Lecture 10 - Linearly Constrained Problems: Separation $\rightarrow$ Alternative Theorems $\rightarrow$ Optimality Conditions

- A hyperplane

$$H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\} \;\; (\mathbf{a} \in \mathbb{R}^n \backslash \{\mathbf{0}\}, b \in \mathbb{R})$$

is said to strictly separate a point $\mathbf{y} \notin S$ from S if

$$\mathbf{a}^T\mathbf{y} > b$$

and

$$\mathbf{a}^T\mathbf{x} \leq b \text{ for all } \mathbf{y} \in S.$$

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is said to strictly separate a point $\mathbf{y} \notin S$ from S if

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Theorem (separation of a point from a closed and convex set) Let $C \subseteq \mathbb{R}^n$ be a nonempty closed and convex set, and let $\mathbf{y} \notin C$. Then there exists $\mathbf{p} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$ such that

$$\mathbf{p}^T\mathbf{y} > \alpha \text{ and } \mathbf{p}^T\mathbf{x} \leq \alpha \text{ for all } \mathbf{x} \in C.$$

Proof of the Separation Theorem

- By the second orthogonal projection theorem, the vector $\bar{\mathbf{x}} = P_C(\mathbf{y}) \in C$ satisfies

$$(\mathbf{y} - \bar{\mathbf{x}})^T(\mathbf{x} - \bar{\mathbf{x}}) \leq 0 \text{ for all } \mathbf{x} \in C,$$

which is the same as

$$(\mathbf{y} - \bar{\mathbf{x}})^T\mathbf{x} \leq (\mathbf{y} - \bar{\mathbf{x}})^T\bar{\mathbf{x}} \text{ for all } \mathbf{x} \in C.$$

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- Denote $\mathbf{p} = \mathbf{y} - \bar{\mathbf{x}} \neq \mathbf{0}$ and $\alpha = (\mathbf{y} - \bar{\mathbf{x}})^T\bar{\mathbf{x}}$. Then

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- On the other hand,

$$\mathbf{p}^T\mathbf{y} = (\mathbf{y} - \bar{\mathbf{x}})^T\mathbf{y} = (\mathbf{y} - \bar{\mathbf{x}})^T(\mathbf{y} - \bar{\mathbf{x}}) + (\mathbf{y} - \bar{\mathbf{x}})^T\bar{\mathbf{x}} = \|\mathbf{y} - \bar{\mathbf{x}}\|^2 + \alpha > \alpha.$$

Farkas Lemma - an Alternative Theorem

Farkas Lemma. Let $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then exactly one of the following systems has a solution

I. $\mathbf{A}\mathbf{x} \leq \mathbf{0}, \mathbf{c}^T\mathbf{x} > 0$.

II. $\mathbf{A}^T\mathbf{y} = \mathbf{c}, \mathbf{y} \geq 0$.

Another equivalent formulation is the following.

Farkas Lemma - second Formulation Let $\mathbf{c} \in \mathbb{R}^n$ and $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then the following two claims are equivalent:

(A) The implication $\mathbf{A}\mathbf{x} \leq \mathbf{0} \Rightarrow \mathbf{c}^T\mathbf{x} \leq 0$ holds true.

(B) There exists $\mathbf{y} \in \mathbb{R}^m_+$ such that $\mathbf{A}^T\mathbf{y} = \mathbf{c}$.

What does it mean?

Example. $\mathbf{A} = \begin{pmatrix} 1 & 5 \\ -1 & 2 \end{pmatrix}, \mathbf{c} = \begin{pmatrix} -1 \\ 9 \end{pmatrix},$

Proof of Farkas Lemma

- Suppose that system (B) is feasible: $\exists \mathbf{y} \in \mathbb{R}^m_+$ such that $\mathbf{A}^T\mathbf{y} = \mathbf{c}$.

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- Multiplying this inequality from the left by $\mathbf{y}^T$:

$$\mathbf{y}^T\mathbf{A}\mathbf{x} \leq 0.$$

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- Suppose that the implication (A) is satisfied, and let us show that the system (B) is feasible. Suppose in contradiction that system (B) is infeasible.

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- Consider the following closed and convex (why?) set

$$S = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} = \mathbf{A}^T\mathbf{y} \text{ for some } \mathbf{y} \in \mathbb{R}^m_+\}$$

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- $\mathbf{c} \notin S$.

Proof Contd.

- By the separation theorem $\exists \mathbf{p} \in \mathbb{R}^n \backslash \{\mathbf{0}\}$ and $\alpha \in \mathbb{R}$ such that $\mathbf{p}^T \mathbf{c} > \alpha$ and

$$\mathbf{p}^T \mathbf{x} \leq \alpha \text{ for all } \mathbf{x} \in S. \tag{1}$$

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- $\mathbf{0} \in S \Rightarrow \alpha \geq 0 \Rightarrow \mathbf{p}^T \mathbf{c} > 0$.

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- $\mathbf{0} \in S \Rightarrow \alpha \geq 0 \Rightarrow \mathbf{p}^T\mathbf{c} > 0$.
- (1) is equivalent to

$$\mathbf{p}^T\mathbf{A}^T\mathbf{y} \leq \alpha \text{ for all } \mathbf{y} \geq \mathbf{0}$$

  or to

$$(\mathbf{A}\mathbf{p})^T\mathbf{y} \leq \alpha \text{ for all } \mathbf{y} \geq \mathbf{0}, \tag{2}$$

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- Therefore, $\mathbf{A}\mathbf{p} \leq \mathbf{0}$.

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- $\mathbf{0} \in S \Rightarrow \alpha \geq 0 \Rightarrow \mathbf{p}^T \mathbf{c} > 0$.
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or to
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- Therefore, $\mathbf{A}\mathbf{p} \leq \mathbf{0}$.
- Contradiction to the assertion that implication (A) holds.

Gordan's Alternative Theorem

Theorem. Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. Then exactly one of the following two systems has a solution.

(A) $\mathbf{Ax} < \mathbf{0}$.

(B) $\mathbf{p} \neq 0, \mathbf{A}^T\mathbf{p} = \mathbf{0}, \mathbf{p} \geq \mathbf{0}$.

Proof.

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- Suppose that system (A) does not have a solution.

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- Suppose that system (A) does not have a solution.
- System (A) is equivalent to (s is a scalar) to $\mathbf{A}\mathbf{x} + s\mathbf{e} \leq \mathbf{0}, s > 0$.

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- Suppose that system (A) does not have a solution.
- System (A) is equivalent to (s is a scalar) to $\mathbf{Ax} + s\mathbf{e} \leq \mathbf{0}, s > 0$.
- or to $\tilde{\mathbf{A}}\begin{pmatrix}\mathbf{x}\\ s\end{pmatrix} \leq \mathbf{0}, \mathbf{c}^T\begin{pmatrix}\mathbf{x}\\ s\end{pmatrix} > 0$, where $\tilde{\mathbf{A}} = \begin{pmatrix}\mathbf{A} & \mathbf{e}\end{pmatrix}$ and $\mathbf{c} = \mathbf{e}_{n+1}$.

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- System (A) is equivalent to (s is a scalar) to $\mathbf{A}\mathbf{x} + s\mathbf{e} \leq \mathbf{0}, s > 0$.
- or to $\tilde{\mathbf{A}} \begin{pmatrix} \mathbf{x} \\ s \end{pmatrix} \leq \mathbf{0}, \mathbf{c}^T \begin{pmatrix} \mathbf{x} \\ s \end{pmatrix} > 0$, where $\tilde{\mathbf{A}} = \begin{pmatrix} \mathbf{A} & \mathbf{e} \end{pmatrix}$ and $\mathbf{c} = \mathbf{e}_{n+1}$.
- The infeasibility of (A) is thus equivalent to the infeasibility of the system

$$\tilde{\mathbf{A}}\mathbf{w} \leq \mathbf{0}, \mathbf{c}^T\mathbf{w} > 0, \mathbf{w} \in \mathbb{R}^{n+1}.$$

Proof of Gordan Contd.

- By Farkas' lemma, $\exists \mathbf{z} \in \mathbb{R}^m_+$ such that

$$\begin{pmatrix} \mathbf{A}^T \\ \mathbf{e}^T \end{pmatrix} \mathbf{z} = \mathbf{c}$$

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- $\Leftrightarrow \exists \mathbf{z} \in \mathbb{R}^m_+ : \mathbf{A}^T\mathbf{z} = \mathbf{0}, \mathbf{e}^T\mathbf{z} = 1.$
- $\Leftrightarrow \exists \mathbf{0} \neq \mathbf{z} \in \mathbb{R}^m_+ : \mathbf{A}^T\mathbf{z} = \mathbf{0}.$

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- $\Leftrightarrow \exists \mathbf{0} \neq \mathbf{z} \in \mathbb{R}^m_+ : \mathbf{A}^T\mathbf{z} = \mathbf{0}.$
- $\Rightarrow$ System (B) is feasible.

KKT Conditions for Linearly Constrained Problems

Theorem (KKT conditions for linearly constrained problems - necessary optimality conditions)

Consider the minimization problem

$$\text{(P)} \quad \begin{array}{ll} \min & f(\mathbf{x}), \\ \text{s.t.} & \mathbf{a}_i^T \mathbf{x} \le b_i, i = 1, 2, \ldots, m \end{array}$$

where f is continuously differentiable over $\mathbb{R}^n$, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n, b_1, b_2, \ldots, b_m \in \mathbb{R}$ and let $\mathbf{x}^*$ be a local minimum point of (P). Then there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \ge 0$ such that

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i \mathbf{a}_i = \mathbf{0}. \tag{3}$$

and

$$\lambda_i (\mathbf{a}_i^T \mathbf{x}^* - b_i) = 0, \quad i = 1, 2, \ldots, m. \tag{4}$$

Proof of KKT Theorem

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- $\mathbf{x}^*$ is a local minimum $\Rightarrow$ $\mathbf{x}^*$ is a stationary point.
- $\nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) \geq 0$ for every $\mathbf{x} \in \mathbb{R}^n$ satisfying $\mathbf{a}_i^T\mathbf{x} \leq b_i$ for any $i = 1, 2, \ldots, m$.

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- Denote the set of *active* constraints by

$$I(\mathbf{x}^*) = \{i : \mathbf{a}_i^T\mathbf{x}^* = b_i\}.$$

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- Denote the set of *active* constraints by

$$I(\mathbf{x}^*) = \{i : \mathbf{a}_i^T\mathbf{x}^* = b_i\}.$$

- Making the change of variables $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$, we have

$$\nabla f(\mathbf{x}^*)^T\mathbf{y} \geq 0 \text{ for any } \mathbf{y} \in \mathbb{R}^m \text{ satisfying } \mathbf{a}_i^T(\mathbf{y} + \mathbf{x}^*) \leq b_i, i = 1, 2, \ldots, m.$$

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- or $\nabla f(\mathbf{x}^*)^T \mathbf{y} \geq 0$ for any $\mathbf{y}$ satisfying

$$\begin{array}{ll} \mathbf{a}_i^T \mathbf{y} \leq 0 & i \in I(\mathbf{x}^*), \\ \mathbf{a}_i^T \mathbf{y} \leq b_i - \mathbf{a}_i^T \mathbf{x}^* & i \notin I(\mathbf{x}^*). \end{array}$$

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- The second set of inequalities can be removed, that is, we will prove that

$$\mathbf{a}_i^T\mathbf{y} \leq 0 \text{ for all } i \in I(\mathbf{x}^*) \Rightarrow \nabla f(\mathbf{x}^*)^T\mathbf{y} \geq 0.$$

Proof Contd.

- Suppose then that $\mathbf{y}$ satisfies $\mathbf{a}_i^T\mathbf{y} \leq 0$ for all $i \in I(\mathbf{x}^*)$

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- Defining $\lambda_i = 0$ for all $i \notin I(\mathbf{x}^*)$ we get that $\lambda_i(\mathbf{a}_i^T\mathbf{x}^* - b_i) = 0$ for all $i \in \{1, 2, \ldots, m\}$ and

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i\mathbf{a}_i = \mathbf{0}.$$

The Convex Case

Theorem [KKT conditions for convex linearly constrained problems - necessary and sufficient optimality conditions]
Consider the minimization problem

$$(\mathrm{P}) \quad \begin{array}{ll} \min & f(\mathbf{x}), \\ \text{s.t.} & \mathbf{a}_i^T\mathbf{x} \le b_i, i = 1, 2, \ldots, m \end{array}$$

where f is a convex continuously differentiable function over $\mathbb{R}^n$, $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m \in \mathbb{R}^n, b_1, b_2, \ldots, b_m \in \mathbb{R}$ and let $\mathbf{x}^*$ be a feasible solution of (P). Then $\mathbf{x}^*$ is an optimal solution if and only if there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \ge 0$ such that

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i \mathbf{a}_i = \mathbf{0}. \tag{5}$$

and

$$\lambda_i(\mathbf{a}_i^T\mathbf{x}^* - b_i) = 0, \quad i = 1, 2, \ldots, m. \tag{6}$$

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$$h(\mathbf{x}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i(\mathbf{a}_i^T \mathbf{x} - b_i).$$

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$$f(\mathbf{x}^*) = f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i(\mathbf{a}_i^T\mathbf{x}^* - b_i) \leq f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i(\mathbf{a}_i^T\mathbf{x} - b_i) \leq f(\mathbf{x}),$$

Problems with Equality and Inequality Constraints

Theorem [KKT conditions for linearly constrained problems]
Consider the minimization problem

$$(\mathrm{Q}) \quad \begin{array}{ll} \min & f(\mathbf{x}), \\ \text{s.t.} & \mathbf{a}_i^T\mathbf{x} \le b_i, i = 1, 2, \ldots, m, \\ & \mathbf{c}_j^T\mathbf{x} = d_j, \quad j = 1, 2, \ldots, p. \end{array}$$

where f cont. dif., $\mathbf{a}_i, \mathbf{c}_j \in \mathbb{R}^n$, $b_i, d_j \in \mathbb{R}$.

(i) **(necessity of the KKT conditions)** If $\mathbf{x}^*$ is a local minimum of (Q), then there exist $\lambda_1, \lambda_2, \ldots, \lambda_m \ge 0$ and $\mu_1, \mu_2, \ldots, \mu_p \in \mathbb{R}$ such that

$$\nabla f(\mathbf{x}^*) + \sum_{i=1}^m \lambda_i \mathbf{a}_i + \sum_{j=1}^p \mu_j \mathbf{c}_j = \mathbf{0}, \tag{7}$$

$$\lambda_i(\mathbf{a}_i^T\mathbf{x}^* - b_i) = 0, \quad i = 1, 2, \ldots, m. \tag{8}$$

(ii) **(sufficiency in the convex case)** If f is convex over $\mathbb{R}^n$ and $\mathbf{x}^*$ is a feasible solution of (Q) for which there exist $\lambda_1, \ldots, \lambda_m \ge 0$ and $\mu_1, \ldots, \mu_p \in \mathbb{R}$ such that (7) and (8) are satisfied, then $\mathbf{x}^*$ is an optimal solution of (Q).

Representation Via the Lagrangian

Given the a problem

$$\text{(NLP)} \quad \begin{array}{ll} \min & f(\mathbf{x}) \\ \text{s.t.} & g_i(\mathbf{x}) \leq 0, i = 1, 2, \ldots, m, \\ & h_j(\mathbf{x}) = 0, j = 1, 2, \ldots, p. \end{array}$$

The associated Lagrangian function os

$$L(\mathbf{x}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = f(\mathbf{x}) + \sum_{i=1}^{m} \lambda_i g_i(\mathbf{x}) + \sum_{j=1}^{p} \mu_j h_j(\mathbf{x}).$$

The KKT conditions can be written as

$$\begin{aligned} \nabla_{\mathbf{x}} L(\mathbf{x}^*, \boldsymbol{\lambda}, \boldsymbol{\mu}) &= \nabla f(\mathbf{x}^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\mathbf{x}^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(\mathbf{x}^*) = \mathbf{0} \\ \lambda_i g_i(\mathbf{x}^*) &= 0, \quad i = 1, 2, \ldots, m. \end{aligned}$$

Examples

- $$\begin{array}{ll} \min & \frac{1}{2}(x_1^2 + x_2^2 + x_3^2) \\ \text{s.t.} & x_1 + x_2 + x_3 = 3. \end{array}$$
- $$\begin{array}{ll} \min & x_1^2 + 2x_2^2 + 4x_1x_2 \\ \text{s.t.} & x_1 + x_2 = 1, \\ & x_1, x_2 \geq 0. \end{array}$$

In class

Projection onto Affine Spaces

Lemma. Let C be the affine space

$$C = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{A}\mathbf{x} = \mathbf{b}\},$$

where $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^m$. Then

$$P_C(\mathbf{y}) = \mathbf{y} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{y} - \mathbf{b}).$$

Proof. In class

Orthogonal Projection onto Hyperplanes

Consider the hyperplane

$$H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\} \quad (\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}).$$

Then by the previous slide:

$$P_H(\mathbf{y}) = \mathbf{y} - \mathbf{a}(\mathbf{a}^T\mathbf{a})^{-1}(\mathbf{a}^T\mathbf{y} - b) = \mathbf{y} - \frac{\mathbf{a}^T\mathbf{y} - b}{\|\mathbf{a}\|^2}\mathbf{a}.$$

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Lemma **(distance of a point from a hyperplane)** Let $H = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} = b\}$, where $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$. Then

$$d(\mathbf{y}, H) = \frac{|\mathbf{a}^T\mathbf{y} - b|}{\|\mathbf{a}\|}.$$

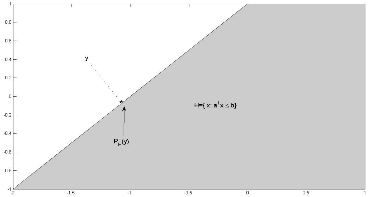
Proof.

$$d(\mathbf{y}, H) = \|\mathbf{y} - P_H(\mathbf{y})\| = \left\|\mathbf{y} - \left(\mathbf{y} - \frac{\mathbf{a}^T\mathbf{y} - b}{\|\mathbf{a}\|^2}\mathbf{a}\right)\right\| = \frac{|\mathbf{a}^T\mathbf{y} - b|}{\|\mathbf{a}\|}.$$

Orthogonal Projection onto Half-Spaces

Let $H^- = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}^T\mathbf{x} \leq b\}$,
where $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n$ and $b \in \mathbb{R}$.
Then

$$P_{H^-}(\mathbf{x}) = \mathbf{x} - \frac{[\mathbf{a}^T\mathbf{x} - b]_+}{\|\mathbf{a}\|^2}\mathbf{a}$$



Proof. In class

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$$H_{\mathbf{x},y} := \left\{ \mathbf{a} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{a} = y \right\}.$$

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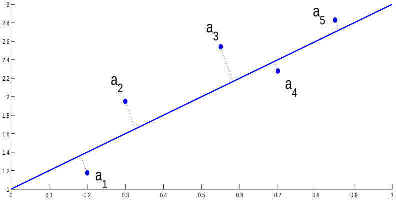

- In the orthogonal regression problem we seek to find a nonzero vector $\mathbf{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$ such that the sum of squared Euclidean distances between the points $\mathbf{a}_1, \ldots, \mathbf{a}_m$ to $H_{\mathbf{x},y}$ is minimal:

$$\min_{\mathbf{x},y} \left\{ \sum_{i=1}^{m} d(\mathbf{a}_i, H_{\mathbf{x},y})^2 : \mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n, y \in \mathbb{R} \right\}.$$

Orthogonal Regression

- $d(\mathbf{a}_i, H_{\mathbf{x},y})^2 = \frac{(\mathbf{a}_i^T\mathbf{x}-y)^2}{\|\mathbf{x}\|^2}, \quad i = 1, \ldots, m.$

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- The Orthogonal Regression problem is the same as

$$\min \left\{ \sum_{i=1}^{m} \frac{(\mathbf{a}_i^T\mathbf{x} - y)^2}{\|\mathbf{x}\|^2} : \mathbf{0} \neq \mathbf{x} \in \mathbb{R}^n, y \in \mathbb{R} \right\}.$$

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- Fixing $\mathbf{x}$ and minimizing first with respect to y we obtain that the optimal y is given by $y = \frac{1}{m}\sum_{i=1}^{m} \mathbf{a}_i^T\mathbf{x} = \frac{1}{m}\mathbf{e}^T\mathbf{A}\mathbf{x}$.

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- Using the above expression for y we obtain that

$$\begin{aligned}
\sum_{i=1}^{m}\left(\mathbf{a}_i^T\mathbf{x}-y\right)^2 &= \sum_{i=1}^{m}\left(\mathbf{a}_i^T\mathbf{x}-\frac{1}{m}\mathbf{e}^T\mathbf{A}\mathbf{x}\right)^2 \\
&= \sum_{i=1}^{m}(\mathbf{a}_i^T\mathbf{x})^2 - \frac{2}{m}\sum_{i=1}^{m}(\mathbf{e}^T\mathbf{A}\mathbf{x})(\mathbf{a}_i^T\mathbf{x}) + \frac{1}{m}(\mathbf{e}^T\mathbf{A}\mathbf{x})^2 \\
&= \sum_{i=1}^{m}(\mathbf{a}_i^T\mathbf{x})^2 - \frac{1}{m}(\mathbf{e}^T\mathbf{A}\mathbf{x})^2 = \|\mathbf{A}\mathbf{x}\|^2 - \frac{1}{m}(\mathbf{e}^T\mathbf{A}\mathbf{x})^2 \\
&= \mathbf{x}^T\mathbf{A}^T\left(\mathbf{I}_m - \frac{1}{m}\mathbf{e}\mathbf{e}^T\right)\mathbf{A}\mathbf{x}.
\end{aligned}$$

Orthogonal Regression

- Therefore, a reformulation of the problem is

$$\min_{\mathbf{x}} \left\{ \frac{\mathbf{x}^T[\mathbf{A}^T(\mathbf{I}_m - \frac{1}{m}\mathbf{e}\mathbf{e}^T)\mathbf{A}]\mathbf{x}}{\|\mathbf{x}\|^2} : \mathbf{x} \neq \mathbf{0} \right\}.$$

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Proposition. An optimal solution of the orthogonal regression problem $(\mathbf{x}, y)$ where $\mathbf{x}$ is an eigenvector of $\mathbf{A}^T(\mathbf{I}_m - \frac{1}{m}\mathbf{e}\mathbf{e}^T)\mathbf{A}$ associated with the minimum eigenvalue and $y = \frac{1}{m}\sum_{i=1}^m \mathbf{a}_i^T\mathbf{x}$. The optimal function value of the problem is $\lambda_{\min}\left[\mathbf{A}^T(\mathbf{I}_m - \frac{1}{m}\mathbf{e}\mathbf{e}^T)\mathbf{A}\right]$.