n$. Then

1. $\mathbf{x}^* \in S$ is a global minimum point of $f$ over $S$ if $f(\mathbf{x}) \geq f(\mathbf{x}^*)$ for any $\mathbf{x} \in S$.
2. $\mathbf{x}^* \in S$ is a strict global minimum point of $f$ over $S$ if $f(\mathbf{x}) > f(\mathbf{x}^*)$ for any $\mathbf{x}^* \neq \mathbf{x} \in S$.
3. $\mathbf{x}^* \in S$ is a global maximum point of $f$ over $S$ if $f(\mathbf{x}) \leq f(\mathbf{x}^*)$ for any $\mathbf{x} \in S$.
4. $\mathbf{x}^* \in S$ is a strict global maximum point of $f$ over $S$ if $f(\mathbf{x}) < f(\mathbf{x}^*)$ for any $\mathbf{x}^* \neq \mathbf{x} \in S$.

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- **global optimum**=global minimum or maximum.
- **maximal value of $f$ over $S$**:

$$\sup\{f(\mathbf{x}) : \mathbf{x} \in S\}$$

- **minimal value of $f$ over $S$**:

$$\inf\{f(\mathbf{x}) : \mathbf{x} \in S\}$$

- minimal and maximal values are always unique.

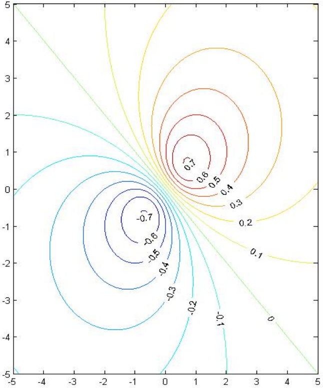
# Example 1:

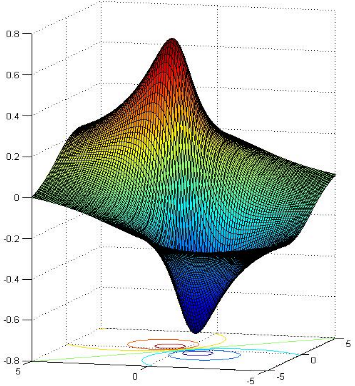
Find the global minimum and maximum points of $f(x, y) = x + y$ over the unit ball $S = B[\mathbf{0}, 1] = \{(x, y)^T : x^2 + y^2 \leq 1\}$

In class.

# Example 2:

$$\min \left\{ f(x,y) = \frac{x+y}{x^2+y^2+1} : x, y \in \mathbb{R} \right\}$$



$(1/\sqrt{2}, 1/\sqrt{2})$ - global maximizer $(-1/\sqrt{2}, -1/\sqrt{2})$ - global minimizer.

# Local Minima and Maxima

**Definition** Let $f : S \to \mathbb{R}$ be defined on a set $S \subseteq \mathbb{R}^n$. Then

1. $\mathbf{x}^* \in S$ is a local minimum of $f$ over $S$ if there exists $r > 0$ for which $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for any $\mathbf{x} \in S \cap B(\mathbf{x}^*, r)$.
2. $\mathbf{x}^* \in S$ is a strict local minimum of $f$ over $S$ if there exists $r > 0$ for which $f(\mathbf{x}^*) < f(\mathbf{x})$ for any $\mathbf{x}^* \neq \mathbf{x} \in S \cap B(\mathbf{x}^*, r)$.
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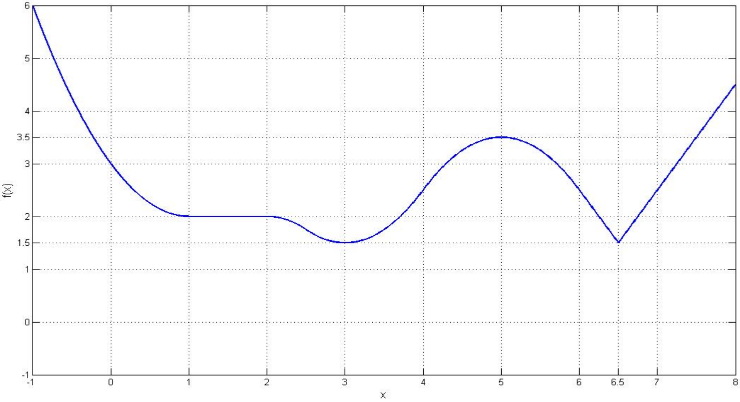
Of course, a global minimum (maximum) point is also a local minimum (maximum) point.

# Example




$f$ described above is defined over $[-1, 8]$. Classify each of the points $x = -1, 1, 2, 3, 5, 6.5, 8$ as strict/nonstrict global/local minimum/maximum points. In class

# Fermat's Theorem - First Order Optimality Condition

**Theorem.** Let $f : U \to \mathbb{R}$ be a function defined on a set $U \subseteq \mathbb{R}^n$. Suppose that $\mathbf{x}^* \in \text{int}(U)$ is a local optimum point and that all the partial derivatives of $f$ exist at $\mathbf{x}^*$. Then $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

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**Proof.**

- Let $i \in \{1, 2, \ldots, n\}$ and consider the 1-D function $g(t) = f(\mathbf{x}^* + t\mathbf{e}_i)$
- $\mathbf{x}^*$ is a local optimum point of $f \Rightarrow t = 0$ is a local optimum of $g \Rightarrow$ $g'(0) = 0$.
- Thus, $\frac{\partial f}{\partial x_i}(\mathbf{x}^*) = g'(0) = 0$.

# Stationary Points

**Definition** Let $f : U \to \mathbb{R}$ be a function defined on a set $U \subseteq \mathbb{R}^n$. Suppose that $\mathbf{x}^* \in \mathrm{int}(U)$ and that all the partial derivatives of $f$ are defined at $\mathbf{x}^*$. Then $\mathbf{x}^*$ is called a stationary point of $f$ if $\nabla f(\mathbf{x}^*) = \mathbf{0}$.

# Example:

$$\min\left\{f(x,y)=\frac{x+y}{x^2+y^2+1} : x,y\in\mathbb{R}\right\}$$

- $$\nabla f(x,y)=\frac{1}{(x^2+y^2+1)^2}\begin{pmatrix}(x^2+y^2+1)-2(x+y)x\\(x^2+y^2+1)-2(x+y)y\end{pmatrix}.$$

# Example:

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- Stationary points are those satisfying:

$$\begin{aligned} -x^2 - 2xy + y^2 &= -1, \\ x^2 - 2xy - y^2 &= -1. \end{aligned}$$

- Hence, the stationary points are $(1/\sqrt{2}, 1/\sqrt{2}), (-1/\sqrt{2}, -1/\sqrt{2})$.
- $(1/\sqrt{2}, 1/\sqrt{2})$ - global maximum, $(-1/\sqrt{2}, -1/\sqrt{2})$ - global minimum.

# Classification of Matrices - Positive Definiteness

1. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called positive semidefinite, denoted by $\mathbf{A} \succeq \mathbf{0}$, if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ for every $\mathbf{x} \in \mathbb{R}^n$.
2. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called positive definite, denoted by $\mathbf{A} \succ \mathbf{0}$, if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for every $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$.

Example 1:

$$\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}.$$

In class

Example 2:

$$\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

In class

Lemma: Let $\mathbf{A}$ be a positive definite (semidefinite) matrix. Then the diagonal elements of $\mathbf{A}$ are positive (nonnegative).

# Negative (Semi)Definiteness, Indefiniteness

1. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called negative semidefinite, denoted by $\mathbf{A} \preceq \mathbf{0}$, if $\mathbf{x}^T\mathbf{A}\mathbf{x} \leq 0$ for every $\mathbf{x} \in \mathbb{R}^n$.
2. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called negative definite, denoted by $\mathbf{A} \prec \mathbf{0}$, if $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$ for every $\mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n$.
3. A symmetric matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is called indefinite if there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0, \mathbf{y}^T\mathbf{A}\mathbf{y} < 0.$.

# Negative (Semi)Definiteness, Indefiniteness

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**Remarks:**

- $\mathbf{A}$ is negative (semi)definite if and only if $-\mathbf{A}$ is positive (semi)definite.
- A matrix is indefinite if and only if it is neither positive semidefinite nor negative semidefinite.

# Eigenvalue Characterization

**Theorem.** Let $\mathbf{A}$ be a symmetric $n \times n$ matrix. Then

(a) $\mathbf{A}$ is positive definite iff all its eigenvalues are positive.

(b) $\mathbf{A}$ is positive semidefinite iff all its eigenvalues are nonnegative.

(c) $\mathbf{A}$ is negative definite iff all its eigenvalues are negative.

(d) $\mathbf{A}$ is negative semidefinite iff all its eigenvalues are nonpositive.

(e) $\mathbf{A}$ is indefinite iff it has at least one positive eigenvalue and at least one negative eigenvalue.

# Eigenvalue Characterization – Proof

**Proof of part (a):** (other parts follows immediately or by similar arguments)

- There exists orthogonal $\mathbf{U} \in \mathbb{R}^{n \times n}$ such that

$$\mathbf{U}^T\mathbf{A}\mathbf{U} = \mathbf{D} \equiv \text{diag}(d_1, d_2, \ldots, d_n)$$

  where $d_i = \lambda_i(\mathbf{A})$.

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- Making the linear change of variables $\mathbf{x} = \mathbf{U}\mathbf{y}$, we have

$$\mathbf{x}^T\mathbf{A}\mathbf{x} = \mathbf{y}^T\mathbf{U}^T\mathbf{A}\mathbf{U}\mathbf{y} = \mathbf{y}^T\mathbf{D}\mathbf{y} = \sum_{i=1}^{n} d_i y_i^2.$$

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- Therefore, $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$ iff

$$\sum_{i=1}^{n} d_i y_i^2 > 0 \text{ for any } \mathbf{y} \neq \mathbf{0}. \quad (1)$$

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- (1) holds iff $d_i > 0$ for all $i$ (why?)

# Trace and Determinant

Corollary. Let $\mathbf{A}$ be a positive semidefinite (definite) matrix. Then $\mathrm{Tr}(\mathbf{A})$ and $\det(\mathbf{A})$ are nonnegative (positive).

**Proof.** In class

Proposition. Let $\mathbf{A}$ be a symmetric $2 \times 2$ matrix. Then $\mathbf{A}$ is positive semidefinite (definite) if and only if $\mathrm{Tr}(\mathbf{A}), \det(\mathbf{A}) \geq 0$ ($\mathrm{Tr}(\mathbf{A}), \det(\mathbf{A}) > 0$).

**Proof.** In class

# Example

Classify the matrices

$$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 1 & 3 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0.1 \end{pmatrix}.$$

In class

# The Principal Minors Criteria

.

Definition Given an $n \times n$ matrix, the determinant of the upper left $k \times k$ submatrix is called the $k$-th principal minor and is denoted by $D_k(\mathbf{A})$.

Example.

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

$$D_1(\mathbf{A}) = a_{11}, D_2(\mathbf{A}) = \det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, D_3(\mathbf{A}) = \det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

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Principal Minors Criteria Let $\mathbf{A}$ be an $n \times n$ symmetric matrix. Then $\mathbf{A}$ is positive definite if and only if $D_1(\mathbf{A}) > 0, D_2(\mathbf{A}) > 0, \ldots, D_n(\mathbf{A}) > 0$.

# Examples

Classify the matrices

$$\mathbf{A} = \begin{pmatrix} 4 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 2 & 4 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} -4 & 1 & 1 \\ 1 & -4 & 1 \\ 1 & 1 & -4 \end{pmatrix}.$$

In class

# Diagonal Dominance

Definition (diagonally dominant matrices) Let **A** be a symmetric $n \times n$ matrix.
(a) **A** is called diagonally dominant if

$$|A_{ii}| \geq \sum_{j \neq i} |A_{ij}| \ \forall i = 1, 2, \ldots, n$$

(b) **A** is called strictly diagonally dominant if

$$|A_{ii}| > \sum_{j \neq i} |A_{ij}| \ \forall i = 1, 2, \ldots, n$$

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Theorem (positive (semi)definiteness of diagonally dominant matrices)

(a) If **A** is symmetric, diagonally dominant with nonnegative diagonal elements, then **A** is positive semidefinite.
(b) If **A** is symmetric, **strictly** diagonally dominant with positive diagonal elements, then **A** is positive definite.

See proof of Theorem 2.25 on pages 22,23.

# Necessary Second Order Optimality Conditions

**Theorem.** Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. Suppose that $f$ is twice continuously differentiable over $U$ and that $\mathbf{x}^*$ is a stationary point. Then

1. if $\mathbf{x}^*$ is a local minimum point, then $\nabla^2 f(\mathbf{x}^*) \succeq \mathbf{0}$.
2. if $\mathbf{x}^*$ is a local maximum point, then $\nabla^2 f(\mathbf{x}^*) \preceq \mathbf{0}$.

**Proof.** of 1:

- There exists a ball $B(\mathbf{x}^*, r) \subseteq U$ for which $f(\mathbf{x}) \geq f(\mathbf{x}^*)$ for all $\mathbf{x} \in B(\mathbf{x}^*, r)$.

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- for any such $\alpha$, $f(\mathbf{x}^*_\alpha) \geq f(\mathbf{x}^*)$.
- On the other hand, there exists a vector $\mathbf{z}_\alpha \in [\mathbf{x}^*, \mathbf{x}^*_\alpha]$ such that

$$f(\mathbf{x}^*_\alpha) - f(\mathbf{x}^*) = \frac{\alpha^2}{2}\mathbf{d}^T \nabla^2 f(\mathbf{z}_\alpha)\mathbf{d}. \quad (2)$$

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- $\Rightarrow$ for any $\alpha \in (0, \frac{r}{\|\mathbf{d}\|})$ the inequality $\mathbf{d}^T \nabla^2 f(\mathbf{z}_\alpha)\mathbf{d} \geq 0$ holds.

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- $\Rightarrow$ for any $\alpha \in (0, \frac{r}{\|\mathbf{d}\|})$ the inequality $\mathbf{d}^T \nabla^2 f(\mathbf{z}_\alpha) \mathbf{d} \geq 0$ holds.
- Since $\mathbf{z}_\alpha \to \mathbf{x}^*$ as $\alpha \to 0^+$, we obtain that $\mathbf{d}^T \nabla f(\mathbf{x}^*) \mathbf{d} \geq 0$.

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- Let $\mathbf{d} \in \mathbb{R}^n$ be a nonzero vector. For any $0 < \alpha < \frac{r}{\|\mathbf{d}\|}$, we have $\mathbf{x}^*_\alpha \equiv \mathbf{x}^* + \alpha \mathbf{d} \in B(\mathbf{x}^*, r)$,
- for any such $\alpha$, $f(\mathbf{x}^*_\alpha) \geq f(\mathbf{x}^*)$.
- On the other hand, there exists a vector $\mathbf{z}_\alpha \in [\mathbf{x}^*, \mathbf{x}^*_\alpha]$ such that

$$f(\mathbf{x}^*_\alpha) - f(\mathbf{x}^*) = \frac{\alpha^2}{2}\mathbf{d}^T \nabla^2 f(\mathbf{z}_\alpha)\mathbf{d}. \tag{2}$$

- $\Rightarrow$ for any $\alpha \in (0, \frac{r}{\|\mathbf{d}\|})$ the inequality $\mathbf{d}^T \nabla^2 f(\mathbf{z}_\alpha)\mathbf{d} \geq 0$ holds.
- Since $\mathbf{z}_\alpha \to \mathbf{x}^*$ as $\alpha \to 0^+$, we obtain that $\mathbf{d}^T \nabla f(\mathbf{x}^*)\mathbf{d} \geq 0$.
- $\Rightarrow \nabla^2 f(\mathbf{x}^*) \succeq \mathbf{0}$.

# Sufficient Second Order Optimality Conditions

**Theorem.** Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. Suppose that $f$ is twice continuously differentiable over $U$ and that $\mathbf{x}^*$ is a stationary point. Then

1. if $\nabla^2 f(\mathbf{x}^*) \succ \mathbf{0}$, then $\mathbf{x}^*$ is a strict local minimum point of $f$ over $U$.
2. if $\nabla^2 f(\mathbf{x}^*) \prec \mathbf{0}$, then $\mathbf{x}^*$ is a strict local maximum point of $f$ over $U$.

**Proof.** of 1: (2 directly follows)

- There exists a ball $B(\mathbf{x}^*, r) \subseteq U$ for which $\nabla^2 f(\mathbf{x}) \succ \mathbf{0}$ for any $\mathbf{x} \in B(\mathbf{x}^*, r)$.

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$$f(\mathbf{x}) - f(\mathbf{x}^*) = \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \nabla^2 f(\mathbf{z}_\mathbf{x})(\mathbf{x} - \mathbf{x}^*).$$

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- $\nabla^2 f(\mathbf{z_x}) \succ \mathbf{0} \Rightarrow$ for any $\mathbf{x} \in B(\mathbf{x}^*, r)$ such that $\mathbf{x} \neq \mathbf{x}^*$, the inequality $f(\mathbf{x}) > f(\mathbf{x}^*)$ holds, implying that $\mathbf{x}^*$ is a strict local minimum point of $f$ over $U$.

# Saddle Points

Definition Let $f : U \to \mathbb{R}$ be a continuously differentiable function defined on an open set $U \subseteq \mathbb{R}^n$. A stationary point $\mathbf{x}^* \in U$ is called a saddle point of $f$ over $U$ if it is neither a local minimum point nor a local maximum point of $f$ over $U$.

# Sufficient Condition for Saddle Points

**Theorem** Let $f : U \to \mathbb{R}$ be a function defined on an open set $U \subseteq \mathbb{R}^n$. Suppose that $f$ is twice continuously differentiable over $U$ and that $\mathbf{x}^*$ is a stationary point. If $\nabla^2 f(\mathbf{x}^*)$ is an indefinite matrix, then $\mathbf{x}^*$ is a saddle point of $f$ over $U$.

[1] quadratic approximation theory

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**Proof.**

- $\nabla^2 f(\mathbf{x}^*)$ has at least one positive eigenvalue $\lambda > 0$, corresponding to a normalized eigenvector denoted by $\mathbf{v}$.

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- $\exists r > 0$ such that $\mathbf{x}^* + \alpha\mathbf{v} \in U$ for any $\alpha \in (0, r)$.

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# Sufficient Condition for Saddle Points

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- $\nabla^2 f(\mathbf{x}^*)$ has at least one positive eigenvalue $\lambda > 0$, corresponding to a normalized eigenvector denoted by $\mathbf{v}$.
- $\exists r > 0$ such that $\mathbf{x}^* + \alpha\mathbf{v} \in U$ for any $\alpha \in (0, r)$.
- BY QAT[1] There exists a function $g : \mathbb{R}_{++} \to \mathbb{R}$ satisfying

$$\frac{g(t)}{t} \to 0 \text{ as } t \to 0,$$

such that for any $\alpha \in (0, r)$

$$f(\mathbf{x}^* + \alpha\mathbf{v}) = f(\mathbf{x}^*) + \frac{\lambda\alpha^2}{2}\|\mathbf{v}\|^2 + g(\|\mathbf{v}\|^2\alpha^2).$$

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# Proof Contd.

- $f(\mathbf{x}^* + \alpha \mathbf{v}) = f(\mathbf{x}^*) + \frac{\lambda \alpha^2}{2} + g(\alpha^2)$.

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- $f(\mathbf{x}^* + \alpha\mathbf{v}) = f(\mathbf{x}^*) + \frac{\lambda\alpha^2}{2} + g(\alpha^2)$.
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# Proof Contd.

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- $\exists \varepsilon_1 \in (0, r)$ such that $g(\alpha^2) > -\frac{\lambda}{2}\alpha^2$ for all $\alpha \in (0, \varepsilon_1)$.
- $\Rightarrow f(\mathbf{x}^* + \alpha\mathbf{v}) > f(\mathbf{x}^*)$ for all $\alpha \in (0, \varepsilon_1)$.

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- $\Rightarrow f(\mathbf{x}^* + \alpha\mathbf{v}) > f(\mathbf{x}^*)$ for all $\alpha \in (0, \varepsilon_1)$.
- $\mathbf{x}^*$ *cannot* be a local maximum point of $f$ over $U$.

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- $\Rightarrow f(\mathbf{x}^* + \alpha\mathbf{v}) > f(\mathbf{x}^*)$ for all $\alpha \in (0, \varepsilon_1)$.
- $\mathbf{x}^*$ *cannot* be a local maximum point of $f$ over $U$.
- Similarly, *cannot* be a local minimum point of $f$ over $U \Rightarrow$ saddle point.

# Attainment of Minimal/Maximal Points

**Weierstrass Theorem** Let $f$ be a continuous function defined over a nonempty compact set $C \subseteq \mathbb{R}^n$. Then there exists a global minimum point of $f$ over $C$ and a global maximum point of $f$ over $C$.

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- When the underlying set is not compact, Weierstrass theorem does not guarantee the attainment of the solution, but certain properties of the function $f$ can imply attainment of the solution.

**Definition** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function over $\mathbb{R}^n$. $f$ is called coercive if

$$\lim_{\|\mathbf{x}\| \to \infty} f(\mathbf{x}) = \infty.$$

**Theorem[Attainment of Global Optima Points for Coercive Functions]** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous and coercive function and let $S \subseteq \mathbb{R}^n$ be a nonempty closed set. Then $f$ attains a global minimum point on $S$.

**Proof.** In class

# Example

Classify the stationary points of the function $f(x_1, x_2) = -2x_1^2 + x_1x_2^2 + 4x_1^4$.

$$\nabla f(\mathbf{x}) = \begin{pmatrix} -4x_1 + x_2^2 + 16x_1^3 \\ 2x_1x_2 \end{pmatrix},$$

$\Rightarrow$ Stationary points are the solutions to

$$\begin{aligned} -4x_1 + x_2^2 + 16x_1^3 &= 0, \\ 2x_1x_2 &= 0. \end{aligned}$$

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$$\nabla^2 f(x_1, x_2) = \begin{pmatrix} -4 + 48x_1^2 & 2x_2 \\ 2x_2 & 2x_1 \end{pmatrix}.$$

$$\nabla^2 f(0.5, 0) = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix}$$

strict local minimum

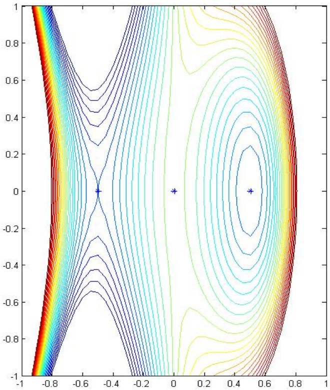
$$\nabla^2 f(-0.5, 0) = \begin{pmatrix} 8 & 0 \\ 0 & -1 \end{pmatrix}$$

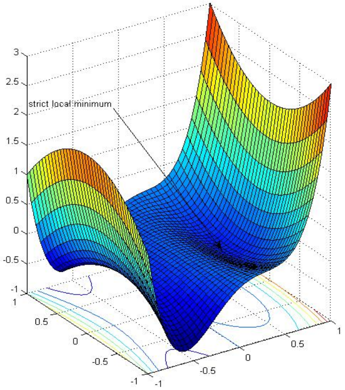
saddle point

$$\nabla^2 f(0, 0) = \begin{pmatrix} -4 & 0 \\ 0 & 0 \end{pmatrix}$$

saddle point (why?)

# Illustration



# Global Optimality Conditions

No free meals...

**Theorem.** Let $f$ be a twice continuously defined over $\mathbb{R}^n$. Suppose that $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for any $\mathbf{x} \in \mathbb{R}^n$. Let $\mathbf{x}^* \in \mathbb{R}^n$ be a stationary point of $f$. Then $\mathbf{x}^*$ is a global minimum point of $f$.

**Proof.** In class

# Example

$$f(\mathbf{x}) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 + (x_1^2 + x_2^2 + x_3^2)^2.$$

$$\nabla f(\mathbf{x}) = \begin{pmatrix} 2x_1 + x_2 + x_3 + 4x_1(x_1^2 + x_2^2 + x_3^2) \\ 2x_2 + x_1 + x_3 + 4x_2(x_1^2 + x_2^2 + x_3^2) \\ 2x_3 + x_1 + x_2 + 4x_3(x_1^2 + x_2^2 + x_3^2) \end{pmatrix}.$$

$$\nabla^2 f(\mathbf{x}) = \begin{pmatrix} 2 + 4(x_1^2 + x_2^2 + x_3^2) + 8x_1^2 & 1 + 8x_1x_2 & 1 + 8x_1x_3 \\ 1 + 8x_1x_2 & 2 + 4(x_1^2 + x_2^2 + x_3^2) + 8x_2^2 & 1 + 8x_2x_3 \\ 1 + 8x_1x_3 & 1 + 8x_2x_3 & 2 + 4(x_1^2 + x_2^2 + x_3^2) + 8x_3^2 \end{pmatrix}.$$

- $\mathbf{x} = \mathbf{0}$ is a stationary point.
- $\nabla^2 f(\mathbf{x}) \succeq \mathbf{0}$ for all $\mathbf{x}$. (why?)
- Consequence: $\mathbf{x} = \mathbf{0}$ is the global minimum point.

# Quadratic Functions

- A quadratic function over $\mathbb{R}^n$ is a function of the form

$$f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c,$$

  where $\mathbf{A} \in \mathbb{R}^{n \times n}$ is symmetric, $\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.
- 

$$\begin{aligned} \nabla f(\mathbf{x}) &= 2\mathbf{A}\mathbf{x} + 2\mathbf{b}, \\ \nabla^2 f(\mathbf{x}) &= 2\mathbf{A}. \end{aligned}$$

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- 

$$\begin{aligned} \nabla f(\mathbf{x}) &= 2\mathbf{A}\mathbf{x} + 2\mathbf{b}, \\ \nabla^2 f(\mathbf{x}) &= 2\mathbf{A}. \end{aligned}$$

Consequently,

**Lemma** Let $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$ ($\mathbf{A} \in \mathbb{R}^{n\times n}$ sym.,$\mathbf{b} \in \mathbb{R}^n, c \in \mathbb{R}$).

1. $\mathbf{x}$ is a stationary point of $f$ iff $\mathbf{A}\mathbf{x} = -\mathbf{b}$.
2. if $\mathbf{A} \succeq \mathbf{0}$, then $\mathbf{x}$ is a global minimum point of $f$ iff $\mathbf{A}\mathbf{x} = -\mathbf{b}$.
3. if $\mathbf{A} \succ \mathbf{0}$, then $\mathbf{x} = -\mathbf{A}^{-1}\mathbf{b}$ is a strict global minimum point of $f$.

**Proof.** In class

# Two Important Theorems on Quadratic Functions

**Lemma [coerciveness of quadratic functions]** Let $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$ where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric, $\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then $f$ is coercive if and only if $\mathbf{A} \succ \mathbf{0}$.

Lemma 2.42 in the book (proof in page 33).

**Theorem [characterization of the nonnegativity of quadratic functions]** $f(\mathbf{x}) = \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c$, where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is symmetric, $\mathbf{b} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then the following two claims are equivalent

(i) $f(\mathbf{x}) \equiv \mathbf{x}^T\mathbf{A}\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + c \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

(ii) $\begin{pmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{b}^T & c \end{pmatrix} \succeq \mathbf{0}$.

Theorem 2.43 in the book (proof in pages 33,34).