Topological singularities and supersymmetry breaking

Y. Aharonov, A. Casher, and S. Yankielowicz
Department of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel
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The influence of topological singularities (fluxons) on two- and three-dimensional supersymmetric quantum mechanics is investigated. It is proved that the Aharonov-Bohm effect is incompatible with supersymmetry. The violation is a quantum effect and its quantitative amount is determined by the topology.

I. INTRODUCTION

Among the global symmetries operating on quantum systems, supersymmetry (SUSY)[1] possesses two related unique features (apart from the obvious one of mixing bosons and fermions). The SUSY generators satisfy an algebra which contains the Hamiltonian, and consequently the SUSY transformations mix the canonical coordinates with their velocities. As a result of this property, it is in general difficult to define closed supersymmetric systems, since the boundary conditions tend to violate the SUSY current conservation. In general such effects are not very interesting since the amount of violation depends on the size of the boundary, and exact conservation is restored when this dimensional parameter disappears from the problem.

The latter circumstance is changed drastically when the boundary conditions are imposed on the system by the action of a gauge field whose topology is nontrivial. The field-free space in this case encloses some finite-size region (which may be shrunk to zero) which contains a nonzero flux. The requirement that finite-energy wave functions be single-valued in the relevant angular variable then imposes a particular behavior on the wave functions as the boundary is approached from the outside. If this behavior causes a nonvanishing SUSY flux at the boundary, which persists when the relevant dimension is shrunk to zero, a breakdown of supersymmetry will be generated. This violation is a topological quantum effect in a system whose classical equations of motion are invariant.

In what follows, we investigate the effect of magnetic fluxons on two-dimensional SUSY quantum mechanics, with two complex supersymmetries. Specifically, we prove that the Aharonov-Bohm effect[2] is inconsistent with supersymmetry. The SUSY breakdown is manifested through split supermultiplets and can be characterized quantitatively by a flux-dependent anomalous commutator between the supercharges and the Hamiltonian. The results may be derived both by solving directly the Schrödinger equation for a singular fluxon and by taking the limit $R\to 0$ of a regular fluxon of radius R. It should be emphasized that (as will be seen) the SUSY violation is not due to the global topology at ∞ but rather depends on the presence of a distribution of "holes" in configuration space. Thus, the violations due to a fluxon and an antifluxon of equal and opposite strengths do not cancel but introduce the distance between the two as a scale in the nonconservation equation. The restriction to two (space) dimensions is not crucial and we shall show that the phenomenon also occurs in three spatial dimensions.

The paper is organized as follows. In Sec. II we discuss two-dimensional SUSY quantum mechanics in the presence of singular fluxons. The case of regular fluxons is investigated in Sec. III while Sec. IV deals with the generalization to three dimensions. We conclude with a discussion of the relevance of the preceding results to supersymmetric gauge field theories. In Appendix A we exhibit the exact solution of the Aharonov-Bohm effect[2] for the two-dimensional SUSY harmonic oscillator. This serves to illustrate the phenomenon discussed in the paper. Appendix B demonstrates that the introduction of singular fluxons into any two-dimensional supersymmetric theory shifts the ground-state energy away from zero.

II. SUSY QUANTUM MECHANICS AND SINGULAR FLUXONS

We shall investigate supersymmetric two-dimensional quantum-mechanical Hamiltonians which can be derived by reducing $N=1$ $(3+1)$-dimensional field theories to $(0+1)$ dimensions. In the language of quantum mechanics we thus deal with *two* complex SUSY generators. We introduce complex canonical variables:

$$[p,q]=0,\quad [p,q^\dagger]=[p^\dagger,q]=\frac{1}{i}\ , \tag{1}$$

$$q=\frac{1}{\sqrt{2}}(q_1+iq_2)=\frac{1}{\sqrt{2}}re^{i\theta}\ ,$$
$$p=\frac{1}{\sqrt{2}}(p_1+ip_2)=\frac{e^{i\theta}}{\sqrt{2}}\frac{1}{i}\left[\partial_r-\frac{l}{r}\right],\quad l=\frac{1}{i}\partial_\theta\ . \tag{2}$$

Corresponding to the complex boson "fields" q there are two fermionic variables:

$$\sigma=1,2:\ \{\chi_\sigma,\chi_\tau\}=0,\quad \{\chi_\sigma,\chi_\tau^\dagger\}=\delta_{\sigma\tau}\ . \tag{3}$$

The SUSY generators are thus

$$Q_\sigma=\chi_\sigma p+i\chi_\tau^\dagger\epsilon_{\tau\sigma}w'(q)\ , \tag{4}$$

where $w(q)$ is an analytic function of q only. The SUSY algebra is therefore[3]

$$\{Q_\sigma,Q_\tau\}=\{Q_\sigma^\dagger,Q_\tau^\dagger\}=0\ , \tag{5}$$

$$\{Q_\sigma,Q_\tau^\dagger\}=\delta_{\sigma\tau}H\ , \tag{6}$$

where the Hamiltonian is

$$H=pp^\dagger+w'w'^\dagger+(\chi_1\chi_2w''^\dagger+\chi_2^\dagger\chi_1^\dagger w'')\ . \tag{7}$$

Equations (5) and (6) insure the conservation of Q:

$$[Q_\sigma,H]=[Q_\sigma^\dagger,H]=0\ . \tag{8}$$

Consider now the addition of an external gauge field characterized by a vector potential a_k and define the velocity operators $p-a$:

$$\Pi=p-a,\ \ [\Pi,\Pi^\dagger]=\partial_1a_2-\partial_2a_1=b\ , \tag{9}$$

where

$$a=\frac{1}{\sqrt{2}}(a_1+ia_2)\ , \tag{10}$$

and b is the magnetic field. Suppose now that a is the vector potential due to a distribution of singular fluxons,

$$a=\frac{i}{2}\sum_A\frac{\nu_A}{(q^\dagger-q_A^\dagger)}\ . \tag{11}$$

We easily find that the magnetic field vanishes:

$$b=0\ , \tag{12}$$

while the flux enclosing the Ath singularity is

$$\oint_{C_A}d\vec{1}\cdot\vec{a}=2\pi\nu_A\ . \tag{13}$$

We *define* the Hamiltonian by replacing in Eq. (7) p by Π. Viewed as differential operators acting on wave functions, the generators Q_σ still satisfy Eqs. (6) and (7) since for $q\neq q_A$ the vector potentials may be canceled by a gauge transformation.[4] We shall show, however, that the SUSY charges cease to be conserved, and derive their nonconservation equation.[5]

Consider first the Schrödinger equation in the vicinity of a given fluxon, which we put at the origin. Expanding the wave function in angular momentum states,

$$\psi(q)=\frac{1}{\sqrt{2\pi}}\sum_{l=-\infty}^{\infty}e^{il\theta}\psi_l(r)\ , \tag{14}$$

the most singular terms of the equation are given by the kinetic part of H:

$$(H\psi)_l=\frac{1}{2}\left[-\partial_r{}^2-\frac{1}{r}\partial_r+\frac{(l-\nu)^2}{r^2}\right]\psi_l+\cdots\ . \tag{15}$$

The condition for a finite-energy solution which belongs to the domain of H is

$$\psi_l(r)\underset{r\to0}{\sim}\psi_l^{(0)}r^{|l-\nu|}\ , \tag{16}$$

where $\psi_l^{(0)}$ is a constant vector in the four-state Hilbert space defined by the fermion operators $(\chi_\sigma,\chi_\tau^\dagger)$. (Although the solutions which behave as $r^{-|l-\nu|}$ are normalized for $|l-\nu|<1$, they are discarded by the requirement that H be self-adjoint.)

Let us calculate the rate of change of a matrix element of Q:

$$i\frac{d}{dt}\langle\psi,Q_\sigma\phi\rangle=\langle\psi,Q_\sigma H\phi\rangle-\langle H\psi,Q_\sigma\phi\rangle\ . \tag{17}$$

Integrating by parts and using the commutativity of Q and H when they act as differential operators on a given wave function leads to

$$\begin{aligned}\frac{d}{dt}\langle\psi,Q_\sigma\phi\rangle&=\frac{i}{2}\sum_{l=-\infty}^{\infty}\lim_{r\to0}r[\psi_l^*(r)'(Q_\sigma\phi)_l(r)-\psi_l^*(r)(Q_\sigma\phi_l(r))']\\&=\frac{1}{2\sqrt{2}}\sum_l\lim_{r\to0}r^{(|l-\nu|+|l-1-\nu|-1)}[\,|l-1-\nu|-(l-1-\nu)][\,|l-\nu|-|l-1-\nu|+1]\\&\quad\times(\psi_l^{(0)}|\chi_\sigma|\phi_l^{(0)})=\sqrt{2}(\psi_1^{(0)}|\chi_\sigma|\phi_0^{(0)})\nu(1-\nu)\ .\end{aligned} \tag{18}$$

In deriving Eq. (18) we have defined ν to satisfy

$$0\le\nu\le1 \tag{19}$$

and used the polar representation for Π:

$$\Pi=\frac{e^{i\theta}}{i\sqrt{2}}\left[\partial_r-\frac{l-\nu}{r}\right] \tag{20}$$

which explicitly exhibits the fact that the kinetic part of Q shifts the angular momentum by one unit. Equation (18) shows that although Q is classically conserved, quantum effects induce an anomaly in the time derivative of its matrix elements between finite-energy states. Note that $\dot{Q}$ is periodic in ν and vanishes when $\nu\to0,1$; a shift of ν by an integer is evidently canceled by a corresponding shift in l. An integral value of ν means that the fluxon is quantized in Dirac units and thus its effects can be gauged away. The constants $\psi_1^{(0)}$, $\phi_0^{(0)}$ depend only on the flux ν and on the infrared properties of the wave function and the superpotential w. In Appendix A we evaluate explicitly these constants for the SUSY harmonic oscillator and derive the induced split in the supermultiplet spectrum. In Appendix B we prove generally that the ground-state energy is nonzero and is in fact positive. Finally, the generalization of Eq. (18) for the multifluxon case is evidently

$$\frac{d}{dt}\langle\psi,Q_\sigma\phi\rangle=\sqrt{2}\sum_A(\psi_1^{(0)}(q_A)|\chi_\sigma|\phi_0^{(0)}(q_A))\nu_A(1-\nu_A)\ , \tag{21}$$

where $\psi_l^{(0)}(q_A)$ is the coefficient of $|\sqrt{2}(q-q_A)|^{|l-\nu_A|}$ when $q\to q_A$. It is clear that the violation persists even if the total flux is zero (say, an equal number of fluxons and antifluxons) since the coefficients $\psi_l^{(0)}(q_A)$ depend on the relative positions of the singularities. Hence, it is not the global topology at infinity, but rather the existence of holes in configuration space which is responsible for the anomaly in $\dot{Q}_\sigma$.

III. REGULAR FLUXONS

We shall now prove that the anomalous time derivative of q calculated in Sec. II can be gotten as the singular limit of a system with regular fluxons. A theorem proved by Crombrugghe and Rittenberg[6] shows that it is impossible to construct a gauge-invariant quantum mechanics with more than one complex supersymmetry generator for a particle in an arbitrary external gauge field.[7] It is amusing to note that the exceptions to this rule in two and three dimensions are the fields of singular fluxons and magnetic monopoles. The preceding statements may be verified explicitly in our case since when $p\to p-a=\Pi$ the algebra [Eqs. (5), (6), and (8)] necessarily breaks down due to the noncommutativity of Π and $\Pi^\dagger$. Let us define the regularized Hamiltonian and supercharge by keeping Eqs. (4) and (7) with $p\to\Pi$. Assume that the magnetic field $b(r)$ is smooth so that the Schrödinger equation is nonsingular at $r\to 0$. As a result, the SUSY violation is now explicit and due to

$$[Q_\sigma,H]=\tfrac{1}{2}\{\Pi,b\}\chi_\sigma\ . \tag{22}$$

Let us choose (a,b) in the following way,

$$a=\frac{ie^{i\theta}}{\sqrt{2}r}\nu(r),\quad b=\frac{\nu'(r)}{r} \tag{23}$$

where

$$\nu(r)=\nu,\quad b=0{:}r>R,\quad \nu(r)\to 0{:}r\to 0\ . \tag{24}$$

Consider now a wave function of definite angular momenta which for $r>R$ contains only momenta small compared to R^{-1}. The behavior of $\psi_l(r)$ inside the circle $r<R$ and at the surface $r\gtrsim R$ is thus

$$\begin{aligned} &r<R:\psi_l(r)\sim A_l r^l R^{(|l-\nu|-l)}\ ,\\ &r\gtrsim R:\psi_l(r)\sim \psi_l^{(0)}(r^{|l-\nu|}+\gamma_l r^{-|l-\nu|}R^{2|l-\nu|})\ , \end{aligned} \tag{25}$$

where $\psi_l^{(0)}$ is by definition the constant used for the singular limit. The matching conditions for ψ and ψ' at $r=R$ lead to

$$A_l=\psi_l^{(0)}(1+\gamma_l)=\psi_l^{(0)}\frac{2|l-\nu|}{|l-\nu|+l}\ . \tag{26}$$

The matrix element of Eq. (22) between ψ_l and ϕ_{l-1} leads to (only ψ_1, ϕ_0 contributes)

$$\begin{aligned} i\frac{d}{dt}\langle\psi|Q_\sigma|\phi\rangle&=\frac{1}{2i\sqrt{2}}(1+\gamma_0)(1+\gamma_1)(\psi_1^{(0)}|\chi_\sigma|\phi_0^{(0)})\\ &\quad\times\int_0^R dr\nu'(r)[2\nu(r)-2]\\ &=i\sqrt{2}(\psi_1^{(0)}|\chi_\sigma|\phi_0^{(0)})\nu(1-\nu) \end{aligned} \tag{27}$$

which is exactly Eq. (18) derived previously for the singular case. We conclude that the SUSY violation may be considered either as an effect of the singular topology or as the regularization-independent limit of the explicit nonconservation induced by a smooth external magnetic field.

IV. GENERALIZATION TO THREE DIMENSIONS

Consider any SUSY quantum-mechanical system whose generators are of the form

$$Q_\sigma=\chi_A M^i_{A\sigma}p_i+\cdots\ , \tag{28}$$

where p_i are canonical momenta conjugate to q_i, and χ are fermion operators. Suppose we add a closed or open fluxon centered around the line $q_i=q_i(s)$ where s is a parameter. When $|q_i-q_i(s)|\to 0$ we can choose locally a coordinate system in which the Hamiltonian would be

$$H=\tfrac{1}{2}\{\Pi,\Pi^\dagger\}+{p_{\|}}^2+\cdots\ , \tag{29}$$

where Π is the same as in the two-dimensional case and $p_{\|}$ are the momenta in the direction perpendicular to the plane on which the flux is defined. Clearly the requirement of finite energy leads to

$$\psi_l(q_1,q_2,q_{\|})\sim(q_1^2+q_2^2)^{|l-\nu|/2}f_l(q)\ , \tag{30}$$

where $f_l(q)$ is regular at $q_i=q_i(s)\equiv 0$. We may now repeat the procedure of Secs. II and III to prove

$$\begin{aligned} &\frac{d}{dt}\langle\psi,Q_\sigma\phi\rangle\\ &\quad=\sqrt{2}\nu(1-\nu)\int ds(\psi_1^{(0)}(q(s))|\chi_\sigma|\phi_0^{(0)}(q(s)))\ , \end{aligned} \tag{31}$$

where ds is the line element along the fluxon and $\psi_l^{(0)}(q(s))$ is the coefficient of $|q-q(s)|^{|l-\nu|}$ when q approaches $q(s)$ in a plane orthogonal to the flux line. Again note that there is no cancellation between positive and negative flux. In particular, a closed fluxon will contribute to the anomaly a term proportional to its length. Also, Eq. (30) may evidently be derived as the limit of the corresponding regularized equation by repeating the procedure of Sec. III.

V. SUMMARY AND DISCUSSION

In the present paper we have investigated the influence of topological singularities on quantum-mechanical SUSY systems in two and three dimensions. The topological singularities can be viewed as holes in configurational space (fluxons) with appropriate boundary conditions which necessarily lead to the breaking of SUSY. The amount of breaking is independent of the size of the hole and is just affected by the total amount of flux going through it. The violation is a quantum effect which is manifested only because the existence of nontrivial flux generates a finite SUSY current which flows into the singular points decreed by the topology. The algebraic relations between the differential operators which represent the supercharges and the Hamiltonian are left unchanged. Furthermore it was proved that the results obtained for the singular-fluxon case can be viewed as a limiting case of the nonconservation of supersymmetry in the presence of a regular fluxon whose size shrinks to zero. This limiting process leaves behind a well-defined spectrum and finite-energy wave functions which vanish on the singular fluxon. The rate at which the wave functions go to zero is determined by the total flux and for an improperly quantized fluxon leads to a nonanalytic behavior which causes the SUSY violation.

The interesting question is what happens in SUSY gauge field theories which admit (with a nonvanishing probability) regular fluxon configurations with a finite mass and contain matter fields whose charge is not quantized in units of the inverse fluxon strength. Truncation of the system to a finite number of degrees of freedom (such as retaining only the zero modes) seems to yield a quantum-mechanical system which is of the type considered in this paper.

It would be interesting to find out whether SUSY is indeed broken. Of course a much more careful analysis should be performed before any definite conclusions are reached. In particular the crucial question is whether in a given SUSY gauge theory fluxon configurations contribute with nonvanishing weight to the functional integral. A more detailed investigation will be presented elsewhere.

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APPENDIX A: THE SUSY OSCILLATOR

The SUSY harmonic oscillator is defined by choosing

$$w'(q)=q\ . \tag{A1}$$

Define the states

$$\chi_\sigma|00\rangle=0,\quad |10\rangle=\chi_1^\dagger|00\rangle,\quad |01\rangle=\chi_2^\dagger|00\rangle\ ,$$
$$|11\rangle=\chi_2^\dagger\chi_1^\dagger|00\rangle\ . \tag{A2}$$

In this basis H becomes

$$H=\begin{vmatrix} H_b & 0 & 0 & 0 \\ 0 & H_b & 0 & 0 \\ 0 & 0 & H_b & 1 \\ 0 & 0 & 1 & H_b \end{vmatrix}\begin{matrix}10\\01\\00\\11\end{matrix}\ , \tag{A3}$$

where

$$H_b=\frac{1}{2}\left[-\partial_r{}^2-\frac{1}{r}\partial_r+\frac{(l-\nu)^2}{r^2}+r^2\right]. \tag{A4}$$

The eigenfunction and eigenvalues of H_b are found by solving the Schrödinger equation

$$H_b\psi_{lN}(r)=(\omega_{lN}+1)\psi_{lN}(r)\ , \tag{A5}$$

and we readily find

$$\psi_{lN}(r)=u_{lN}e^{-r^2/2}\ , \tag{A6}$$

$$\omega_{lN}=|l-\nu|+2N,\quad N=0,1,2,\ldots, \tag{A7}$$

$$u_{lN}=c_{lN}\sum_{k=0}^{N}(-)^k\frac{r^{2k+|l-\nu|}}{(k!)^2\Gamma(1+|l-\nu|+k)}\ . \tag{A8}$$

The constants c_{lN} are determined by

$$1=|c_{lN}|^2\int_0^\infty dr\, re^{-r^2}|u_{lN}|^2\ , \tag{A9}$$

which leads to

$$|c_{lN}|^{-1}=\sum_{k,m=0}^{N}(-)^{k+m}\frac{\Gamma(1+|l-\nu|+m+k)}{(k!m!)^2\Gamma(1+|l-\nu|+k)\Gamma(1+|l-\nu|+m)}\ . \tag{A10}$$

The case $N=0$ is particularly simple and we find

$$|c_{l0}|^{-1}=1/\Gamma(1+|l-\nu|)\ . \tag{A11}$$

We thus have

$$\psi_{10}^{*(0)}\psi_{00}^{(0)}=\left[\frac{\sin\pi\nu}{\pi\nu(1-\nu)}\right]^{1/2}, \tag{A12}$$

and thus,

$$\langle\psi_{10}|\dot{Q}_\sigma|\psi_{00}\rangle=\sqrt{2}\chi_\sigma\left[\frac{\nu(1-\nu)\sin\pi\nu}{\pi}\right]^{1/2}. \tag{A13}$$

The spectrum is determined by noting that the linear combinations $(|00\rangle\mp|11\rangle)$ diagonalize H and cause a shift by ∓1 in the "boson" energies. We have

$$\begin{aligned} l\le 0:\ & \epsilon_{lN}^{(10)}=\epsilon_{lN}^{(01)}=\nu+|l|+2N+1\ ,\\ & \epsilon_{lN}^{(-)}=\epsilon_{lN}^{(+)}-2=\nu+|l|+2N\ ,\\ l\ge 1:\ & \epsilon_{lN}^{(10)}=\epsilon_{lN}^{(01)}=l-\nu+2N+1\ ,\\ & \epsilon_{lN}^{(-)}=\epsilon_{lN}^{(+)}-2=l-\nu+2N\ . \end{aligned} \tag{A14}$$

The SUSY generators Q_σ connect (for $\nu=0$) the states, say, $|(-),l=0,N\rangle$ to the states $|(01),l=1,N-1\rangle$, and annihilate the states $|(-),l=0,N=0\rangle$. These supermultiplets are now split by an amount $\Delta\epsilon=2\nu$. Note also that the ground state energy is now

$$\epsilon_0=\min(\epsilon_{00}^{(-)},\epsilon_{10}^{(-)})=\min(\nu,1-\nu)>0\ . \tag{A15}$$

Thus, the ground state is manifestly nonsupersymmetric but still nondegenerate (except for $\nu=\frac{1}{2}$). It is of interest to compute explicitly the Witten index[8] for this system. The index is defined through

$$\Delta(\nu)=\operatorname{str}e^{-\beta H(\nu)}=\operatorname{tr}[e^{-\beta H}(-)^F]\ , \tag{A16}$$

where F is 0 (1) for bosonic (fermionic) states. Clearly Δ is a β-independent integer $[=N_b(\epsilon=0)-N_f(\epsilon=0)]$ if supersymmetry is conserved. For the spectrum defined in Eq. (A14) we easily find

$$\Delta(\nu)=\frac{\cosh\beta(\frac{1}{2}-\nu)}{\frac{1}{2}\beta}\ . \tag{A17}$$

Thus, in the presence of a nontrivial fluxon the index is not an integer.

APPENDIX B: THE GROUND-STATE ENERGY FOR GENERAL POTENTIALS

The variational principle which leads to the Schrödinger equation is the same for the cases $\nu=0$ and $\nu\neq 0$, namely,

$$\langle\psi|H|\psi\rangle=\tfrac{1}{2}(\|Q_\sigma\psi\|^2+\|Q_\sigma^\dagger\psi\|^2)\ , \tag{B1}$$

where Q_σ is the differential operator

$$Q_\sigma=\chi_\sigma\Pi+i\chi_\tau^\dagger\epsilon_{\tau\sigma}w'\ . \tag{B2}$$

The violation of SUSY and the nonzero ground-state energy in the presence of a topologically nontrivial flux is due to the change in the domain of H. In particular, we shall show that although there exist zero-energy normalizable states for $\nu\neq 0$, they are not in the spectrum.

The equation $H\psi=0$ leads [in the basis defined by Eq. (A2)] to

$$\frac{\partial}{\partial q^\dagger}\psi_{00}-w'(q)\psi_{11}=0\ , \tag{B3}$$

$$\frac{\partial}{\partial q}\psi_{11}-w'^\dagger(q^\dagger)\psi_{00}=0\ .$$

Define

$$\psi_{00}=w'U,\quad \psi_{11}=w'^\dagger V \tag{B4}$$

and change variables to w:

$$\frac{\partial U}{\partial w^\dagger}-V=0,\quad \frac{\partial V}{\partial w}-U=0\ , \tag{B5}$$

hence,

$$\partial_w\partial_{w^\dagger}U-U=0=\partial_w\partial_{w^\dagger}V-V\ . \tag{B6}$$

Assume that $w(q)$ (which is analytic) behaves at $q\rightarrow\infty$ as

$$w(q)\underset{q\rightarrow\infty}{\sim} q^N\ . \tag{B7}$$

The requirement that U and V be single valued in the presence of the fluxon translates into

$$U(e^{2\pi iN}w)=e^{-2\pi i\nu}U(w)\ , \tag{B8}$$

and thus leads to

$$U(w)=\sum_{n=0}^{N-1}e^{i(n/N-\nu)\theta}u_n(\rho)\ , \tag{B9}$$

where u_n satisfies the Bessel equation $[\,|w|=(1/\sqrt{2})\rho]$. The requirement that u be normalizable at $\rho\rightarrow\infty$ leads to

$$u_n(\rho)=H^{(1)}_{|n/N-\nu|}(i\rho)\ . \tag{B10}$$

Clearly Eq. (B6) leads to similar results for V. The normalizability of (u_n,v_n) at $\rho\rightarrow 0$ then restricts n to $(0<\nu<1)$

$$1\leq n\leq N\ . \tag{B11}$$

Note that $H^{(1)}_\alpha=-e^{i\pi\alpha}H^{(1)}_{-\alpha}$ so that $n/N-\nu$ may be restricted to positive values. Equation (B5), however, shows that a given u_n generates a solution V_{n+N} whose index lies outside the range allowed by (B11). Thus, all the zero-energy solutions of the Schrödinger equation do not belong to the spectrum. We remark that for $\nu=0$ a careful examination of the relevant inequalities proves the existence of N-independent zero-energy states.

Finally, it is worth remarking that in the case $w=0$ with a regular magnetic field (the case considered in Sec. III), there exist $[\nu-1]$ zero-energy states[9] ($[\nu-1]$ is the closest integer to ν). These states become nonnormalizable at the origin when the singular limit is taken and disappear from the spectrum.

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