

### Strong Subdifferential Results

| $f(\mathbf{x})$                                     | $\partial f(\mathbf{x})$   | assumptions  |
|---|--|--|
| $\ \mathbf{x}\ $                                    | $B_{\ \cdot\ _*}[\mathbf{0}, 1]$   | $\mathbf{x} = \mathbf{0}$  |
| $\ \mathbf{x}\ _1$                                  | $\left\{ \sum_{i \in I_{\neq}(\mathbf{x})} \text{sgn}(x_i) \mathbf{e}_i + \sum_{i \in I_0(\mathbf{x})} [-\mathbf{e}_i, \mathbf{e}_i] \right\}$   | $\mathbb{E} = \mathbb{R}^n$ , $I_{\neq}(\mathbf{x}) = \{i : x_i \neq 0\}$ , $I_0(\mathbf{x}) = \{i : x_i = 0\}$ .  |
| $\ \mathbf{x}\ _2$                                  | $\left\{ \begin{array}{l} \left\{ \frac{\mathbf{x}}{\ \mathbf{x}\ _2} \right\}, \quad \mathbf{x} \neq \mathbf{0}, \\ B_{\ \cdot\ _2}[\mathbf{0}, 1], \quad \mathbf{x} = \mathbf{0}. \end{array} \right.$   | $\mathbb{E} = \mathbb{R}^n$  |
| $\ \mathbf{x}\ _{\infty}$                           | $\left\{ \sum_{i \in I(\mathbf{x})} \lambda_i \text{sgn}(x_i) \mathbf{e}_i : \begin{array}{l} \sum_{i \in I(\mathbf{x})} \lambda_i = 1 \\ \lambda_i \geq 0 \end{array} \right.$  | $\mathbb{E} = \mathbb{R}^n$ , $I(\mathbf{x}) = \{i : \ \mathbf{x}\ _{\infty} =  x_i \}$ , $\mathbf{x} \neq \mathbf{0}$   |
| $\max(\mathbf{x})$                                  | $\left\{ \sum_{i \in I(\mathbf{x})} \lambda_i \mathbf{e}_i : \sum_{i \in I(\mathbf{x})} \lambda_i = 1, \lambda_i \geq 0 \right.$   | $\mathbb{E} = \mathbb{R}^n$ , $I(\mathbf{x}) = \{i : \max(\mathbf{x}) = x_i\}$   |
| $\max(\mathbf{x})$                                  | $\Delta_n$   | $\mathbb{E} = \mathbb{R}^n$ , $\mathbf{x} = \alpha \mathbf{e}$ for some $\alpha \in \mathbb{R}$  |
| $\delta_S(\mathbf{x})$                              | $N_S(\mathbf{x})$  | $\emptyset \neq S \subseteq \mathbb{E}$  |
| $\delta_{B[0,1]}(\mathbf{x})$                       | $\left\{ \begin{array}{l} \{\mathbf{y} \in \mathbb{E}^* : \ \mathbf{y}\ _* \leq \langle \mathbf{y}, \mathbf{x} \rangle\}, \quad \ \mathbf{x}\  \leq 1, \\ \emptyset, \quad \ \mathbf{x}\  > 1. \end{array} \right.$  |  |
| $\ \mathbf{Ax} + \mathbf{b}\ _1$                    | $\sum_{i \in I_{\neq}(\mathbf{x})} \text{sgn}(\mathbf{a}_i^T \mathbf{x} + b_i) \mathbf{a}_i + \sum_{i \in I_0(\mathbf{x})} [-\mathbf{a}_i, \mathbf{a}_i]$  | $\mathbb{E} = \mathbb{R}^n$ , $\mathbf{A} \in \mathbb{R}^{m \times n}$ , $\mathbf{b} \in \mathbb{R}^m$ , $I_{\neq}(\mathbf{x}) = \{i : \mathbf{a}_i^T \mathbf{x} + b_i \neq 0\}$ , $I_0(\mathbf{x}) = \{i : \mathbf{a}_i^T \mathbf{x} + b_i = 0\}$         |
| $\ \mathbf{Ax} + \mathbf{b}\ _2$                    | $\left\{ \begin{array}{l} \frac{\mathbf{A}^T(\mathbf{Ax} + \mathbf{b})}{\ \mathbf{Ax} + \mathbf{b}\ _2}, \quad \mathbf{Ax} + \mathbf{b} \neq \mathbf{0}, \\ \mathbf{A}^T B_{\ \cdot\ _2}[\mathbf{0}, 1], \quad \mathbf{Ax} + \mathbf{b} = \mathbf{0}. \end{array} \right.$ | $\mathbb{E} = \mathbb{R}^n$ , $\mathbf{A} \in \mathbb{R}^{m \times n}$ , $\mathbf{b} \in \mathbb{R}^m$   |
| $\ \mathbf{Ax} + \mathbf{b}\ _{\infty}$             | $\left\{ \sum_{i \in I_{\mathbf{x}}} \lambda_i \text{sgn}(\mathbf{a}_i^T \mathbf{x} + b_i) \mathbf{a}_i : \begin{array}{l} \sum_{i \in I_{\mathbf{x}}} \lambda_i = 1 \\ \lambda_i \geq 0 \end{array} \right.$  | $\mathbb{E} = \mathbb{R}^n$ , $\mathbf{A} \in \mathbb{R}^{m \times n}$ , $\mathbf{b} \in \mathbb{R}^m$ , $I_{\mathbf{x}} = \{i : \ \mathbf{Ax} + \mathbf{b}\ _{\infty} =  \mathbf{a}_i^T \mathbf{x} + b_i \}$ , $\mathbf{Ax} + \mathbf{b} \neq \mathbf{0}$ |
| $\ \mathbf{Ax} + \mathbf{b}\ _{\infty}$             | $\mathbf{A}^T B_{\ \cdot\ _1}[\mathbf{0}, 1]$  | same as above but with $\mathbf{Ax} + \mathbf{b} = \mathbf{0}$   |
| $\max_i \{\mathbf{a}_i^T \mathbf{x} + \mathbf{b}\}$ | $\left\{ \sum_{i \in I(\mathbf{x})} \lambda_i \mathbf{a}_i : \sum_{i \in I(\mathbf{x})} \lambda_i = 1, \lambda_i \geq 0 \right.$   | $\mathbb{E} = \mathbb{R}^n$ , $\mathbf{a}_i \in \mathbb{R}^n$ , $b_i \in \mathbb{R}$ , $I(\mathbf{x}) = \{i : f(\mathbf{x}) = \mathbf{a}_i^T \mathbf{x} + b_i\}$   |
| $\frac{1}{2} d_C(\mathbf{x})^2$                     | $\{\mathbf{x} - P_C(\mathbf{x})\}$   | $C$ - nonempty closed and convex, $\mathbb{E}$ - Euclidean   |
| $d_C(\mathbf{x})$                                   | $\left\{ \begin{array}{l} \left\{ \frac{\mathbf{x} - P_C(\mathbf{x})}{d_C(\mathbf{x})} \right\}, \quad \mathbf{x} \notin C, \\ N_C(\mathbf{x}) \cap B[\mathbf{0}, 1] \quad \mathbf{x} \in C. \end{array} \right.$  | $C$ - nonempty closed and convex, $\mathbb{E}$ - Euclidean   |