Orthogonal Projections

set (C)	$P_C(\mathbf{x})$	assumptions
$\mathbb{R}^n_+$	$[\mathbf{x}]_+$	–
$\text{Box}[\boldsymbol{\ell}, \mathbf{u}]$	$P_C(\mathbf{x})_i = \min\{\max\{x_i, \ell_i\}, u_i\}$	$\ell_i \leq u_i$
$B_{\|\cdot\|_2}[\mathbf{c}, r]$	$\mathbf{c} + \frac{r}{\max\{\|\mathbf{x}-\mathbf{c}\|_2, r\}}(\mathbf{x} - \mathbf{c})$	$\mathbf{c} \in \mathbb{R}^n, r > 0$
$\{\mathbf{x} : \mathbf{A}\mathbf{x} = \mathbf{b}\}$	$\mathbf{x} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x} - \mathbf{b})$	$\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{b} \in \mathbb{R}^m$, $\mathbf{A}$ full row rank
$\{\mathbf{x} : \mathbf{a}^T\mathbf{x} \leq b\}$	$\mathbf{x} - \frac{[\mathbf{a}^T\mathbf{x}-b]_+}{\|\mathbf{a}\|^2}\mathbf{a}$	$\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n, b \in \mathbb{R}$
Δ_n	$[\mathbf{x} - \mu^*\mathbf{e}]_+$ where $\mu^* \in \mathbb{R}$ satisfies $\mathbf{e}^T[\mathbf{x} - \mu^*\mathbf{e}]_+ = 1$	
$H_{\mathbf{a},b} \cap \text{Box}[\boldsymbol{\ell}, \mathbf{u}]$	$P_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x} - \mu^*\mathbf{a})$ where $\mu^* \in \mathbb{R}$ satisfies $\mathbf{a}^T P_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x} - \mu^*\mathbf{a}) = b$	$\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $b \in \mathbb{R}$
$H^-_{\mathbf{a},b} \cap \text{Box}[\boldsymbol{\ell}, \mathbf{u}]$	$\begin{cases} P_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x}), & \mathbf{a}^T\mathbf{v}_\mathbf{x} \leq b, \\ P_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x} - \lambda^*\mathbf{a}), & \mathbf{a}^T\mathbf{v}_\mathbf{x} > b, \end{cases}$ $\mathbf{v}_\mathbf{x} = P_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x})$, $\mathbf{a}^T P_{\text{Box}[\boldsymbol{\ell},\mathbf{u}]}(\mathbf{x} - \lambda^*\mathbf{a}) = b, \lambda^* > 0$	$\mathbf{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$, $b \in \mathbb{R}$
$B_{\|\cdot\|_1}[\mathbf{0}, \alpha]$	$\begin{cases} \mathbf{x}, & \|\mathbf{x}\|_1 \leq \alpha, \\ \mathcal{T}_{\lambda^*}(\mathbf{x}), & \|\mathbf{x}\|_1 > \alpha, \end{cases}$ $\|\mathcal{T}_{\lambda^*}(\mathbf{x})\|_1 = \alpha$, $\lambda^* > 0$	$\alpha > 0$
$\{\mathbf{x} : \boldsymbol{\omega}^T\|\mathbf{x}\| \leq \beta,$ $-\boldsymbol{\alpha} \leq \mathbf{x} \leq \boldsymbol{\alpha}\}$	$\begin{cases} \mathbf{v}_\mathbf{x}, & \boldsymbol{\omega}^T\|\mathbf{v}_\mathbf{x}\| \leq \beta, \\ \mathcal{S}_{\lambda^*\boldsymbol{\omega},\boldsymbol{\alpha}}(\mathbf{x}), & \boldsymbol{\omega}^T\|\mathbf{v}_\mathbf{x}\| > \beta, \end{cases}$ $\mathbf{v}_\mathbf{x} = P_{\text{Box}[-\boldsymbol{\alpha},\boldsymbol{\alpha}]}(\mathbf{x})$, $\boldsymbol{\omega}^T\|\mathcal{S}_{\lambda^*\boldsymbol{\omega},\boldsymbol{\alpha}}(\mathbf{x})\| = \beta, \lambda^* > 0$	$\boldsymbol{\omega} \in \mathbb{R}^n_{++}$, $\boldsymbol{\alpha} \in [0, \infty]^n$, $\beta \in \mathbb{R}_{++}$
$\{\mathbf{x} > \mathbf{0} : \Pi x_i \geq \alpha\}$	$\begin{cases} \mathbf{x}, & \mathbf{x} \in C, \\ \left(\frac{x_j + \sqrt{x_j^2 + 4\lambda^*}}{2}\right)_{j=1}^n, & \mathbf{x} \notin C, \end{cases}$ $\Pi_{j=1}^n \left((x_j + \sqrt{x_j^2 + 4\lambda^*})/2\right) = \alpha, \lambda^* > 0$	$\alpha > 0$
$\{(\mathbf{x}, s) : \|\mathbf{x}\|_2 \leq s\}$	$\left(\frac{\|\mathbf{x}\|_2+s}{2\|\mathbf{x}\|_2}\mathbf{x}, \frac{\|\mathbf{x}\|_2+s}{2}\right)$ if $\|\mathbf{x}\|_2 \geq \|s\|$ $(\mathbf{0}, 0)$ if $s < \|\mathbf{x}\|_2 < -s$, $(\mathbf{x}, s)$ if $\|\mathbf{x}\|_2 \leq s$.	
$\{(\mathbf{x}, s) : \|\mathbf{x}\|_1 \leq s\}$	$\begin{cases} (\mathbf{x}, s), & \|\mathbf{x}\|_1 \leq s, \\ (\mathcal{T}_{\lambda^*}(\mathbf{x}), s + \lambda^*), & \|\mathbf{x}\|_1 > s, \end{cases}$ $\|\mathcal{T}_{\lambda^*}(\mathbf{x})\|_1 - \lambda^* - s = 0$, $\lambda^* > 0$	

Orthogonal Projections onto Symmetric Spectral Sets in $\mathbb{S}^n$

set (C)	$P_C(\mathbf{X})$	assumptions
$\mathbb{S}^n_+$	$\mathbf{U}\text{diag}([\boldsymbol{\lambda}(\mathbf{X})]_+)\mathbf{U}^T$	–
$\{\mathbf{X} : \ell\mathbf{I} \preceq \mathbf{X} \preceq u\mathbf{I}\}$	$\mathbf{U}\text{diag}(\mathbf{v})\mathbf{U}^T$, $v_i = \min\{\max\{\lambda_i(\mathbf{X}), \ell\}, u\}$	$\ell \leq u$
$B_{\\|\cdot\\|_F}[\mathbf{0}, r]$	$\frac{r}{\max\{\\|\mathbf{X}\\|_F, r\}}\mathbf{X}$	$r > 0$
$\{\mathbf{X} : \text{Tr}(\mathbf{X}) \leq b\}$	$\mathbf{U}\text{diag}(\mathbf{v})\mathbf{U}^T$, $\mathbf{v} = \boldsymbol{\lambda}(\mathbf{X}) - \frac{[\mathbf{e}^T\boldsymbol{\lambda}(\mathbf{X})-b]_+}{n}\mathbf{e}$	$b \in \mathbb{R}$
Υ_n	$\mathbf{U}\text{diag}(\mathbf{v})\mathbf{U}^T$, $\mathbf{v} = [\boldsymbol{\lambda}(\mathbf{X}) - \mu^*\mathbf{e}]_+$ where $\mu^* \in \mathbb{R}$ satisfies $\mathbf{e}^T[\boldsymbol{\lambda}(\mathbf{X}) - \mu^*\mathbf{e}]_+ = 1$	-
$B_{\\|\cdot\\|_{S_1}}[\mathbf{0}, \alpha]$	$\begin{cases} \mathbf{X}, & \\|\mathbf{X}\\|_{S_1} \leq \alpha, \\ \mathbf{U}\mathcal{T}_{\lambda^*}(\boldsymbol{\lambda}(\mathbf{X}))\mathbf{U}^T, & \\|\mathbf{X}\\|_{S_1} > \alpha, \end{cases}$ $\\|\mathcal{T}_{\lambda^*}(\boldsymbol{\lambda}(\mathbf{X}))\\|_1 = \alpha$, $\lambda^* > 0$	$\alpha > 0$

Orthogonal Projections onto Symmetric Spectral Sets in $\mathbb{R}^{m \times n}$

set (C)	$P_C(\mathbf{X})$	assumptions
$B_{\\|\cdot\\|_{S_\infty}}[\mathbf{0}, \alpha]$	$\mathbf{U}\text{diag}(\mathbf{v})\mathbf{V}^T$, $v_i = \min\{\sigma_i(\mathbf{X}), \alpha\}$	$\alpha > 0$
$B_{\\|\cdot\\|_F}[\mathbf{0}, r]$	$\frac{r}{\max\{\\|\mathbf{X}\\|_F, r\}}\mathbf{X}$	$r > 0$
$B_{\\|\cdot\\|_{S_1}}[\mathbf{0}, \alpha]$	$\begin{cases} \mathbf{X}, & \\|\mathbf{X}\\|_{S_1} \leq \alpha, \\ \mathbf{U}\mathcal{T}_{\lambda^*}(\sigma(\mathbf{X}))\mathbf{U}^T, & \\|\mathbf{X}\\|_{S_1} > \alpha, \end{cases}$ $\\|\mathcal{T}_{\lambda^*}(\boldsymbol{\sigma}(\mathbf{X}))\\|_1 = \alpha$, $\lambda^* > 0$	$\alpha > 0$