Measuring energy, estimating Hamiltonians, and the time-energy uncertainty relation

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(Received 11 October 2001; published 22 November 2002)

Suppose that the Hamiltonian acting on a quantum system is unknown and one wants to determine which is the Hamiltonian. We show that, in general, this requires a time Δt that obeys the uncertainty relation $\Delta t \Delta H \gtrsim 1$, where ΔH is a measure of how accurately the unknown Hamiltonian must be estimated. We apply this result to the problem of measuring the energy of an unknown quantum state. It has been previously shown that if the Hamiltonian is known, then the energy can, in principle, be measured with arbitrarily large precision in an arbitrarily short time. On the other hand, we show that if the Hamiltonian is not known then an energy measurement necessarily takes a minimum time Δt which obeys the uncertainty relation $\Delta t \Delta E \gtrsim 1$, where ΔE is the precision of the energy measurement. Several examples are studied to address the question of whether it is possible to saturate these uncertainty relations. Their interpretation is discussed in detail.

DOI: 10.1103/PhysRevA.66.052107 PACS number(s): 03.65.Ta, 03.67.−a

I. INTRODUCTION

The uncertainty relations play a central role in quantum mechanics. Their importance lies in the fact that they express in a succinct manner the fundamental limitations on the measurements imposed by quantum mechanics. In particular, it is these limitations that guarantee that the mathematical formalism of the theory is free from contradictions with the experiment.

The time-energy uncertainty, however, has a particular status because time is an external parameter in the theory, and the energy operator plays a special role since it determines the temporal evolution. Thus the time-energy uncertainty does not follow from the commutation relations of two operators, but is determined indirectly, for instance, from the mathematical properties of the Fourier transform with respect to the time variable. The interpretation and status of the time-energy uncertainty should therefore be examined with particular care.

In this paper, we shall concentrate on energy measurements. By analogy with other measurements, one expects that the time-energy uncertainty expresses a fundamental constraint which energy measurements must satisfy. The aim of this paper is to clarify whether such a constraint exists, and what should be its interpretation.

Suppose that one must measure the energy of an unknown quantum state. One's first intuition [1,2] in this case is that the energy of an unknown state can be determined to an accuracy ΔE, only if the duration Δt of the measurement is larger than $1/\Delta E$ (since $\Delta t \Delta E \gtrsim 1$), where we set $\hbar = 1$. Here and throughout, by accuracy of the measurement, we mean how much the result of the measurement differs from the result of an ideal von Neumann measurement of the Hamiltonian operator H. This will be defined with precision below.

Surprisingly, this intuition is incorrect. It is possible to measure the energy of an unknown quantum state to arbitrarily high accuracy in an arbitrarily short time [3]. As an illustration consider a spin-1/2 particle with a magnetic moment μ in a magnetic field $\vec{B}_0 = B_0 \vec{1}_z$ pointing along the z direction. In order to measure the energy of the spin, one can apply a strong magnetic field $\vec{B}(z) = B(z) \vec{1}_z$ pointing in the z direction with a gradient in the z direction. This realizes a Stern-Gerlach measurement of σ_z, hence of the energy of the particle. The time necessary for this measurement depends on the magnitude of the additional magnetic field $B(z)$, not on the original magnetic field B_0. Since $B(z)$ can be arbitrarily large, the energy can be determined in an arbitrarily short time. Another example, which is discussed in detail in the original article [3], is the measurement of the energy $H = p^2/2m$ of a free particle.

Thus the time-energy uncertainty seems not to apply to energy measurements. Is this result universal? Or are there cases where the time-energy uncertainty does apply, and measuring the energy to an accuracy ΔE does require a time Δt limited by $\Delta t \Delta E \gtrsim 1$?

We first note that if one attempts to devise an energy measurement, it is often the case that this measurement will take a time that satisfies the time-energy uncertainty. We refer for instance, to the example considered by Landau and Peierls [1], see also Landau and Lifschitz [2], in which the kinetic energy of a particle is measured by allowing it to collide with another particle initially at rest. In this example

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if the energy is to be measured with precision, the initial momentum of the particle at rest must be well known, but then its position is spread out and the time at which the measurement takes place is uncertain. Another well-known example is the radiative decay of an excited state. The emission of a photon by an excited atom that decays to its ground state can be viewed as a measurement of the energy of the electronic state of the atom. The mean time it takes for the photon to be emitted, that is the lifetime of the excited state, is then interpreted as the mean time it takes for the measurement. This lifetime is related to the linewidth, that is to the uncertainty of the energy measurement, by the time-energy uncertainty. Following Ref. [3], one would argue that such energy measurements, which satisfy the time-energy uncertainty, are simply badly designed, and that one can, in principle, make energy measurements that do not obey the time-energy uncertainty. Nevertheless, it is surprising that so many energy measurements do obey the time-energy uncertainty. We would like to know what makes some energy measurements less efficient than others.

There are also some particular circumstances in which the time-energy uncertainty must be obeyed in an energy measurement. Specifically, we consider the situation in which one wants to measure the energy of an isolated system. Since it is isolated, i.e., uncoupled to any exterior degrees of freedom, the measuring apparatus must be internal to the system itself. In Ref. [4], it is argued that this constraint implies that measuring the total energy of an isolated system to an accuracy ΔE requires a time Δt that satisfies $\Delta t \Delta E \geqslant 1$. Contrary to the examples mentioned in the preceding paragraph, where a more sophisticated strategy could, in principle, measure the energy in a smaller time than that given by the uncertainty relation, in this case the uncertainty relation must be obeyed.

Thus the present status of the time-energy uncertainty in the context of energy measurements is unsatisfactory. Different examples suggest conflicting interpretations.

As we shall prove below, the resolution of this puzzle is the following. When the Hamiltonian of the system is *known*, the conclusions of Ref. [3] hold, namely we can measure the energy as precisely as we want in a time as short as we want. On the other hand, *whenever the Hamiltonian of a system is completely unknown, determining what is the Hamiltonian to precision* ΔH *requires a time* Δt *given by* $\Delta t \Delta H \geqslant 1$.

The origin of the conflicting interpretations of the time-energy uncertainty relation in the context of estimating the energy of an unknown state is now clear. In some cases, the energy measurement can also serve to estimate an unknown Hamiltonian. In these cases, because the measurement can serve this dual purpose, it requires a time Δt which is limited by the accuracy with which it could estimate an unknown Hamiltonian. On the other hand, the measurements envisaged in Ref. [3] cannot estimate an unknown Hamiltonian. These measurements measure an operator A (σ_z in the example above, or the momentum p in the example of Ref. [3]). If the Hamiltonian is a function of this operator $H = f(A)$ (for instance, $H = \mu B_0 \sigma_z$ or $H = p^2/2m$), then the measurement of A can be used to determine the energy of an unknown state. Such a von Neumann measurement of an operator A can be realized in an arbitrarily short time. But if the time is very short, the measurement is brutal, that is the interaction between the measuring apparatus and the system dominates the evolution. It is therefore impossible to use such a measurement to estimate an unknown Hamiltonian since the action of the original Hamiltonian is completely masked by the interaction.

Therefore in order to understand the interpretation of the time-energy uncertainty relation in the context of energy measurements, it is mandatory to first understand the limitations quantum mechanics imposes on the estimation of the Hamiltonian acting on a system when the Hamiltonian is unknown. This problem is, in a certain sense, the dual of the standard problem of quantum information in which one must estimate the state of a quantum system. Here one must estimate the dynamics. This is a fundamental problem in quantum mechanics. In fact many experimental situations can be phrased in this language. For instance, there could be an unknown force acting on a particle that one wants to estimate. Or the Hamiltonian could depend on an unknown parameter that one wants to estimate.

We show in the present paper that the precision with which one can estimate the dynamics and the time required for this estimation are related by an uncertainty relation

$$\Delta t \Delta H \geqslant 1, \tag{1}$$

where the precise meaning of ΔH depends on the details of the problem.

It is intuitively obvious that the dynamics of a quantum system cannot be estimated instantaneously. Indeed suppose that initially the state is ψ_0. Then after a time t the state has evolved to $\psi(t,H) = e^{-itH}\psi_0 \simeq \psi_0 + itH\psi_0$. From this expression it is clear that in order to estimate the dynamics, i.e., to estimate H, sufficient time must elapse so that the zeroth-order term ψ_0 in the expansion in t does not dominate. Were this the case, the states $\psi(t,H)$ would be almost identical to ψ_0 and hence undistinguishable. The results reported in this paper make this qualitative statement precise.

The problem of estimating the dynamics has been considered recently by Childs, Preskill, and Renes [5]. Their results constitute a starting point for our discussion.

The remainder of the paper is organized as follows. We first discuss the problem of distinguishing with certainty between the two Hamiltonians. Next we consider the case where one only wants to estimate with finite error probability which of the two Hamiltonians is the true Hamiltonian. Then we turn to the problem of estimating what is the Hamiltonian when one has no prior knowledge about the Hamiltonian. Finally, we go back to the problem of estimating the energy of an unknown state. We discuss how it is related to estimating an unknown Hamiltonian. We prove that if the Hamiltonian is completely unknown then the time necessary to estimate the energy and the precision with which the energy is estimated must obey a time-energy uncertainty relation. We illustrate the problem of estimating the energy of a state when the Hamiltonian is completely or partially unknown by several examples.

II. DISTINGUISHING BETWEEN THE TWO HAMILTONIANS

A. Minimum time necessary to distinguish between the two unknown Hamiltonians

In this section, we consider the problem of estimating the Hamiltonian acting on a system when the Hamiltonian is unknown. We consider in this section the special case in which there are only two possible Hamiltonians that could act on the system, H_1 or H_2, and one must distinguish with certainty which it is.

This particular problem has been considered previously by Childs, Preskill, and Renes [5]. They show that the minimum time Δt required to determine with certainty the Hamiltonian must satisfy the constraint

$$\Delta t\, D_0(H_1,H_2) \geqslant \pi, \tag{2}$$

where $D_0(H_1,H_2)$ measures how much the two Hamiltonians differ and is defined as follows. Take the difference of the two Hamiltonians $H^d = H_1 - H_2$. Denote by E^d_{max} and E^d_{min} the largest and smallest eigenvalue of H^d, respectively. Then $D_0(H_1,H_2)$ is given by

$$D_0(H_1,H_2) = \max\{E^d_{max} - E^d_{min}, |E^d_{max}|, |E^d_{min}|\}. \tag{3}$$

(This expression generalizes a result of Ref. [5].)

In the Appendix, it is shown that D_0 is a distance on the space of Hamiltonian operators (i.e., it is symmetric, positive and equal to zero only if the two Hamiltonians coincide, and it obeys the triangle inequality). This important property is central to some of the arguments below.

In order to make the problem considered by Childs, Preskill, and Renes more concrete, consider the following example. You are given a box in which there is one of the two unknown magnetic fields $\vec{B}_1$ or $\vec{B}_2$. Your task is to determine which kind of box you have. The only way you can probe the box is to send through the box a specific kind of particle of magnetic moment μ. Thus the two possible kinds of boxes differ in that they act as two different Hamiltonians $H_{1,2} = \mu \vec{B}_{1,2} \cdot \vec{\sigma}$. Given sufficient time or a sufficient supply of particles you can always determine which is the magnetic field. (We suppose that the time that the particles pass in the box can be freely chosen, for instance by choosing their initial velocity. And one can send the particle through the box as many times as one wants.) However, if you are given only one particle, what is the minimum time necessary to accomplish this task? Childs, Preskill, and Renes show that this time is limited by Eq. (2).

We now present a proof of Eqs. (2), (3) that is inspired by the techniques developed to study interaction-free measurements in Ref. [6]. The proof is also closely connected to the bounds on oracle query complexity obtained in Ref. [7] and even more so to a continuous-time analog obtained in Ref. [8]. We first describe the most general strategy that could be used. First of all, we consider that we may let the particle go into the box or not, i.e., for certain values of its position, the particle can pass through the box and for others not. This allows us to make a superposition of particle passing through the box and not passing through the box. Mathematically, we describe this by the decomposition of the Hilbert space into the sum $\mathcal{H} = \mathcal{H}_{box} \oplus \mathcal{H}_{Nbox}$, where $\mathcal{H}_{box}$ corresponds to the particle passing through the box and $\mathcal{H}_{Nbox}$ corresponds to the particle passing next to the box. Second, we consider that the particle can also be entangled with some other particle called ancilla onto which the Hamiltonian does not act (e.g., the ancilla is kept out of the box). The Hilbert space is thus further increased to $\mathcal{H} = (\mathcal{H}_{box} \oplus \mathcal{H}_{Nbox}) \otimes \mathcal{H}_{ancilla}$.

In the particular problem considered in this section, namely distinguishing between two possible Hamiltonians, using an ancilla turns out to be irrelevant. (However, when one must distinguish between more than two Hamiltonians, using an ancilla can be helpful, see Ref. [5] for a simple example.) To simplify the proof we will first consider the case with no ancilla and then show in Sec. II C that the presence of the ancilla makes no difference.

The most general strategy consists of sending the particle several times through the box and making an arbitrary unitary transformation on the particle after each passage. We describe this as follows. Initially the particle is in state $|\psi_0\rangle$. Before the kth passage through the box, the particle is in state $|\psi^i_{k-1}\rangle$, where the superscript $i = 1,2$ corresponds to which Hamiltonian $H_{1,2}$ is acting on the particle. We decompose the state as $|\psi^i_{k-1}\rangle = |u^i_k\rangle + |v^i_k\rangle$, where $|u^i_k\rangle \in \mathcal{H}_{box}$ and $|v^i_k\rangle \in \mathcal{H}_{Nbox}$. After passing through the box the particle's state is $|\psi'^i_k\rangle = e^{-iH_i t_k}|u^i_k\rangle + |v^i_k\rangle$, where t_k is the time the particle is in the box (and we have supposed without loss of generality that if the particle does not pass through the box the Hamiltonian is zero). We then make an arbitrary unitary transformation on the particle $|\psi'^i_k\rangle \to |\psi^i_k\rangle = U_k|\psi'^i_k\rangle$. We can describe the whole evolution succinctly by

$$|\psi^i_k\rangle = U_k e^{-i\tilde{H}_i t_k}|\psi^i_{k-1}\rangle, \tag{4}$$

where $\tilde{H}_i$ is the extension of H_i to $\mathcal{H}_{box} \oplus \mathcal{H}_{Nbox}$. It is equal to H_i on $\mathcal{H}_{box}$ and equal to zero on $\mathcal{H}_{Nbox}$.

We therefore have

$$\langle \psi^1_k | \psi^2_k \rangle = \langle \psi^1_{k-1} | e^{i\tilde{H}_1 t_k} e^{-i\tilde{H}_2 t_k} | \psi^2_{k-1} \rangle, \tag{5}$$

or expressed differently

$$\langle \psi^1_k | \psi^2_k \rangle - \langle \psi^1_{k-1} | \psi^2_{k-1} \rangle = \langle \psi^1_{k-1} | e^{i\tilde{H}_1 t_k} e^{-i\tilde{H}_2 t_k} - 1 | \psi^2_{k-1} \rangle. \tag{6}$$

The procedure to distinguish between H_1 and H_2 can only improve if the total time T is kept fixed, but the particle is allowed to pass more times through the box. That is the discrimination can only improve if one takes smaller time intervals t_k. In the limit of infinitesimal t_k, one obtains

$$\frac{d\langle \psi^1 | \psi^2 \rangle}{dt} = i\langle \psi^1 | (\tilde{H}_1 - \tilde{H}_2) | \psi^2 \rangle = i\langle \psi^1 | \tilde{H}^d | \psi^2 \rangle, \tag{7}$$

where $\tilde{H}^d = \tilde{H}_1 - \tilde{H}_2$ is equal to $H^d = H_1 - H_2$ on $\mathcal{H}_{box}$ and equal to zero on $\mathcal{H}_{Nbox}$. The absolute value of the overlap therefore changes as

$$\frac{d|\langle\psi^1|\psi^2\rangle|^2}{dt}=4\,\mathrm{Im}[\langle\psi^2|\psi^1\rangle\langle\psi^1|\tilde{H}^d|\psi^2\rangle]. \tag{8}$$

We can always write the states $|\psi^{1,2}\rangle$ as

$$|\psi^1\rangle=e^{+i\chi/2}[\cos(\theta/2)|\psi^{\|}\rangle+\sin(\theta/2)|\psi^{\perp}\rangle],$$

$$|\psi^2\rangle=e^{-i\chi/2}[\cos(\theta/2)|\psi^{\|}\rangle-\sin(\theta/2)|\psi^{\perp}\rangle],$$

$$\text{where}\quad 0\leqslant\theta\leqslant\pi/2,\quad \langle\psi^{\|}|\psi^{\perp}\rangle=0. \tag{9}$$

This enables us to write Eq. (8) as

$$\frac{d\cos^2\theta}{dt}=2\cos\theta\sin\theta\,\mathrm{Im}[\langle\psi^{\|}|\tilde{H}^d|\psi^{\perp}\rangle] \tag{10}$$

or equivalently

$$\frac{d\theta}{dt}=\mathrm{Im}[\langle\psi^{\|}|\tilde{H}^d|\psi^{\perp}\rangle]. \tag{11}$$

Now we note that for any operator A,

$$|\mathrm{Im}[\langle\psi^{\|}|A|\psi^{\perp}\rangle]|\leqslant\frac{a_{max}-a_{min}}{2}, \tag{12}$$

where a_{max} is the largest eigenvalue of A and a_{min} is the smallest. Equality is attained if

$$|\psi^{\|}\rangle=\frac{e^{i\varphi}|a_{max}\rangle+e^{i\varphi'}|a_{min}\rangle}{\sqrt{2}},$$

$$|\psi^{\perp}\rangle=\frac{e^{i(\varphi+\pi/2)}|a_{max}\rangle+e^{i(\varphi'-\pi/2)}|a_{min}\rangle}{\sqrt{2}}, \tag{13}$$

where φ and φ' are arbitrary phases. In the case of $\tilde{H}^d$, its largest eigenvalue is $\max\{E^d_{max},0\}$ and its smallest eigenvalue is $\min\{E^d_{min},0\}$ (where E^d_{max} and E^d_{min} are the largest and smallest eigenvalues of H^d). Hence the absolute value of the right hand side of Eq. (11) is bounded by $|\mathrm{Im}[\langle\psi^{\|}|\tilde{H}^d|\psi^{\perp}\rangle]|\leqslant D_0(H_1,H_2)/2$, where D_0 is given by Eq. (3). We therefore have

$$\left|\frac{d\theta}{dt}\right|\leqslant D_0(H_1,H_2)/2. \tag{14}$$

On integration we have

$$-\frac{D_0(H_1,H_2)t}{2}\leqslant\theta(t)-\theta(0)\leqslant\frac{D_0(H_1,H_2)t}{2}. \tag{15}$$

Initially $\theta(0)=0$ since $|\psi(0)\rangle$ is independent of i. Requiring that one can recognize with certainty which is the Hamiltonian, that is requiring $\theta(t)=\pi/2$, one finds that $tD_0(H_1,H_2)\geqslant\pi$, as announced.

In the above proof the possibility of allowing the particle go through the box as well as outside the box allows us to extend the Hamiltonian H^d, so that it also has the eigenvalue zero. This is described by replacing H_i with $\tilde{H}_i$. It is the possibility that allows the maximum in Eq. (3) to be taken not only over the first term $E^d_{max}-E^d_{min}$, but also over the two other terms $|E^d_{max}|$ and $|E^d_{min}|$, and makes our result (3) differ from the result obtained in Ref. [5]. Indeed if one does not allow for this then Eq. (2) continues to hold, but Eq. (3) is replaced by $D_0(H_1,H_2)=E^d_{max}-E^d_{min}$. Thus in this case the measurement may be less efficient.

In order to illustrate how the ''in/out of the box'' possibility can be used, we consider the following example. We are given one of two black boxes, and must distinguish which box we have. The boxes are conducting and hollow. They are connected to an external electrostatic potential. The only way the two boxes differ is that the potential can take two different values ϕ_1 and ϕ_2. The only way we can probe which box we have is by sending a charged particle through the box. The particle will therefore acquire a phase that depends on the potential. Mathematically, we can describe this as the problem of distinguishing two Hamiltonians $H_{1,2}=H_0+\phi_{1,2}I$, where I is the identity operator. In order to distinguish which box we have, we put the particle in a superposition of two states, one of which passes through the box and the other does not. Thus we prepare the particle in the state $(|\text{through box}\rangle+|\text{not through box}\rangle)/\sqrt{2}$ and send the particle at a speed such that it passes a time $T=\pi/(\phi_1-\phi_2)$ in the box. After this time the particle is in one of the two orthogonal states $(|\text{not through box}\rangle\pm e^{i\pi\phi_1/(\phi_1-\phi2)}|\text{through box}\rangle)/\sqrt{2}$, which can easily be distinguished. On the other hand, if we had not been allowed to use the ''out of the box'' alternative, we could not have distinguished between the two Hamiltonians since they differ only by a constant term that adds an unobservable phase to the wave function.

There are also situations where it is not necessary to use the ''in/out of the box'' possibility. Suppose one must distinguish, using a spin-1/2 particle of magnetic moment μ, between two magnetic fields of equal magnitude but pointing in opposite directions $B_{1,2}=\pm B_0\vec{1}_z$. The strategy in this case is simply to prepare the spin in state $|\uparrow_x\rangle$ and let it evolve in the magnetic field a time $T=\pi/\mu B_0$, and then to measure the operator σ_y.

B. Distinguishing between the two time-dependent Hamiltonians

In the preceding section, we considered the case where the two Hamiltonians that must be distinguished are time independent. One can easily generalize this result to the case where the Hamiltonians are time dependent.

Let us suppose one must distinguish between the two time-dependent Hamiltonians $H_1(t)$ and $H_2(t)$. The arguments of Sec. II A can be followed unchanged until Eq. (14) which becomes

$$\left|\frac{d\theta}{dt}\right|\leqslant D_0(H_1(t),H_2(t))/2, \tag{16}$$

where $D_0(H_1(t),H_2(t))$ is the instantaneous value of the distance (3). On integration, we have

$$-\int_0^t dt\,\frac{D_0(H_1(t),H_2(t))}{2}\leqslant\theta(t)-\theta(0)\leqslant\int_0^t dt\,\frac{D_0(H_1(t),H_2(t))}{2}. \qquad (17)$$

Initially $\theta(0)=0$ since $|\psi(0)\rangle$ is independent of i. Requiring that one can recognize with certainty which is the Hamiltonian, that is requiring $\theta(t)=\pi/2$, one finds that

$$\int_0^t dt\,D_0(H_1(t),H_2(t))\geqslant\pi, \qquad (18)$$

which is the generalization of the constraint (2) to the case of time-dependent Hamiltonians.

C. Using an ancilla does not help to distinguish between the two Hamiltonians

In the proof of Eqs. (2), (3), and (18) given in Secs. II A and II B, we did not consider the possibility that the particle passing through the box is entangled with another particle (ancilla). We shall now show that if we assume the ancilla is not allowed into the box, then Eqs. (2), (3), and (18) continue to hold.

To this end it is helpful to view the particle and ancilla as a single larger system with Hamiltonian $H_i^{total}=H_i(t)\otimes H_{ancilla}(t)$, where H_i is unknown and $H_{ancilla}$ is known but arbitrary, and to reformulate the task as the problem of distinguishing H_1^{total} from H_2^{total}. Note that this reformulation includes apparently more involved strategies where, for instance, the ancilla is repeatedly measured and the evolution made conditional on the results of these intermediate measurements. Indeed by including the measuring device in an even bigger ancilla, one recovers the above formulation.

Now the distance D_0 between two such Hamiltonians obeys

$$D_0(H_1^{total}(t),H_2^{total}(t))=D_0(H_1(t)\otimes H_{ancilla}(t),H_2(t)\otimes H_{ancilla}(t))=D_0(H_1(t),H_2(t)). \qquad (19)$$

Thus the time required to distinguish between the two Hamiltonians (18) does not depend on the presence of an ancilla.

The basic reason why an ancilla does not help in distinguishing between two unknown Hamiltonians is that all the argument of Sec. II A depends on the eigenvalues of the difference between the two Hamiltonians H^d but not on the degeneracy of the eigenvalues. Including an ancilla does not change the eigenvalues of H^d, but changes their degeneracy.

D. Attaining the bound in the dichotomic case

We shall now show that one can always attain the bound (2). In general this will require putting the particle in a superposition of ''in the box'' and ''out of the box'' states. Equation (13) shows that to attain the bound the particle must be kept in a superposition with equal weights of the eigenstates of $\tilde{H}^d=\tilde{H}_1^d-\tilde{H}_2^d$, with maximal and minimal eigenvalue $|\tilde{E}_{max}^d\rangle$ and $|\tilde{E}_{min}^d\rangle$. In the preceding section, we gave two examples of how to do this when the two Hamiltonians H_1 and H_2 commute. When the Hamiltonians do not commute one must use a more complicated strategy. Let us first rewrite the Hamiltonians $\tilde{H}_1$ and $\tilde{H}_2$ as

$$\tilde{H}_1=\tilde{H}^+ +\tilde{H}^d/2,\quad \tilde{H}_2=\tilde{H}^+ -\tilde{H}^d/2,\qquad \tilde{H}^+=(\tilde{H}_1+\tilde{H}_2)/2,\quad \tilde{H}^d=\tilde{H}_1-\tilde{H}_2. \qquad (20)$$

The evolution during a small time interval τ can then be written as

$$\exp(-i\tilde{H}_1\tau)=\exp[-i(\tilde{H}^+ +\tilde{H}^d/2)\tau]\simeq\exp(-i\tilde{H}^+\tau)\exp(-i\tilde{H}^d\tau/2)\exp[O(\tau^2)],$$
$$\exp(-i\tilde{H}_2\tau)=\exp[-i(\tilde{H}^+ -\tilde{H}^d/2)\tau]\simeq\exp(-i\tilde{H}^+\tau)\exp(+i\tilde{H}^d\tau/2)\exp[O(\tau^2)]. \qquad (21)$$

To distinguish between the two Hamiltonians, we initially prepare the system in the state $|\psi(0)\rangle=(|\tilde{E}_{max}^d\rangle+|\tilde{E}_{min}^d\rangle)/\sqrt{2}$. The evolution is the following. We let the unknown Hamiltonian act for a small time $\tau=T/N$, where $T=\pi/D_0(H_1,H_2)$ and N is a large integer. We then act on the system with the unitary transformation $U=\exp(-i\nu\tilde{H}^d)\exp(+i\tilde{H}^+\tau)$, where ν is an arbitrary real number. The term on the right in U cancels the term on the left in Eq. (21). This unitary evolution U can, in principle, be done in an arbitrarily short time. After N repetitions, the evolution is

$$|\psi(T)\rangle=\{\exp[-i(\nu\pm 1/2)\tilde{H}^d\tau]\exp[O(\tau^2)]\}^N|\psi(0)\rangle\simeq\exp(-i\pi(\nu\pm 1/2)\tilde{H}^d)|\psi(0)\rangle, \qquad (22)$$

where we have written an expression valid in the limit of large N. Thus we obtain two orthogonal states that can be distinguished with certainty.

Note that if we take the arbitrary real number ν to be either $\pm 1/2$, then $U=\exp(-i\tau\tilde{H}_{1,2})$, that is we have exactly canceled the evolution of one of the Hamiltonians. This is the technique that is proposed in Ref. [5].

III. ESTIMATING AN UNKNOWN HAMILTONIAN

A. Formulation of the problem

In the preceding section, we considered the situation where one must *distinguish with certainty* between *two* possible Hamiltonians. In the present section, we shall consider the problem where one must *estimate with finite precision* which is the Hamiltonian.

In order to give a precise formulation to this problem, let us suppose that the possible Hamiltonians are denoted by H_i and can occur with *a priori* probabilities $p(H_i)$. After an evolution that lasts for a time Δt, a measurement is carried out which yields result j. The probability of the result j given that the Hamiltonian is H_i is denoted by $p(j|H_i)$. The result of the measurement allows one to estimate which is the Hamiltonian. We shall suppose that one makes a guess of which is the true Hamiltonian. We call the guess H_j^{guess}. Note that H_j^{guess} can be one of the initial Hamiltonians H_i, or it could be a different Hamiltonian altogether. The quality of the guess is measured by an uncertainty

$$\Delta H=\sum_i p(H_i)\sum_j p(j|H_i)D(H_i,H_j^{guess}). \tag{23}$$

In this equation $D(H_i,H_j^{guess})$ is a distance on the space of Hamiltonians that measures how close the guess is to the true Hamiltonian H_i. It is natural to normalize the distance D, for instance, by requiring that $D(H,H+E\mathbb{I})=E$, where $\mathbb{I}$ is the identity operator.

There are many different distances on the space of Hamiltonians. For instance $D(H_1,H_2)=\sqrt{\mathrm{Tr}(H_1-H_2)^2}/d$, where d is the dimension of the Hilbert space. In general the distance that one will use will depend on the specific problem one considers. In the present paper we shall use the distance D_0 introduced in Eq. (3). The reason is that this is the distance that appears in the results of Sec. II, and these results are used in the arguments below. We expect that using another distance would change quantitatively, but not qualitatively, our results.

One of the most fundamental questions concerning the estimation of an unknown Hamiltonian is the relation between the precision ΔH with which the Hamiltonian is known and the time Δt used to carry out the estimation. The remainder of this section is devoted to addressing this question.

B. Estimating a Hamiltonian which can only take two values, H_1 or H_2

As a first application of the general problem of estimating an unknown Hamiltonian, we consider the particular situation in which their are only two possible Hamiltonians H_1 and H_2, which are equally probably $[p(H_1)=p(H_2)=1/2]$ and the task is to estimate which it is in a finite time Δt. The quality of this estimate shall be expressed by using the distance D_0 as the distance on the space of Hamiltonians defined in Eq. (3). The techniques developed in Sec. II will allow us to solve this problem exactly.

In Sec. II, it was shown that if $\Delta t\geqslant \pi/D_0(H_1,H_2)$, then the two Hamiltonians can be distinguished perfectly and therefore ΔH is zero. On the other hand, when $\Delta t\to 0$, it is impossible to obtain any information about the Hamiltonians and the best strategy is to randomly guess either H_1 or H_2. Hence in this limit $\Delta H\to D_0(H_1,H_2)/2$. For intermediate times, ΔH will decrease from $D_0(H_1,H_2)/2$ to zero as a function of Δt. We shall show that for the optimal estimation strategy, ΔH is given by

$$\Delta H=\max\left\{0,\frac{D_0(H_1,H_2)}{2}\left[1-\sin\left(\frac{D_0(H_1,H_2)\Delta t}{2}\right)\right]\right\}. \tag{24}$$

To prove this let us first show that for an optimal guessing strategy, it is sufficient that the guessed Hamiltonian be either H_1 or H_2. We recall that in the estimation problem, as formulated in Sec. III A, we left open the possibility of guessing a Hamiltonian that is not one of the possible Hamiltonians H_i. In the present case, it is not necessary to consider such possibilities.

To show this let us consider the contribution of guess j to ΔH [we denote this contribution $\Delta H(j)$]:

$$\Delta H(j)=\frac{1}{2}p(j|H_1)D(H_1,H_j^{guess}) + \frac{1}{2}p(j|H_2)D(H_2,H_j^{guess}). \tag{25}$$

Using the triangle inequality, we can write

$$\Delta H(j)\geqslant\frac{1}{2}p(j|H_1)D(H_1,H_2) + \frac{1}{2}[p(j|H_2)-p(j|H_1)]D(H_2,H_j^{guess}). \tag{26}$$

If $p(j|H_2)-p(j|H_1)\geqslant 0$, we finally have

$$\Delta H(j)\geqslant\frac{1}{2}p(j|H_1)D(H_1,H_2) \tag{27}$$

with equality if and only if $H_j^{guess}=H_2$. Thus if the *a posteriori* probability that the Hamiltonian H_2 is greater than the *a posteriori* probability that the Hamiltonian was H_1, one should guess that the Hamiltonian is H_2. And conversely if the *a posteriori* probability that the Hamiltonian was H_1 is greater than the a posteriori probability that the Hamiltonian was H_2, one should guess that the Hamiltonian is H_1.

Let us now consider the optimal evolution and measurement strategy. The estimation strategy starts with a given quantum state ψ_0. If the Hamiltonian is H_1 this state evolves into $\psi_1(t)$, whereas if the Hamiltonian is H_2 the state evolves into $\psi_2(t)$. In Sec. II A, it was shown that the overlap between these two states must obey the inequality

$$|\langle\psi_1(t)|\psi_2(t)\rangle|\geqslant\cos\frac{D_0t}{2}, \tag{28}$$

with equality attained for the optimal strategy [we have denoted $D_0=D_0(H_1,H_2)$]. Furthermore, it is shown in Ref. [10] that the probability p_E of making an error when trying to distinguish two equiprobable states ψ_1 and ψ_2 is bounded by

$$p_E\geqslant\frac{1-\sqrt{1-|\langle\psi_1|\psi_2\rangle|^2}}{2}\geqslant\frac{1-\sin(D_0t/2)}{2}, \tag{29}$$

with equality attained if one carries out a von Neumann measurement of the basis $(|\psi^{\parallel}\rangle \pm |\psi^{\perp}\rangle)/2$. Hence we find that for the optimal strategy,

$$\Delta H = p_E D_0 = \frac{D_0}{2}\left[1 - \sin\left(\frac{D_0 \Delta t}{2}\right)\right]. \tag{30}$$

When $\Delta t \geq \pi/D_o$, then ΔH in Eq. (30) is negative. Since ΔH should always be positive, we take the maximum of Eq. (30) and zero, which gives Eq. (24).

C. Estimating a completely unknown Hamiltonian

We now consider the situation where the Hamiltonian is completely unknown. Once more we shall use the distance D_0 as the distance on the space of Hamiltonians of Eq. (3). We shall show that in this case the precision ΔH with which the Hamiltonian is estimated and the time used to estimate the Hamiltonian must obey the constraint

$$\Delta H \Delta t \geq \frac{1}{4}. \tag{31}$$

This constitutes one of the fundamental results of this paper.

To prove this, we will contrast two situations. In the first situation (which is the one we are interested in), the experimenter has no information about the Hamiltonian. In the second situation, we imagine that there is a ''spy'' that knows the true Hamiltonian, call it H_0. The spy then tells the experimenter that the true Hamiltonian is either H_0 or some other Hamiltonian H_1. The *a priori* probabilities that the spy chooses Hamiltonian H_0 or H_1 are equal.

Denote by ΔH the maximum precision with which the Hamiltonian can be known in the first case, and by ΔH_{dicho} the maximum precision with which the Hamiltonian can be known with the help of the spy. Obviously, $\Delta t \Delta H \geq \Delta t \Delta H_{dicho}$ since the time intervals are the same in the two situations and the information provided by the spy can only increase the precision with which one can estimate the Hamiltonian. This means that $\Delta t \Delta H \geq \max \Delta t \Delta H_{dicho}$, where the maximum is taken over all possible choices of the spy. The results of Sec. III B can be used to show that $\max \Delta t \Delta H_{dicho} \geq 1/4$, which proves Eq. (31).

To show that $\max \Delta t \Delta H_{dicho} \geq 1/4$, note that in the dichotomic case, the product $\Delta H \Delta t$ takes the form

$$\Delta H \Delta t = \frac{D_0 \Delta t}{2}\left[1 - \sin\left(\frac{D_0 \Delta t}{2}\right)\right] \tag{32}$$

for $0 \leq \Delta t \leq \pi/D_0$. For small times this tends to zero since ΔH is bounded and $\Delta t \to 0$. And for $\Delta t \geq \pi/D_0$, the product is zero since $\Delta H = 0$. There is an intermediate time where the product attains its maximum. One easily shows [using $\sin x \leq x$ for $x \geq 0$, which implies that $x(1-\sin x) \geq x(1-x)$] that the maximum value of this product is greater than a quarter,

$$\max \Delta H \Delta t \geq \frac{1}{4}. \tag{33}$$

D. Estimating a completely unknown Hamiltonian acting in a *d*-dimensional space

Equation (31) gives a lower bound on the product of the precision with which a completely unknown Hamiltonian is measured and the time taken to estimate it. We believe that this lower bound is not tight and that in general a stronger lower bound should hold. We do not know at present what form this stronger lower bound will take, but we believe that it should depend on the dimensionality of the Hilbert space on which the Hamiltonian acts.

An indication that this should be the case is provided by an example due to Farhi and Gutmann [8] inspired by Grover's search algorithm [9]. In this example, one must distinguish between d Hamiltonians of the form $H_k = E|k\rangle\langle k|$, where the d states $|k\rangle$ form an orthonormal basis. Farhi and Gutmann show that in order to distinguish these Hamiltonians perfectly, a minimum time of $\Delta t \geq c d^{1/2}/E$ is necessary (where c is some positive constant).

This example shows that there are situations where estimating an unknown Hamiltonian becomes increasingly difficult as the dimension d of the Hilbert space on which it acts increases. However, in the Fahri-Gutmann example, the unknown Hamiltonian has a very specific form which is known before hand. We have obtained preliminary indications that when the Hamiltonian is completely unknown, estimating it should take substantially more time than suggested by the Fahri-Gutmann example. We hope to report on this issue in a future publication.

IV. MEASURING ENERGY WHEN THE HAMILTONIAN IS UNKNOWN

A. Introduction

The results presented in the preceding section concerning the estimation of Hamiltonians have important implications for energy measurements. As shown in Ref. [3], the energy of the state can be measured in an arbitrarily short time. However a careful scrutiny of the arguments of Ref. [3] shows that a quick energy measurement is possible only if the Hamiltonian is known. In the example discussed in the Introduction, it is possible to carry out a quick energy measurement only because we know that the particle is in a magnetic field $\vec{B}_0$ of known magnitude pointing along the $+z$ direction. Suppose, however, that the magnetic field is pointing initially either along the $+z$ or the $-z$ axis. Then a measurement of σ_z yields no information about the energy. Thus in order to determine the energy of the particle, we must also determine the magnetic field. That is we must also determine which is the Hamiltonian. But as we discussed above, determining the Hamiltonian will take a minimum time Δt. (We suppose that the only way we can probe the magnetic field is with a particle of magnetic moment μ. Of course, if we could use a particle of larger magnetic moment, the measurement of the magnetic field could be done faster.) Thus in this example measuring the energy of the state cannot be done instantaneously because the Hamiltonian is not perfectly known.

In fact this is a very general result. We shall show below that *if the Hamiltonian acting on a system is completely unknown, then the precision* ΔE *with which one can estimate the energy of the state in a time interval* Δt *obeys the constraint*

$$\Delta t\, \Delta E \geqslant 1/4. \tag{34}$$

This assertion follows easily from the results obtained in the preceding section. However, before proving it we first need to define with precision what we mean by an accuracy ΔE of an energy measurement.

B. Accuracy of an energy measurement

An ideal energy measurement is a von Neumann measurement of the Hamiltonian operator $H=\Sigma_E E|E\rangle\langle E|$. If the quantum state is $|\psi\rangle$, the measurement gives result E with probability $p(E|\psi)=|\langle\psi|E\rangle|^2$.

Let us consider an imperfect measurement. This measurement will predict that the energy is E' with probability $p(E')$. Neither the energies E' nor the probabilities $p(E')$ need to coincide with the energies and probabilities for an ideal energy measurement. Nevertheless, we would like to define in a precise way the accuracy of an energy measurement.

The simplest situation in which to define the accuracy of an energy measurement is when the quantum state is an energy eigenstate $|\psi\rangle=|E\rangle$. In this case the true energy of the state is well defined. Hence the accuracy of the imperfect energy measurement is simply the amount by which the energies E' differ from the true energy E:

$$\Delta E=\sum_{E'} p(E'|E)|E'-E|, \tag{35}$$

where $p(E'|E)$ is the probability that the estimated energy is E' when the quantum state is $\psi=|E\rangle$.

If the state is not an energy eigenstate then we define the accuracy of the energy measurement as the average over the probability $|\langle E|\psi\rangle|^2$ that an ideal energy measurement gives result E times of the accuracy of the measurement if the state is $|E\rangle$:

$$\Delta E=\sum_{E} |\langle E|\psi\rangle|^2 \sum_{E'} p(E'|E)|E'-E|. \tag{36}$$

C. Proof of the time-energy uncertainty relation for energy measurements when the Hamiltonian is completely unknown

We start by noting that the proof of Eq. (34) when ψ is not an eigenstate of the Hamiltonian follows from the case where ψ is an eigenstate of the Hamiltonian, since we have defined in Eq. (36) the uncertainty when ψ is not an eigenstate of the Hamiltonian as an average of the uncertainties when ψ is an eigenstate of the Hamiltonian times the probability that a measurement of the Hamiltonian operator yields the corresponding energy. Therefore we can restrict ourselves to considering the case where $\psi=|E\rangle$ is an eigenstate of the Hamiltonian H. Since the Hamiltonian H is completely unknown, the state ψ is also unknown.

To prove Eq. (34), we fix Δt and contrast, as in the Sec. III C, the two situations. In the first, one has no information about the Hamiltonian. In the second, a spy gives the additional information that the Hamiltonian is either H_0 (the true Hamiltonian) or H_1. We shall suppose that $H_1=H_0+\epsilon\mathbb{1}$ where ϵ is a *c* number and $\mathbb{1}$ is the identity operator. We shall further suppose that with the information provided by the spy there is equal *a priori* probabilities that the Hamiltonian is H_0 or H_1. Let us denote the energy uncertainty in the first situation by ΔE and in the second situation by ΔE_{spy}. Obviously we have $\Delta t\Delta E \geqslant \Delta t\Delta E_{spy}$ since Δt is the same in both situations and the information provided by the spy can only decrease the energy uncertainty. We therefore want to put a bound on ΔE_{spy}.

First, note that since the two Hamiltonians H_0 and $H_1=H_0+\epsilon\mathbb{1}$ commute, the experimenter can immediately determine which is the state ψ by measuring the operator H_0. This measurement can, in principle, be done arbitrarily fast. Hence the experimenter knows that the energy is either E, the true energy, or $E+\epsilon$.

Since the experimenter has only two possibilities between which to choose, an optimal strategy will consist of guessing either that the energy is E or $E+\epsilon$. It is not necessary to consider other possibilities such as guessing that the energy is $E+\epsilon/2$. Furthermore, the energy uncertainty will be $\Delta E_{spy}=\epsilon p_E$, where p_E is the probability of making the wrong guess. The proof of these assertions follows from the fact that ΔE, as defined in Eq. (35), is linear in the probabilities $p(E')$ times a distance $|E'-E|$ on the space of energies. Hence the arguments of Sec. III B, [Eqs. (25)–(27)], can be used in the present case.

But the error probability p_E of mistaking one energy for the other is identical to the error probability of wrongly identifying the Hamiltonians H_0 and H_1 (since knowing the energy is equivalent to knowing the Hamiltonian). Hence in the present case $\Delta E_{spy}=\Delta H$, where ΔH is the uncertainty in estimating the two Hamiltonians H_0 and H_1. But we have shown in Sec. III B that for given Δt there exists a choice of ϵ such that $\Delta t\Delta H \geqslant 1/4$.

D. Saturating the time-energy uncertainty for energy measurements when the Hamiltonian is unknown?

We now address the question of *whether it is possible to devise a universal measurement strategy that can determine the energy of an unknown state even if there is no prior knowledge about the Hamiltonian?* Such universal measurement strategies exist and are well known. We illustrate them by a typical example, namely the emission of electromagnetic radiation by an excited state of an atom.

We then inquire whether this measurement strategy saturates the time-energy uncertainty relation obtained in the preceding section. It turns out that for this measurement strategy, and using the definition of Sec. IV B, $\Delta t\Delta E$ is infinite. However, in a qualitative way this measurement strategy does obey a time-energy uncertainty. This is discussed in detail.

As mentioned in the Introduction, the emission of one or more photons by an excited atomic state can be viewed as a measurement of the energy of the electrons. The coupling between the measuring apparatus (the electromagnetic field A^μ) and the system (the electron) is realized through the interaction

$$H_{int}=\int d^3x A^\mu(x) J_\mu(x), \tag{37}$$

where $J_\mu(x)$ is the electric current. In interaction representation it takes the form

$$H_{int}=\int_0^\infty d\omega \sum_k (a^\dagger_{\omega k} e^{i\omega t}+a_{\omega k} e^{-i\omega t}) J_{k\omega}. \tag{38}$$

Here ω is the energy of the photons; k represents other degrees of freedom of the photons, in addition to their energy (momentum and polarization); and $J_{k\omega}$ are operators acting on the electrons Hilbert space. The photons are taken to be initially in their ground state $a_{\omega k}|0\rangle=0$.

The interaction Hamiltonian (37) is independent of the electron Hamiltonian, i.e., it is independent of whether the electron is bound to a proton, a helium nucleus, a molecule, etc. Therefore such a measurement can determine the energy of an unknown state, independently of the Hamiltonian. It can also determine which is the Hamiltonian, since the energy of the emitted photons will differ if the electron is bound to a proton, a helium nucleus, a molecule, etc. The price to pay for this universality is that the energy resolution of the measurement and the time necessary for the measurement are constrained by the time-energy uncertainty. The arguments presented in this paper show that this will always be the case for a measurement of energy, which does not take into account prior knowledge about the Hamiltonian.

An important limitation of the above measurement scheme is that the emitted photon only reveals the difference in energy between the initial and final state of the atom. If there are several allowed transitions with identical energy differences, then the measurement will not allow these initial states to be differentiated. It would have to be complemented by a second measurement to determine which of the possible final states the atom reached. Nevertheless, the important point of this example is to show that, in principle, it is possible to come close to saturating the time-energy uncertainty relation when estimating the energy of a system whose Hamiltonian is unknown.

In order to see how close we come to saturating the time-energy uncertainty in this scheme, let us examine it in more detail. The probability density that a photon is emitted at time t is

$$P(\text{decay at time } t)=\gamma e^{-\gamma t}. \tag{39}$$

Thus the time it takes to complete the measurement is not well defined. Rather this time is variable but its mean is finite,

$$\int dt\, t\, P(\text{decay at time } t)=\gamma^{-1}. \tag{40}$$

This is to be contrasted with the situation envisaged in the previous sections where we required that the measurement be finished after some time interval Δt.

Let us now consider the energy of the emitted photon. If the true energy of the electronic state is E_0, then the probability density that the emitted photon has energy E is

$$P(\text{emitted photon has energy } E)=\frac{1}{\pi}\left(\frac{\gamma}{\gamma^2+(E-E_0)^2}\right). \tag{41}$$

If we compute the accuracy of the energy measurement using the definition (36), we find

$$\Delta E=\int dE P(\text{emitted photon has energy } E)|E-E_0|=+\infty. \tag{42}$$

Thus this measurement satisfies the time-energy uncertainty relation. In fact the product $\Delta t \Delta E$ is infinite since both the time it takes to complete the measurement and the energy uncertainty are infinite. However, we note that if we modify the definition of Δt to be the mean time γ^{-1} taken to carry out the measurement and if we modify the definition of the energy uncertainty ΔE to be the linewidth γ, then this measurement does obeys a time-energy uncertainty relation.

Therefore it may be possible to devise a better energy measurement that saturates the time-energy uncertainty derived in Sec. IV C. Or this uncertainty relation is too strong, and one can prove a weaker form of the uncertainty relation, for instance using as definition of Δt and ΔE, the mean time taken for the measurement and the linewidth [where the linewidth can be defined operationally as $\gamma=(E_2-E_1)/2$ with $E_{1,2}$ the energies such that P (emitted photon has energy E) has half its maximum value]. In the latter case, the measurement just described would be optimal in the sense that it would saturate the time-energy uncertainty for energy measurements when the Hamiltonian is unknown.

E. Estimating energy when one has partial knowledge about the Hamiltonian

In the preceding section, we considered the situation where one wants to estimate the energy of an unknown state, but one has no prior knowledge about the Hamiltonian. When some prior knowledge is available the situation is considerably more complicated and the relation between the time used for the measurement and the precision with which the energy can be estimated will depend on the details of the problem.

To illustrate this we consider two examples. First, consider the case of two Hamiltonians H_1 and H_2 that have the same eigenstates $H_1\psi_k=E_{1k}\psi_k$, $H_2\psi_k=E_{2k}\psi_k$ and their eigenvalues coincide except for one eigenstate $E_{1k}=E_{2k}(k\neq k_0)$ but $E_{1k_0}\neq E_{2k_0}$. Suppose we must determine the energy of an unknown state Ψ. A strategy to do this in a short time is to first carry out a von Neumann measurement of the basis ψ_k that diagonalizes H_1 and H_2 (this can be done in an arbitrarily short time). If one finds that the outcome k is different from k_0, then one immediately knows the energy. On the other hand, if $k=k_0$, then to know the energy one must determine which is the Hamiltonian. This takes a time $\Delta t=\pi/|E_{1k_0}-E_{2k_0}|$. If the unknown state Ψ was uniformly

distributed in Hilbert space (denoted $\mathcal{H}$), then the probability that Ψ belongs to subspace k_0 is $1/\dim\mathcal{H}$ and the average time necessary to determine the energy of the state is $\pi/\dim\mathcal{H}|E_{1k_0}-E_{2k_0}|$, which is much smaller than the time needed to determine the Hamiltonian. The reason for this difference in time scales is because the particle has most of its support in a part of the Hilbert space where the two Hamiltonians do not differ.

Our second example is superficially similar to the previous one. But more careful consideration show some subtle differences. We consider a particle confined to a (one-dimensional) box. The potential in the box vanishes everywhere, except in a corner where it may be either zero or take a large negative value. In order to measure the energy of the particle in a minimum time the following strategy seems natural. First, we measure whether the particle is in the corner or not. (This position measurement should be slightly fuzzy so as not to disturb the momentum too much.) This measurement can, in principle, be done in an arbitrarily short time. If the particle is not in the corner, we measure its momentum, and hence know its energy. This can also be done in an arbitrarily short time. If the particle is in the corner, then we must determine the value of the potential in order to know the energy of the particle. This takes a finite time of order $1/\Delta V$, where ΔV is the uncertainty in the potential.

Thus in this case it seems that the minimum time required for the energy measurement depends essentially on the probability of the particle being in the corner of the box rather than on the precision ΔE with which one wants to know the energy. However, the situation is more complicated. The above procedure approximates to some extent a von Neumann measurement of the Hamiltonian operator. Indeed the statistics of the measurement outcomes are such that they reproduce correctly the moments of the Hamiltonian operator. Thus, for instance, upon repeating the measurement many times one will obtain a good estimate of the average energy $\langle H\rangle$, or the average value of any power of the Hamiltonian $\langle H^n\rangle$. However, the above procedure is not equivalent to a von Neumann measurement of the Hamiltonian operator (which is the task we set out to perform). Indeed there are some functions of the Hamiltonian, which cannot be estimated correctly with the above procedure.

Suppose, for instance, that the particle is known to be approximately localized at a distance d from the corner of the box (d is taken to be much larger than the size of the region where the potential is unknown), and suppose that it is known that the momentum of the particle is approximately p. Since the particle is far from the corner, by measuring the momentum of the particle one has some information about its energy. For instance, independent repetitions of the measurement will yield estimates of the moments of the Hamiltonian operator $\langle H^n\rangle$. Such measurements of the momentum can be done in an arbitrarily short time. However, suppose that one wants to measure the operator $\cos(mdH/p)=(e^{imdH/p}+e^{-imdH/p})/2$ (where m is the mass of the particle). This is the real part of the operator that evolves the particle from its initial position up to the corner where the potential is unknown. The expectation value of the above operator clearly depends on the value of the potential in the corner, and therefore it can only be determined in a time of order $1/V$.

The origin of this surprising situation is that in the above example the exact spectrum of H depends on the potential in the corner. When the particle is far from the corner, most questions concerning the energy of the particle are independent of the exact spectrum of H. But a variable such as $\cos(mdH/p)$ is sensitive to the exact spectrum and therefore in order to measure it one must know what is the potential in the corner. Operators such as $\cos(mdH/p)$ are called modular variables, and have been introduced in Ref. [11].

V. CONCLUSIONS

In the present paper, we have shown that if the Hamiltonian that governs the evolution of a quantum system is unknown, then the time necessary to estimate the Hamiltonian obeys a time-energy uncertainty relation (2). To this end we first gave a simple proof of the problem considered by Childs, Preskill, and Renes [5] where there are only two possible Hamiltonians between which one must choose. We then showed how to extend this result to the case where there are many Hamiltonians among which one must choose. The bound we obtained is probably not tight when the unknown Hamiltonians act in a space of large dimensionality $d>2$, and it should be possible to refine it by a more detailed analysis.

Our results concerning the time-energy uncertainty relation applied to estimating Hamiltonians have many applications. In particular, they provide new insight about how the time-energy uncertainty applies to energy measurements. It has been shown by Aharonov and Bohm that if the Hamiltonian of the system is known, then the Hamiltonian can, in principle, be measured in an arbitrarily short time [3]. On the other hand, we show that if the Hamiltonian is unknown then the energy measurement cannot be done in an arbitrarily short time. The minimum amount of time required depends on the details of the problem, for instance what is the prior knowledge about the Hamiltonian, what is the prior knowledge about the state, and exactly what one wants to know about the energy of the state. We show that if one has no prior information about the Hamiltonian, then the time taken to carry out the measurement and the precision with which the energy is measured obey a time-energy uncertainty relation.

We also show that one can devise a measurement of the energy of a quantum system that always works, independently of any prior knowledge about the system. Such a measurement is obtained by coupling the system to an external apparatus that oscillates at all frequencies and such that each frequency is coupled to different degrees of freedom of the apparatus. This is illustrated in Eq. (38) in the case where the external apparatus is taken to be the electromagnetic field. Such measurements do not saturate our time-energy uncertainty relation, although they do obey a qualitatively similar uncertainty relation between the lifetime of the state and the linewidth.

To conclude, we find that the real meaning of energy in quantum mechanics is that of governing the time evolution of a system. To measure the energy one has to determine the time evolution, and this takes time. Thus energy measurements require time, and their precision is limited by the time we have at our disposal. On the other hand, in the examples presented by Aharonov and Bohm [3], the Hamiltonian is known in advance, hence one need not spend time to determine the time evolution. Instead, one could find out the value of the energy not by determining the time evolution, i.e., not by measuring the actual energy, but by measuring an operator (the operator to which the Hamiltonian is equal) whose numerical eigenvalue is equal to that of the energy. However, we emphasize that although this procedure does yield the numerical values equal to that of the energy, it is not a proper energy measurement. Indeed, if we believe the Hamiltonian to be H but in reality it is different, say H', then the value obtained by the instantaneous Aharonov-Bohm measuring procedure (which tells us to measure the operator H) would no longer be correct, and, furthermore, we would not know that our measurement is wrong. Thus a proper energy measurement necessarily probes the time evolution and therefore cannot be done instantaneously. Rather the time taken to carry out the measurement and the precision with which one knows the energy are constrained by a time-energy uncertainty relation $\Delta t \Delta E \geqslant 1$.

ACKNOWLEDGMENTS

Y.A. acknowledges support by Grant No. 471/98 of the Israel Science Foundation established by the Israel Academy of Sciences and Humanities. S.M. and S.P acknowledge funding by project EQUIP (IST-FET program) of the European Union. S.M. is supported by the Belgium National Research Foundation (FNRS).

APPENDIX

In this appendix, we show that $D_0(H_1,H_2)$ defined in Eq. (3) is a distance on the space of Hamiltonians. That is D_0 is (1) positive: $D_0(H_1,H_2) \geqslant 0$ with equality if and only if $H_1=H_2$; (2) D_0 is symmetric: $D_0(H_1,H_2)=D_0(H_2,H_1)$; and (3) D_0 obeys the triangle inequality $D_0(H_1,H_2) \leqslant D_0(H_1,H_3)+D_0(H_3,H_2)$.

Let us first introduce a norm

$$||H||_0 = \max\{E^{max}-E^{min}, |E^{max}|, |E^{min}|\}, \tag{A1}$$

where E^{max}, E^{min} are the largest, smallest eigenvalues of H. Now

$$D_0(H_1,H_2) = ||H_1-H_2||_0, \tag{A2}$$

hence if we can prove that $||H||_0$ is indeed a norm, then it follows immediately that D_0 is a distance.

We recall that a norm must satisfy the following properties: (1) positivity: $||H|| \geqslant 0$ with equality if and only if $H=0$; (2) linearity: $||\lambda H|| = |\lambda| ||H||$ for any c number λ; (3) triangle inequality $||H_1+H_2|| \leqslant ||H_1|| + ||H_2||$.

Properties (1) and (2) are immediate. Let us consider property (3). Let us denote by E_1^{max} and $|\psi_1^{max}\rangle$ the largest eigenvalue of H_1 and the corresponding eigenvector; by E_2^{max} and $|\psi_2^{max}\rangle$ the largest eigenvalue of H_2 and the corresponding eigenvector; by E_{12}^{max} and $|\psi_{12}^{max}\rangle$ the largest eigenvalue of H_1+H_2 and the corresponding eigenvector. Let us show that $E_{12}^{max} \leqslant E_1^{max}+E_2^{max}$. We have

$$\begin{aligned} E_{12}^{max} &= \langle \psi_{12}^{max} | (H_1+H_2) | \psi_{12}^{max} \rangle \\ &= \langle \psi_{12}^{max} | H_1 | \psi_{12}^{max} \rangle + \langle \psi_{12}^{max} | H_2 | \psi_{12}^{max} \rangle \\ &\leqslant \langle \psi_1^{max} | H_1 | \psi_1^{max} \rangle + \langle \psi_2^{max} | H_2 | \psi_2^{max} \rangle \\ &= E_1^{max} + E_2^{max}. \end{aligned} \tag{A3}$$

Similarly we have $E_{12}^{min} \geqslant E_1^{min} + E_2^{min}$, where $E_{12}^{min}, E_1^{min}, E_2^{min}$ are the smallest eigenvalues of H_1+H_2, H_1, H_2, respectively. The triangle inequality follows from these relations between eigenvalues and from the definition of $||H||_0$.

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