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Multiple-time properties of quantum-mechanical systems

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We describe several novel experimental procedures for measuring the "multiple-time" properties of various quantum-mechanical systems, and an important curiosity in the behavior of those properties under Lorentz transformations is pointed out.

I. INTRODUCTION

Recent investigations of the measuring process[1] (investigations motivated by a certain perplexity in the relativistic theory of measurement, of which we will discuss more later) have suggested that the familiar picture of time evolution is simply too narrow to encompass everything that can happen to a quantum-mechanical system, and have brought to light a new and tantalizing species of basic quantum phenomena. If these phenomena are to be adequately explored, they will require a new species of experiments, and the present work is a first attempt to imagine what sorts of experiments these might be.

The phenomena in question arise in quantum-mechanical systems within the time interval between two measurements, and they are purely quantum phenomena: they have no classical analog whatsoever. In classical physics, the complete specification of the state of a closed physical system at any time (t_0, say) serves to determine (via the equations of motion) the result of *any* other measurement on the system, carried out at *any* time, either before or after t_0. Measurements carried out on the system at times other than t_0 are therefore in a certain sense redundant: since their results can be deduced from the result of the t_0 experiment, they produce, in principle, no additional information about the system (either about its past or about its future).

An essential difference (perhaps *the* essential one) between quantum and classical theories lies precisely here; because every new complete measurement on a quantum-mechanical system will, in general, augment our information about that system. Suppose, for example, that a certain particle may be located in either of two small impenetrable boxes, positioned, respectively, at x_1 and x_2; and suppose that the particle is measured at time t_i to be in the state

$$|\alpha\rangle \equiv \frac{1}{\sqrt{2}}(|x_1\rangle + |x_2\rangle) . \tag{1}$$

What can be said about the outcome of a measurement carried out on the system at time t_1 ($t_1 > t_i$)? Assuming that the particle is undisturbed in the interval $t_i < t < t_1$, we can predict from the outcome of the t_i measurement that, if an observable A of which $|\alpha\rangle$ is an eigenstate with eigenvalue α is measured at t_1, the result is certain to be α. If, on the other hand, the particle's position is measured at t_1, the result may be either $X = x_1$ or $X = x_2$.

But now suppose that, in addition to the outcome of the t_i measurement, we are informed of the outcome of a measurement of the particle's position at time t_f ($t_f > t_1 > t_i$), and suppose that the result of the t_f measurement is $X = x_1$. Then we can say, as before, that if A is measured at t_1, the result must with certainty be that $A = \alpha$; but now we can also say that if X (rather than A) is measured at t_1, the result must with certainty be $X = x_1$. Given the result of the measurement at t_f, then, we know *more* about the particle at t_1 than we knew on the basis of the t_i measurement alone.

These considerations can easily be generalized to measurements of arbitrary complete sets of observables on arbitrary systems. If a system is measured at t_i to be in the state $|A = a\rangle$ and at t_f to be in the state $|B = b\rangle$ (where $[A,B] \neq 0$), then the probability that a measurement of a nondegenerate observable C at time t_1 will yield $C = c_n$ (assuming for simplicity that A, B, and C are all constants of the motion) is given by[2]

$$P(c_n) = \frac{|\langle a | c_n \rangle|^2 |\langle c_n | b \rangle|^2}{\sum_{i=1}^{m} |\langle a | c_i \rangle|^2 |\langle c_i | b \rangle|^2} , \tag{2}$$

where $|c_1\rangle, \ldots, |c_m\rangle$ comprise a complete basis for the system's Hilbert space. Notwithstanding that the state of the system is completely determined by the measurement of A at t_i, then, the measurement of B at t_f produces additional information about the system at t_1. This is possible in quantum mechanics because the results of the measurements at t_i and t_f do not (as they do in the classical theory) determine one another. In a sense (albeit a limited sense, which we shall describe more carefully below) it is as if, with each new measurement, we sample another degree of freedom of the system.

To reiterate: more can be said with certainty about a quantum-mechanical system within the interval between two complete measurements than can be said of a system in any single particular quantum state. In what follows we will describe novel experiment procedures whereby *all* of that can be verified. In addition, we will describe situations in which a new type of observable— *a multiple-time* observable—takes on a well-defined value in the interval between t_i and t_f, and we will discuss the measurement of such observables. Finally, we will discuss the implications of these phenomena for relativistic quantum theory, where the time ordering of spacelike-separated measurement events (such as simultaneous measurements on particles placed at x_1 and x_2) is not Lorentz invariant.

II. COMPENSATING MEASUREMENTS OF NONCOMMUTING OBSERVABLES

A *complete* description of a quantum-mechanical system within the interval between two measurements, then, ought to take account of the outcome of the measurement at the end of the interval as well as the outcome of the one at the beginning. Consider how such a description (that is, *everything* in such a description) might be verified.

Consider a spin-half particle in circumstances in which the Hamiltonian of the particle is independent of spin. Suppose that a measurement of the x component of the particle's spin at a time t_i determines that $\sigma_x(t_i)=+1$, where $\sigma_x \equiv (2/\hbar)S_x$ is one of the Pauli matrices. Suppose, moreover, that a measurement of the z component of the particle's spin at a later time t_f determines that $\sigma_z(t_f)=+1$. Then we can predict from the t_i measurement that, if a measurement of σ_x is carried out at an intermediate time t_2 ($t_1 < t_2 < t_f$), the result is certain to be $\sigma_x(t_2)=+1$. Furthermore, we can retrodict from the t_f measurement that if σ_z, rather than σ_x, is measured at an intermediate time (say, t_1, where $t_i < t_1 < t_f$), the result is certain to be $\sigma_z(t_1)=+1$ (see Fig. 1).

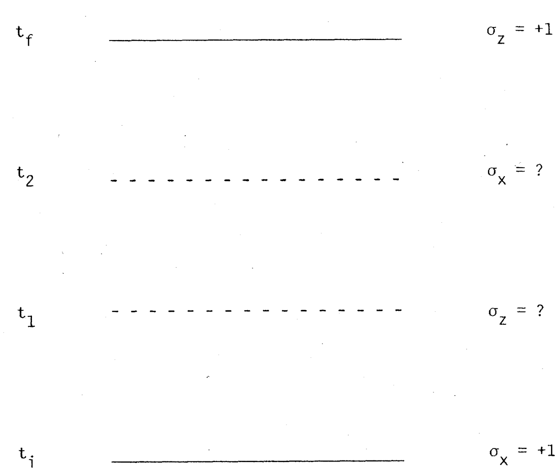

FIG. 1. A measurement of σ_z or σ_x is carried out in the interval between t_i and t_f. A measurement of σ_z at t_1 is certain to yield $\sigma_z(t_1)=+1$; a measurement of σ_x at t_2 is certain to yield $\sigma_x(t_2)=+1$.

Now suppose that we attempt to check the validity of our predictive and retrodictive statements by carrying out *both* intermediate measurements, that is, by measuring both $\sigma_z(t_1)$ and $\sigma_x(t_2)$. Then the two measurements will disrupt one another, and we will not be certain to find either that $\sigma_x(t_2)=\sigma_x(t_i)=+1$ or that $\sigma_z(t_1)=\sigma_z(t_f)=+1$. Yet it is not impossible to perform two measurements in the interval between t_i and t_f in such a way as to verify both that a measurement of σ_z at t_1 (if conducted alone) would yield $+1$ and that a measurement of σ_x at t_2 (if conducted alone) would yield $+1$. We can do this as follows.

Suppose that we have two measuring devices which interact impulsively with the particle at times t_1 and t_2 via a Hamiltonian of the form[3]

$$H_{\text{int}} = g(t-t_1)q_1\sigma_z + g(t-t_2)q_2(\sigma_x \cos 2q_1 + \sigma_y \sin 2q_1) \ . \tag{3}$$

(See Fig. 2.) In this expression, the variables q_1 and q_2 are internal coordinates associated with the first and second measuring devices, respectively. The function $g(t-t_1)$ is a coupling which is large and nonzero only during the short time interval about t_1 during which the first measuring device is switched on. Similarly, $g(t-t_2)$ is large and nonzero only during a short time interval about t_2 (an interval which does not overlap with the short interval about t_1). We can assume for simplicity that $g(t-t_1)$ and $g(t-t_2)$ are δ functions in time, that is, that $g(t-t_1)=G\delta(t-t_1)$ and $g(t-t_2)=G\delta(t-t_2)$, where G is a constant of unit magnitude with the dimensions of momentum.

Using the Heisenberg formalism to obtain the equations of motion for the observables σ_z, σ_x, q_1, q_2, Π_1, and Π_2 (where Π_j is the momentum canonically conjugate to the measuring device coordinate q_j), we find that

$$\begin{aligned} dq_1/dt &= d\sigma_z/dt = 0 \ , \\ d\Pi_1/dt &= -\delta(t-t_1)\sigma_z \ , \\ d\sigma_x/dt &= -2\delta(t-t_1)q_1\sigma_y \ , \end{aligned} \tag{4}$$

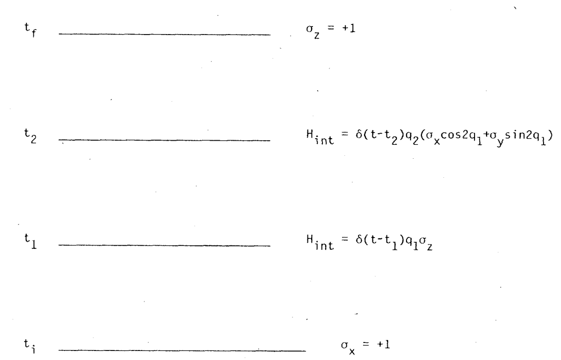

FIG. 2. Compensating measurements of σ_z and σ_x are carried out at t_1 and t_2, with the result that $\Delta\Pi_1 = -\sigma_z(t_f)$ and $\Delta\Pi_2 = -\sigma_x(t_i)$.

and

$$d\sigma_y/dt = 2\delta(t-t_1)q_1\sigma_x$$

during the interval in which the first measuring device is switched on, and that

$$\begin{aligned}
&dq_2/dt = dq_1/dt = 0 \ ,\\
&d\Pi_2/dt = -\delta(t-t_2)(\sigma_x \cos 2q_1 + \sigma_y \sin 2q_1) \ ,\\
&d\Pi_1/dt = -2\delta(t-t_2)q_2(-\sigma_x \sin 2q_1 + \sigma_y \cos 2q_1) \ ,\\
&d\sigma_z/dt = 2\delta(t-t_2)q_2(-\sigma_x \sin 2q_1 + \sigma_y \cos 2q_1) = -d\Pi_1/dt \ ,\\
&d\sigma_x/dt = 2\delta(t-t_2)q_2\sigma_z \sin 2q_1 \ ,
\end{aligned} \tag{5}$$

and

$$d\sigma_y/dt = -2\delta(t-t_2)q_2\sigma_z \cos 2q_1$$

during the interval in which the second device is switched on. The equations for q_1 and q_2 can be integrated immediately to give $q_1(t)=\text{constant}=q_1$ and $q_2(t)=\text{constant}=q_2$.

Notice that Eqs. (4) indicate that the measurement of σ_z at t_1 rotates the particle's spin in the xy plane, so that

$$\sigma_x(t_1+\epsilon) = \sigma_x(t_1-\epsilon)\cos 2q_1 - \sigma_y(t_1-\epsilon)\sin 2q_1 \ , \tag{6a}$$

and

$$\sigma_y(t_1+\epsilon) = \sigma_y(t_1-\epsilon)\cos 2q_1 + \sigma_x(t_1-\epsilon)\sin 2q_1 \ . \tag{6b}$$

Also notice that, during the second measurement, the quantity $(\sigma_x \cos 2q_1 + \sigma_y \sin 2q_1)$ is constant:

$$\begin{aligned}
\sigma_x(t_2)\cos 2q_1 + \sigma_y(t_2)\sin 2q_1
&= \sigma_x(t_2-\epsilon)\cos 2q_1 + \sigma_y(t_2-\epsilon)\sin 2q_1\\
&= \sigma_x(t_1+\epsilon)\cos 2q_1 + \sigma_y(t_1+\epsilon)\sin 2q_1 \ .
\end{aligned}$$

This is significant because, as can be seen from the time-reversed version of Eq. (6), the quantity $[\sigma_x(t_1+\epsilon)\cos 2q_1 + \sigma_y(t_1+\epsilon)\sin 2q_1]$ is equal to $\sigma_x(t_1-\epsilon)$, which is just the initial value of σ_x. Thus the second term in the interaction Hamiltonian couples q_2 to $\sigma_x(t_i)$.

Let us now integrate the equations of motion for the measuring devices' momenta. We obtain

$$\begin{aligned}
&\Pi_1(t_1+\epsilon) - \Pi_1(t_1-\epsilon) \equiv \Delta\Pi_1(t_1) = -\sigma_z(t_1) \ ,\\
&\Pi_1(t_2+\epsilon) - \Pi_1(t_2-\epsilon)\\
&\qquad \equiv \Delta\Pi_1(t_2) = -[\sigma_z(t_2+\epsilon) - \sigma_z(t_2-\epsilon)] \ ,\\
&\Pi_2(t_2+\epsilon) - \Pi_2(t_2-\epsilon) \equiv \Delta\Pi_2(t_2) = -\sigma_x(t_i) \ .
\end{aligned} \tag{7}$$

Combining the two equations that involve Π_1, we see that the total change in Π_1 is given by $\Delta\Pi_1 \equiv \Delta\Pi_1(t_1) + \Delta\Pi_1(t_2) = -\sigma_z(t_f)$, since $\sigma_z(t_2-\epsilon) = \sigma_z(t_1)$ and $\sigma_z(t_2+\epsilon) = \sigma_z(t_f)$. Thus Eqs. (7) can be rewritten as

$$\begin{aligned}
&\Delta\Pi_1 = -\sigma_z(t_f) \ ;\\
&\Delta\Pi_2 = -\sigma_x(t_i) \ .
\end{aligned} \tag{8}$$

Notice that Eqs. (8) do *not* imply either that $\sigma_z(t_1) = \sigma_z(t_f)$ or that $\sigma_x(t_2) = \sigma_x(t_i)$. In contrast to an interaction Hamiltonian of the form $H_{\text{direct}} = g(t-t_1)q_1\sigma_z + g(t-t_2)q_2\sigma_x$, the interaction Hamiltonian of Eq. (3) does not provide for direct, nondemolition measurements of $\sigma_z(t_1)$ and $\sigma_x(t_2)$; that is, it does not lead to a proportional relationship between $\Delta\Pi_1$ and the value of σ_z at t_1 or between $\Delta\Pi_2$ and the value of σ_x at t_2. Instead, the present Hamiltonian couples $\Delta\Pi_1$ to the value of σ_z at t_f, which is the value that $\sigma_z(t_1)$ *would have had,* if no measurement of σ_x had occurred between t_1 and t_f. Similarly, it couples $\Delta\Pi_2$ to the value of σ_x at t_i, which is the value that $\sigma_x(t_2)$ *would have had,* if no measurement of σ_z had occurred between t_i and t_2. Thus the two measurements dictated by Eq. (3) constitute a pair of *compensating measurements*: the measurement at t_2, which involves σ_x, compensates for the disturbance of σ_x caused by the measurement of σ_z at t_1, and vice versa.

It is interesting to note that it is possible to carry out compensating measurements of the x and z components of a particle's spin in a sequence different from that dictated by the interaction Hamiltonian of Eq. (3). The Hamiltonian defined by Eq. (9), for example, also generates the relations $\Delta\Pi_1 = -\sigma_z(t_f)$ and $\Delta\Pi_2 = -\sigma_x(t_i)$, even though it entails the measurement of a rotated quantity before (rather than after) a direct measurement of one of the particle's spin components:

$$H'_{\text{int}} = \delta(t-t_1)q_1(\sigma_z \cos 2q_2 + \sigma_y \sin 2q_2) + \delta(t-t_2)q_2\sigma_x \ . \tag{9}$$

It is also interesting to note that repeated pairs of compensating measurements of σ_z and σ_x carried out on a spin-half particle can reveal the presence of an external magnetic field of *arbitrary* orientation. In contrast, ordinary spin measurements of σ_z or σ_x (or σ_y) carried out repeatedly on a spin-half particle reveal the presence of an external field only if the field has a component perpendicular to the direction of the spin component being measured.

Let us show explicitly that compensating measurements carried out on a particle at times t_1, t_2, t_3, and t_4 provide a means of detecting an external field switched on in the interval between the first pair of measurements and the second (that is, between t_2 and t_3). Suppose that a particle is made to interact with four measuring devices at times t_1-t_4 in such a way that the Hamiltonian of the system during the measurements is given by

$$H''_{\rm int}=\delta(t-t_1)q_1\sigma_z+\delta(t-t_2)q_2(\sigma_x\cos 2q_1+\sigma_y\sin 2q_1)+\delta(t-t_3)q_3\sigma_z$$
$$+\delta(t-t_4)q_4[\sigma_x\cos(2q_3+2q_1)+\sigma_y\sin(2q_3+2q_1)]\ , \tag{10}$$

as illustrated in Fig. 3. Then it can easily be shown via calculations in the Heisenberg representation that the total changes in the momenta of the four measuring devices, after all four measurements have been completed, will be

$$\Delta\Pi_1\equiv\Delta\Pi_1(t_1)+\Delta\Pi_1(t_2)+\Delta\Pi_1(t_4)=-\sigma_z(t_2+\epsilon)+\sigma_z(t_3-\epsilon)-\sigma_z(t_f)\ ,$$
$$\Delta\Pi_2=-\sigma_x(t_i)=-[\sigma_x(t_2+\epsilon)\cos 2q_1+\sigma_y(t_2+\epsilon)\sin 2q_1]\ ,$$
$$\Delta\Pi_3\equiv\Delta\Pi_3(t_3)+\Delta\Pi_3(t_4)=-\sigma_z(t_f)\ ,$$
$$\Delta\Pi_4=-[\sigma_x(t_3-\epsilon)\cos 2q_1+\sigma_y(t_3-\epsilon)\sin 2q]\ .$$

Now consider the possibility that an external magnetic field is switched on in the interval between t_2 and t_3. If *no* magnetic field is switched on in this interval, then, after all measurements have been completed, it will with certainty be found that $\Delta\Pi_1=\Delta\Pi_3=-\sigma_z(t_f)$ and that $\Delta\Pi_2=\Delta\Pi_4=-\sigma_x(t_i)$, since in that case, all three spin components will have the same value at $t_3-\epsilon$ as at $t_2+\epsilon$. On the other hand, if it is found after all measurements have been completed either that $\Delta\Pi_1\neq\Delta\Pi_3$ or that $\Delta\Pi_2\neq\Delta\Pi_4$, then it will be known with certainty that there *was* a magnetic field switched on between t_2 and t_3 which rotated one or more of the particle's spin components. Thus the procedure dictated by the measurement Hamiltonian of Eq. (10) does, indeed, provide a means of detecting an external magnetic field. Notice that *no* external field (switched on for an arbitrary length of time) can escape detection by this method, since there is no orientation of an external field that will leave all three components of the particle's spin unaffected.

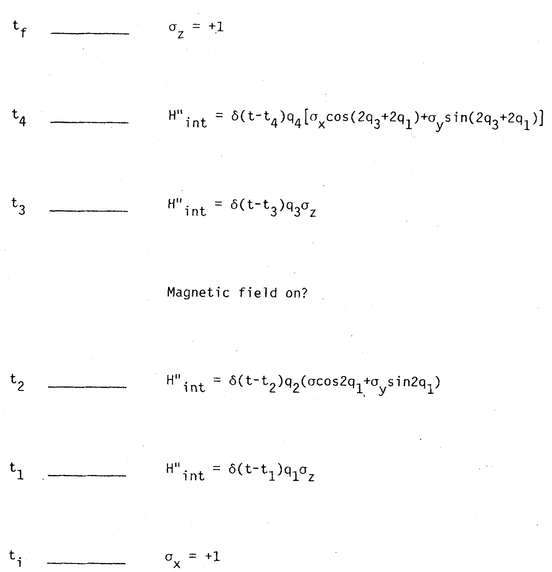

FIG. 3. Two pairs of compensating measurements of σ_z and σ_x are conducted at t_1, t_2, t_3, and t_4. If no magnetic field is switched on between t_2 and t_3, then after all measurements are complete it will be found that $\Delta\Pi_1=\Delta\Pi_3=-\sigma_z(t_f)$ and that $\Delta\Pi_2=\Delta\Pi_4=-\sigma_x(t_i)$. If, on the other hand, it is found either that $\Delta\Pi_1\neq\Delta\Pi_3$ or that $\Delta\Pi_2\neq\Delta\Pi_4$, then it will be known with certainty that a magnetic field was switched on in the interval between t_2 and t_3.

It is possible to extend the procedures that have been developed in this section to a system of two spin-half particles. Consider two such particles (in circumstances in which the particles' Hamiltonian is effectively zero) for which it is known that the state of the system at t_i is a spin singlet and that the state of the system at t_f is some eigenstate of single-particle spin operators. Specifically, consider two particles whose initial state is $|\psi(t_1)\rangle=|J=0;J_z=0\rangle$ (where J is the total spin angular momentum of the two particles and J_z is the z component of J), and whose final state is $|\psi(t_f)\rangle=|\sigma_z=+1\rangle_1|\sigma_x=-1\rangle_2$.

What can be said about the result of a single-particle spin measurement carried out on the system in the interval between t_i and t_f? Suppose that either $\sigma_z^{(2)}$ or $\sigma_x^{(1)}$ is measured at some intermediate time. If $\sigma_z^{(2)}$ is measured at t_1 (where $t_i<t_1<t_f$), then, since the particles' spins are correlated at t_i and since $\sigma_z^{(1)}(t_f)=+1$, the result of the t_2 measurement must be that $\sigma_z^{(2)}(t_1)=-1$. If $\sigma_x^{(1)}$, rather than $\sigma_z^{(2)}$, is measured at an intermediate time (t_2, say), then, since $\sigma_x^{(2)}(t_f)=-1$, the result of the t_2 measurement must be that $\sigma_x^{(1)}(t_2)=+1$. (See Fig. 4.) In other words, if a measuring device is coupled directly to $\sigma_z^{(2)}$ at t_1, then there must be a change in the momentum of the device given by

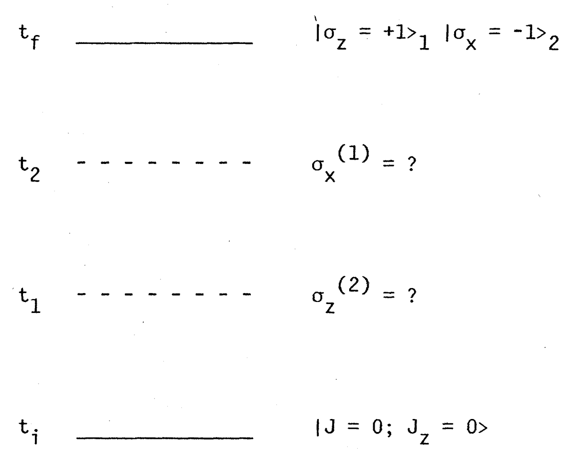

FIG. 4. A measurement of $\sigma_z^{(2)}$ or $\sigma_x^{(1)}$ is carried out in the interval between t_i and t_f. A measurement of $\sigma_z^{(2)}$ at t_1 is certain to yield $\sigma_z^{(2)}(t_1)=-1$; a measurement of $\sigma_x^{(1)}$ at t_2 is certain to yield $\sigma_x^{(1)}(t_2)=+1$.

$$\Delta\Pi_1 = -\sigma_z^{(2)}(t_1) = \sigma_z^{(1)}(t_f)\ . \tag{11}$$

If, instead, a measuring device is coupled directly to $\sigma_x^{(1)}$ at t_2, then there must be a change in the momentum of the second device given by

$$\Delta\Pi_2 = -\sigma_x^{(1)}(t_2) = \sigma_x^{(2)}(t_f)\ . \tag{12}$$

Now suppose that we couple two measuring devices to the system in an attempt to measure both $\sigma_z^{(2)}(t_1)$ and $\sigma_x^{(1)}(t_2)$. Then the two measurements will disrupt one another (even if the measurement events are spacelike-separated), since the first measurement will destroy the correlation between the particles' spins and the second measurement will change the value of $\sigma_z^{(1)}$. As a result, we will not necessarily find either that $\Delta\Pi_1 = \sigma_z^{(1)}(t_f)$ or that $\Delta\Pi_2 = \sigma_x^{(2)}(t_f)$. Yet it is possible to carry out two measurements on the system at t_1 and t_2 in such a way that each measurement compensates for the disruptive effect of the other, as follows: Let three measuring devices interact with the particles via the Hamiltonian

$$H_1 = \delta(t-t_1)q_1\sigma_z^{(2)} + \delta(t-t_2)q_2(\sigma_x^{(1)}\cos 2q_3 - \sigma_y^{(1)}\sin 2q_3)\ ,$$

where q_1, q_2, and q_3 are internal variables associated with the three measuring devices. (See Fig. 5.) Furthermore, let the initial conditions on the first and third measuring devices be[4,5]

$$q_{10} - q_{30} = \Pi_{10} + \Pi_{30} = 0\ .$$

Then the Heisenberg formalism can be used to show that, after the measurements at t_1 and t_2, the measuring device variables Π_1, Π_2, and Π_3 will have changed in such a way that

$$\Delta\Pi_1 = -\sigma_z^{(2)}(t_1) = -\sigma_z^{(2)}(t_i) = \sigma_z^{(1)}(t_i)\ , \tag{13a}$$

$$\Delta\Pi_3 = \Delta\sigma_z^{(1)}(t_2) = \sigma_z^{(1)}(t_2+\epsilon) - \sigma_z^{(1)}(t_2-\epsilon) = \sigma_z^{(1)}(t_f) - \sigma_z^{(1)}(t_i) \tag{13b}$$

and

$$\Delta\Pi_2 = -[\sigma_x^{(1)}(t_2)\cos 2q_3 - \sigma_y^{(1)}(t_2)\sin 2q_3]\ . \tag{13c}$$

t_f ———— $|\sigma_z = +1\rangle_1\ |\sigma_x = -1\rangle_2$

t_2 ———— $H_1 = \delta(t-t_2)q_2(\sigma_x^{(1)}\cos 2q_3 - \sigma_y^{(1)}\sin 2q_3)$

t_1 ———— $H_1 = \delta(t-t_1)q_1\sigma_z^{(2)}$

t_i ———— $|J = 0;\ J_z = 0\rangle$

FIG. 5. Compensating measurements are carried out on two spin-correlated particles at t_1 and t_2. After the measurements have been completed, it will be found that $\Delta\Pi_2 = \sigma_x^{(2)}(t_f) = -1$ and that $\Delta\Pi_1 + \Delta\Pi_3 = \sigma_z^{(1)}(t_f) = +1$.

Notice that, since $\sigma_x^{(1)}(t_2) = \sigma_x^{(1)}(t_i) = -\sigma_x^{(2)}(t_i)$ and $\sigma_y^{(1)}(t_2) = \sigma_y^{(1)}(t_i) = -\sigma_y^{(2)}(t_i)$, the right-hand side of Eq. (13c) represents the value of $\sigma_x^{(2)}$ after its rotation by the measurement of $\sigma_z^{(2)}$ at t_1; i.e., Eq. (13c) can be rewritten as

$$\Delta\pi_2 = \sigma_x^{(2)}(t_i)\cos 2q_3 - \sigma_y^{(2)}(t_i)\sin 2q_3 = \sigma_x^{(2)}(t_f)\ . \tag{14}$$

Furthermore, although $\Delta\Pi_1$ and $\Delta\Pi_3$ are not separately well defined because of the initial conditions on the first and third measuring devices, their sum is well defined and is given by

$$\Delta\Pi_1 + \Delta\Pi_3 = \sigma_z^{(1)}(t_f)\ . \tag{15}$$

A comparison of Eqs. (14) and (15) with Eqs. (12) and (11), respectively, reveals that the present scheme allows us to learn from the second measuring device (that is, from the value of $\Delta\Pi_2$) what the result of a direct measurement of $\sigma_x^{(1)}$ at t_2 *would have been,* if no measurement involving $\sigma_z^{(2)}$ had been carried out at t_1. Similarly, it allows us to learn from the other measuring devices (from the value of $\Delta\Pi_1 + \Delta\Pi_3$) what the result of a direct measurement of $\sigma_z^{(2)}$ at t_1 *would have been,* if no measurement involving $\sigma_x^{(1)}$ had been performed at t_2. Thus H_1 provides (as was intended) a means of carrying out compensating measurements of $\sigma_z^{(2)}$ and $\sigma_x^{(1)}$.

Finally, we note that it is possible to combine compensating measurements of the sort that have been described in such a way as to relate the results of measurements carried out on a two-particle system at t_1 and t_2 to the final state of the system and results of measurements carried out on the system at t_3 and t_4 (where $t_i < t_1 < t_2 < t_3 < t_4 < t_f$) to the initial state of the system. The sequence of measurements illustrated in Fig. 6, for example, is designed to couple measuring device variables to the particles' spins in such a way that, after all measurements have been completed, it will be found that $\Delta\Pi_1 + \Delta\Pi_3 = \sigma_z^{(1)}(t_f)$, that $\Delta\Pi_2 = \sigma_x^{(2)}(t_f)$, and that

t_f ———— $|\sigma_z = +1\rangle_1\ |\sigma_x = -1\rangle_2$

t_4 ———— $H_2 = \delta(t-t_4)q_5(\sigma_y^{(1)}\cos 2q_1 + \sigma_x^{(1)}\sin 2q_1)$

t_3 ———— $H_2 = \delta(t-t_3)q_4(\sigma_y^{(2)}\cos 2q_2 - \sigma_z^{(2)}\sin 2q_2)$

t_2 ———— $H_2 = \delta(t-t_2)q_2(\sigma_x^{(1)}\cos 2q_3 - \sigma_y^{(1)}\sin 2q_3)$

t_1 ———— $H_2 = \delta(t-t_1)q_1\sigma_z^{(2)}$

t_i ———— $|J = 0;\ J_z = 0\rangle$

FIG. 6. Compensating measurements are performed on a two-particle system in such a way as to relate $(\Delta\Pi_1 + \Delta\Pi_3)$ and $\Delta\Pi_2$ to the final state of the system and $(\Delta\Pi_4 + \Delta\Pi_5)$ to its initial state. The initial conditions on the measuring devices are that $q_{10} - q_{30} = \Pi_{10} + \Pi_{30} = q_{40} - q_{50} = \Pi_{40} + \Pi_{50} = 0$. After all measurements have been completed, it will be found that $\Delta\Pi_1 + \Delta\Pi_3 = \sigma_z^{(1)}(t_f) = +1$, $\Delta\Pi_2 = \sigma_x^{(2)}(t_f) = -1$, and $\Delta\Pi_4 + \Delta\Pi_5 = 0$.

$$\Delta\Pi_4+\Delta\Pi_5=-[\sigma_y^{(1)}(t_i)+\sigma_y^{(2)}(t_i)]\cos2q_1\cos2q_2 -[\sigma_x^{(1)}(t_i)+\sigma_x^{(2)}(t_i)]\sin2q_1\cos2q_2 +[\sigma_z^{(1)}(t_i)+\sigma_z^{(2)}(t_i)]\sin2q_2=0\ ,$$

assuming that the initial state of the particles is a spin singlet. More complex combinations of compensating measurements can also be imagined.

III. MEASUREMENTS INVOLVING TWO-TIME OBSERVABLES

The *multiple-time* properties of quantum-mechanical systems (properties such as are described in detail in Ref. 1) refer, rather than to the value of any given observable at any given time, to *correlations* between the value of one observable at one time and the value of another (one which does not, in general, commute with the first) at another time. If, for example, the initial and final states of a spin-half particle (in circumstances in which its Hamiltonian is independent of spin) are $|\chi(t_i)\rangle=|\sigma_x=+1\rangle$ and $|\chi(t_f)\rangle=|\sigma_z=+1\rangle$, and if a measurement of the two-time observable $\sigma_z(t_1)+\sigma_x(t_4)\equiv\sigma_{zx}(t_1,t_4)$ is carried out on the particle at t_1 and t_4 with the result that $\sigma_{zx}(t_1,t_4)=0$, then it can be said with certainty that a measurement of the two-time observable $\sigma_x(t_2)-\sigma_z(t_3)\equiv\sigma_{x,-z}(t_2,t_3)$ carried out on the particle at t_2 and t_3 (where $t_i<t_1<t_2<t_3<t_4<t_f$) will show that $\sigma_{x,-z}(t_2,t_3)=0$. The two-time measurement carried out at t_1 and t_4, given that its result is zero, creates a correlation between the x and z components of the particle's spin at different times, and this correlation is such that a measurement of $\sigma_{x,-z}(t,t')$ "nested inside" the (t_1/t_4) measurement must give zero. It is as if the process of measuring $\sigma_{zx}(t_1,t_4)$ and finding that $\sigma_{zx}(t_1,t_4)=0$ places the particle in a *two-time state* that is an eigenstate (with eigenvalue zero) of the two-time operators $\sigma_{zx}(t'',t''')$ and $\sigma_{x-z}(t,t')$.

In the discussion that follows, we will show that it is possible to carry out two-time measurements on a system of *two* particles in such a way as to create correlations between the values of observables associated with one particle at one time and the values of observables associated with the second particle at a different time. (Using the language of two-time states, we will show that it is possible for a system of two particles to be in an eigenstate of various two-particle, two-time operators.) These interparticle correlations can be used to "predict" (that is, to make definite statements about) the results of two-time experiments carried out at intermediate times if the initial and final states of the particles are known.[6] Conversely, the correlations can be used to predict the final state of a two-particle system from a knowledge of its initial state and the results of intermediate two-time measurements. The second application is the one we wish to develop here.

Let us consider first a system of two spin-half particles (in circumstances in which the Hamiltonian of the system is independent of spin) whose initial state is given by $|\chi(t_i)\rangle_1|\chi(t_i)\rangle_2=|\sigma_x=+1\rangle_1|\sigma_y=-1\rangle_2$. Suppose that "crossed" measurements of the two-particle, two-time observables $\sigma_x(t_1,t_2)\equiv\sigma_x^{(1)}(t_1)+\sigma_x^{(2)}(t_2)$ and $\sigma_y(t_2,t_1)\equiv\sigma_y^{(1)}(t_2)+\sigma_y^{(2)}(t_1)$ are carried out on the system at times t_1 and t_2, where $t_i<t_1<t_2<t_f$, using the procedure described in Ref. 5. [We refer to these measurements as crossed measurements, since a line connecting the measurement of $\sigma_x^{(1)}(t_1)$ with the measurement of $\sigma_x^{(2)}(t_2)$ on a spacetime diagram crosses the line connecting the measurement of $\sigma_y^{(1)}(t_2)$ with the measurement of $\sigma_y^{(2)}(t_1)$.] If the result of this pair of two-time measurements is that $\sigma_x(t_1,t_2)=\sigma_y(t_2,t_1)=0$, then, clearly, the final state of the particles must be $|\chi(t_f)\rangle_1|\chi(t_f)\rangle_2=|\sigma_y=+1\rangle_1|\sigma_x=-1\rangle_2$, since the state of particle 1 must satisfy the relation $\sigma_y^{(1)}(t_f)=-\sigma_y^{(2)}(t_i)$ and the final state of particle 2 must satisfy $\sigma_x^{(2)}(t_f)=-\sigma_x^{(1)}(t_i)$.

This result, illustrate in Fig. 7, suggests a more general one: that crossed two-time measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ carried out on a two-particle system in an *arbitrary* initial state function as a spin-flip operator in the xy plane if the result of each measurement is zero. In other words, the effect of crossed measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ [given that $\sigma_x(t_1,t_2)=\sigma_y(t_2,t_1)=0$] is to "exchange and flip" the x and y components of the particles' spins in such a way that the final spin of particle 1 is correlated to the initial spin of particle 2, and vice versa.

The validity of the more general rule can easily be shown. Let the initial state of particle 1 be some arbitrary spin state $|\chi(t_i)\rangle_1$, represented as a linear combination of the eigenstates of $\sigma_x^{(1)}$:

$$|\chi(t_i)\rangle_1=a_1|\sigma_x=+1\rangle_1+b_1|\sigma_x=-1\rangle_1\ .$$

Similarly, let the initial state of particle 2 be some arbi-

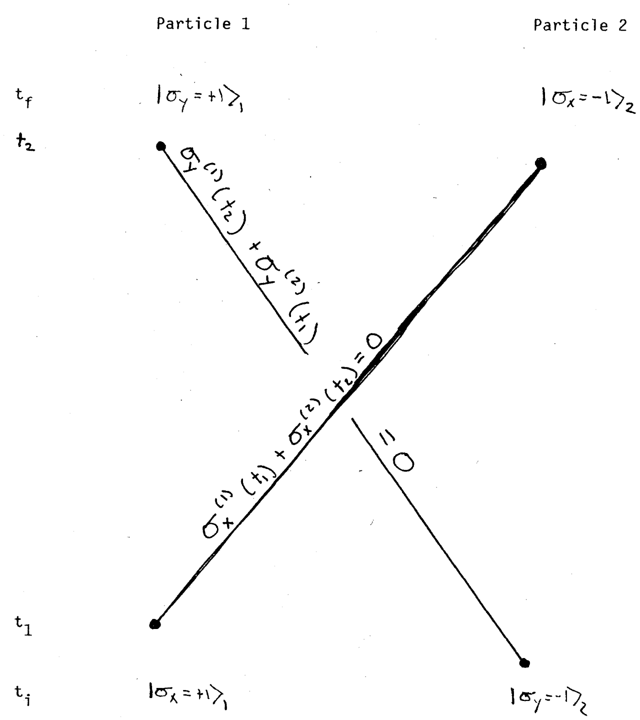

FIG. 7. Crossed measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ on a system whose initial state is $|\sigma_x=+1\rangle_1|\sigma_y=-1\rangle_2$.

trary state $|\chi(t_i)\rangle_2$, represented as a linear combination of the eigenstates of $\sigma_y^{(2)}$:

$$|\chi(t_i)\rangle_2 = a_2\,|\sigma_y=+1\rangle_2 + b_2\,|\sigma_y=-1\rangle_2 \, .$$

Then the initial state of the two-particle system can be written as

$$\begin{aligned} |\psi(t_i)\rangle = {} & a_1a_2\,|\sigma_x=+1\rangle_2\,|\sigma_y=+1\rangle_2 \\ & + a_1b_2\,|\sigma_x=+1\rangle_1\,|\sigma_y=-1\rangle_2 \\ & + b_1a_2\,|\sigma_x=-1\rangle_1\,|\sigma_y=+1\rangle_2 \\ & + b_1b_2\,|\sigma_x=-1\rangle_1\,|\sigma_y=-1\rangle_2 \, . \end{aligned} \tag{16}$$

[Notice that $|\psi(t_i)\rangle$ need not be expressible as a product of single-particle eigenstates: if $a_1a_2=-b_1b_2=-(b_1a_2)^*=(b_2a_1)^*=(1+i)/2$, for example, the initial state of the system is a spin singlet.] Now if crossed measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ are performed on the system, and if the result of each measurement is zero, then the final state of the system can be constructed term-by-term from $|\psi(t_i)\rangle$ by exchanging and flipping the x and y components of the two particles' spins in each term:

$$\begin{aligned} |\psi(t_f)\rangle = {} & a_1a_2\,|\sigma_y=-1\rangle_1\,|\sigma_x=-1\rangle_2 + a_1b_2\,|\sigma_y=+1\rangle_1\,|\sigma_x=-1\rangle_2 \\ & + b_1a_2\,|\sigma_y=-1\rangle_1\,|\sigma_x=+1\rangle_2 + b_1b_2\,|\sigma_y=+1\rangle_1\,|\sigma_x=+1\rangle_2 \\ = {} & (a_2\,|\sigma_y=-1\rangle_1 + b_2\,|\sigma_y=+1\rangle_1)(a_1\,|\sigma_x=-1\rangle_2 + b_1\,|\sigma_x=+1\rangle_2) \, . \end{aligned} \tag{17}$$

Two cases of special interest emerge from this more general analysis. Consider first the case in which $a_1=a_2=(1-i)/2$ and $b_1=b_2=(1+i)/2$, that is, the case in which $|\psi(t_i)\rangle = |\sigma_y=-1\rangle_1\,|\sigma_x=+1\rangle_2$. If crossed measurements are carried out at t_1 and t_2 in this case, with the result that $\sigma_x(t_1,t_2)=\sigma_y(t_2,t_1)=0$, then the final state of the system [as given by Eq. (17)] will be $|\psi(t_f)\rangle = |\sigma_x=-1\rangle_1\,|\sigma_y=+1\rangle_2$. This result, illustrated in Fig. 8, shows that the *order* of crossed measurements of $\sigma_x(t,t')$ and $\sigma_y(t',t)$ is immaterial, as long as the results of both measurements are zero. The state of the system after measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ yield zero is the state that would have resulted if measurements of $\sigma_y(t_1,t_2)$ and $\sigma_x(t_2,t_1)$ had been performed (with results of zero), instead—namely, the state which satisfies the relations $\sigma_x^{(1)}(t_f)=-\sigma_x^{(2)}(t_i)$ and $\sigma_y^{(2)}(t_f)=-\sigma_y^{(1)}(t_i)$.

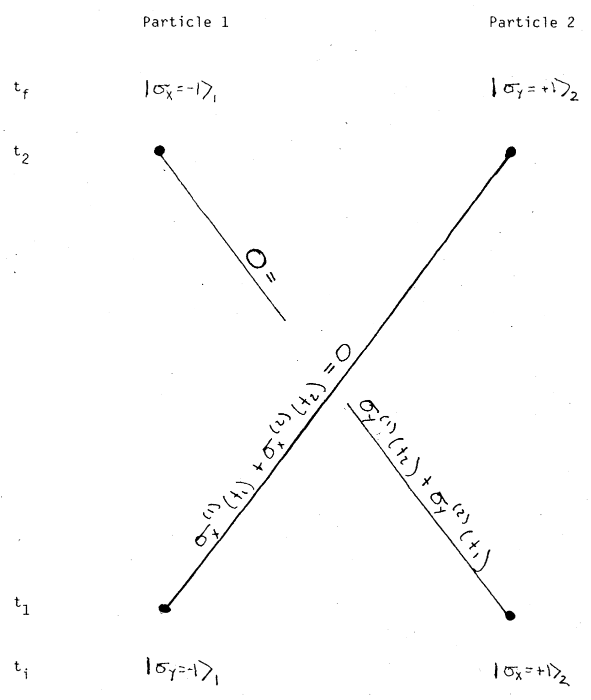

FIG. 8. Crossed measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ on a system whose initial state is $|\sigma_y=-1\rangle_1\,|\sigma_x=+1\rangle_2$.

A second noteworthy result emerges when the initial state of the particles is taken to be $|\psi(t_i)\rangle = |\sigma_z=+1\rangle_1\,|\sigma_z=-1\rangle_2$, corresponding to the choice of constants $a_1=b_1=\sqrt{1/2}$, $a_2=b_2^*=(-i)\sqrt{1/2}$ in Eq. (16). When crossed measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ are performed in this case, we find from Eq. (17) that the final state of the particles is $|\psi(t_f)\rangle = |\sigma_z=-1\rangle_1\,|\sigma_z=+1\rangle_2$. Notice that, as far as the z component of the particles' spins is concerned, the effect of the measurements is to "exchange, but *not* to flip" the spins, in agreement with our earlier statement that the crossed measurements act as a spin-flip operator in the xy plane.

It can be shown that this property of the crossed measurements is a necessary consequence of the uncertainty principle. Suppose that it were possible to carry out crossed measurements on a two-particle system in such a way as to anticorrelate all three components of the final spin of particle 2 to the corresponding components of the initial spin of particle 1, and vice versa; that is, suppose that it were possible for $\mathbf{J}^{(2)}(t_f)+\mathbf{J}^{(1)}(t_i)$ to equal zero and for $\mathbf{J}^{(1)}(t_f)+\mathbf{J}^{(2)}(t_i)$ to equal zero. Then a specification of any of the individual components $\mathbf{J}^{(1)}$ or $\mathbf{J}^{(2)}$ at t_i or t_f would be inconsistent with the uncertainty principle; we could not, for example, claim that the initial state of the system was $|\sigma_z=+1\rangle_1\,|\sigma_z=-1\rangle_2$.

Neither could we claim that the initial state of the particles was a spin singlet or a spin triplet. As illustrated in Fig. 9, a statement that the initial state of the particles was a singlet and that $\mathbf{J}^{(1,2)}(t_f)+\mathbf{J}^{(2,1)}(t_i)=0$ is equivalent to a statement that the measurements of σ_x and σ_y carried out in the interval between t_i and t_f had exactly the same effect on both particles. But this cannot be the case, since the order of the measurements differed for the two particles. (A similar argument can be given for the case of an initial triplet state.). We conclude, then, that it is not possible for crossed measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ to correlate all three components of the particles'

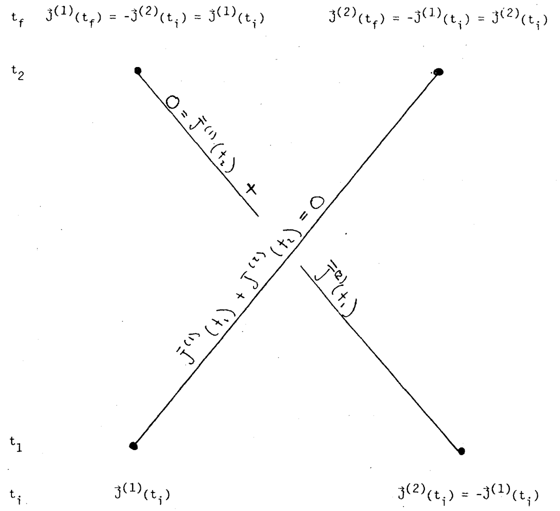

FIG. 9. If crossed measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ conducted on particles initially in the singlet state could yield the results $\mathbf{J}^{(2)}(t_f)+\mathbf{J}^{(1)}(t_i)=0$ and $\mathbf{J}^{(1)}(t_f)+\mathbf{J}^{(2)}(t_i)=0$, then the measurements, though performed in reverse order, would have the same effect on both particles, returning each to its initial state.

spins in such a way that $\mathbf{J}^{(1,2)}(t_f)=-\mathbf{J}^{(2,1)}(t_i)$.

As alluded to earlier, one possible interpretation of the interparticle correlations introduced by a pair of crossed two-time measurements is that the measurements place the system in a *two-time state* that is an eigenvalue-zero eigenstate of the two-particle, two-time operators $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$. Such an interpretation is appealing because it suggests that it is possible to extend the formalism used to describe the results of ordinary, "uncrossed" two-particle measurements [such as measurements of $\sigma_x^{(1)}(t)+\sigma_x^{(2)}(t)$ and $\sigma_y^{(1)}(t')+\sigma_y^{(2)}(t')$] to the case of crossed two-time measurements.

Let us briefly review a familiar example of the former type of measurement. Consider two spin-half particles, each in an arbitrary initial spin state. Suppose we are told that the x component of the total spin of the system is measured at a time t, that the y component of the total spin of the system is measured at a later time t', and that the result of each measurement is zero. Then we can state with certainty that the final state of the two-particle system is a spin singlet, regardless of the identity of the initial states of the particles. In other words, we can state that the final state of the particles is that state whose projection on the eigenvalue-zero eigenstate of $J_x\equiv\sigma_x^{(1)}+\sigma_x^{(2)}$ and $J_y\equiv\sigma_y^{(2)}+\sigma_y^{(2)}$ is one.

Now let us return to the case of two crossed measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$. By analogy with the preceding example, we expect that the final state of two particles for which crossed measurements of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ yield zero will be that state which, together with the initial state of the particles, forms a *two-time state* whose projection on the eigenvalue-zero two-time eigenstate of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ is one.

Let us define[7] the eigenvalue-zero eigenstate of these two-particle operators (to within an overall constant) as

$$\begin{aligned}|\psi_0(t_2)\rangle\langle\psi_0(t_2)|\psi_0(t_1)\rangle\langle\psi_0(t_1)| = {} & |\sigma_y=+1\rangle_1|\sigma_x=+1\rangle_2\cdots\langle\sigma_x=-1|_1\langle\sigma_y=-1|_2\\ &+|\sigma_y=-1\rangle_1|\sigma_x=+1\rangle_2\cdots\langle\sigma_x=-1|_1\langle\sigma_y=+1|_2\\ &+|\sigma_y=+1\rangle_1|\sigma_x=-1\rangle_2\cdots\langle\sigma_x=+1|_1\langle\sigma_y=-1|_2\\ &+|\sigma_y=-1\rangle_1|\sigma_x=-1\rangle_2\cdots\langle\sigma_x=+1|_1\langle\sigma_y=+1|_2\,,\end{aligned}$$

where the ellipses dots in each term represent the scalar product of the ket and bra in that term. [Notice that the four terms in $|\psi_0(t_2)\rangle\cdots\langle\psi_0(t_1)|$ correspond to the four ways in which the conditions $\sigma_x^{(1)}(t_1)+\sigma_x^{(2)}(t_2)=0$ and $\sigma_y^{(1)}(t_2)+\sigma_y^{(2)}(t_1)=0$ may be satisfied.] We require, then, that the final and initial states of the two particles together form a two-time state such that

$$\langle\psi(t_f)|[\,|\psi_0(t_2)\rangle\cdots\langle\psi_0(t_1)|\,]|\psi(t_i)\rangle=1\,.$$

In other words, $|\psi(t_f)\rangle$ must be given by

$$|\psi(t_f)\rangle=|\psi_0(t_2)\rangle\cdots\langle\psi_0(t_1)|\psi(t_i)\rangle\,. \tag{18}$$

As may easily be verified, Eq. (18) correctly predicts that $|\psi(t_f)\rangle=|\sigma_y=+1\rangle_1|\sigma_x=-1\rangle_2$ when $|\psi(t_i)\rangle=|\sigma_x=+1\rangle_1|\sigma_y=-1\rangle_2$, as was shown earlier. Furthermore, since $|\psi_0(t_2)\rangle\cdots\langle\psi_0(t_1)|$ can be reexpressed as

$$\begin{aligned}&|\sigma_x=+1\rangle_1|\sigma_y=+1\rangle_2\cdots\langle\sigma_y=-1|_1\langle\sigma_x=-1|_2+|\sigma_x=+1\rangle_1|\sigma_y=-1\rangle_2\cdots\langle\sigma_y=+1|_1\langle\sigma_x=-1|_2\\ &\quad+|\sigma_x=-1\rangle_1|\sigma_y=+1\rangle_2\cdots\langle\sigma_y=-1|_1\langle\sigma_x=+1|_2+|\sigma_x=-1\rangle|\sigma_y=-1\rangle_2\cdots\langle\sigma_y=+1|_1\langle\sigma_x=+1|_2\end{aligned}$$

or as

$$|\sigma_z=+1\rangle_1|\sigma_z=+1\rangle_2\cdots\langle\sigma_z=+1|_1\langle\sigma_z=+1|_2-|\sigma_z=+1\rangle_1|\sigma_z=-1\rangle_2\cdots\langle\sigma_z=-1|_1\langle\sigma_z=+1|_2$$

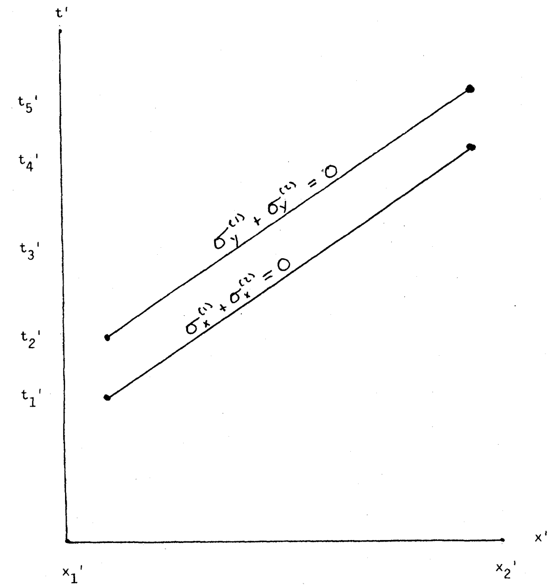

FIG. 10. Observer A's measurements of $\sigma_x^{(1)}+\sigma_x^{(2)}$ and $\sigma_y^{(1)}+\sigma_y^{(2)}$, as seen in observer B's frame of reference.

$$-\,|\sigma_z=-1\rangle_1|\sigma_z=+1\rangle_2\cdots\langle\sigma_z=+1|_1\langle\sigma_z=-1|_2$$
$$+\,|\sigma_z=-1\rangle_1|\sigma_z=-1\rangle_2\cdots\langle\sigma_z=-1|_1\langle\sigma_z=-1|_2\,,$$

Eq. (18) correctly predicts the form of $|\psi(t_f)\rangle$ when the initial state of the system is $|\sigma_y=-1\rangle_1|\sigma_x=+1\rangle_2$ or $|\sigma_z=+1\rangle_1|\sigma_z=-1\rangle_2$.

IV. RELATIVISTIC CONSIDERATIONS

In Sec. III, an analogy was drawn between the traditional formalism used to describe the results of single-time measurements on a two-particle system (such as measurements of J_x and J_y) and a formalism of two-time eigenstates (or, more generally, multiple-time eigenstates) that can be used to describe the results of two-time measurements. Let us now investigate more closely the relationship between single-time and multiple-time information pertaining to a given quantum system by analyzing the descriptions of the system given by two or more Lorentz observers.

Consider first a case in which one Lorentz observer—say, A—decides to carry out single-time measurements of J_x and J_y on two spin-half particles which are at rest in his frame of reference, in order to determine whether or not the particles' spins are anticorrelated. To accomplish this, A arranges for four measuring devices to interact with the particles at t_1 and t_2 in such a way as to record the values of[5] $\sigma_x^{(1)}(t_1)+\sigma_x^{(2)}(t_1)\equiv(2/\hbar)J_x(t_1)$ and $\sigma_y^{(1)}(t_2)+\sigma_y^{(2)}(t_2)\equiv(2/\hbar)J_y(t_2)$. If the result of A's measurements is that $J_x(t_1)=J_y(t_2)=0$, then A will conclude that, irrespective of the initial state of the particles, the final state of the particles is a spin singlet. He therefore will be able to state with certainty that any single-time measurements of J_x, J_y, or J_z carried out on the particles at a time $t>t_2$ will yield zero.

Consider next the description of A's measurements given by a second Lorentz observer B, with respect to whom A moves with a uniform velocity in the $+x$ direction. According to B, A's measurements of $\sigma_x^{(1)}$ and $\sigma_x^{(2)}$ occur at two *different* times (say, t_1' and t_4'), as do A's measurements of $\sigma_y^{(1)}$ and $\sigma_y^{(2)}$ (at times t_2' and t_5', say). B's representation of A's measurements on a spacetime diagram therefore has the appearance of Fig. 10.

Now observer B, in contrast to observer A, is unable to make definite statements about the single-time state of the two-particle system in the interval between t_1' and t_5'. He cannot say, for example, that the spins of the particles are anticorrelated at $t'=t_3'$, where $t_1'<t_2'<t_3'<t_4'<t_5'$. B *can* state, however, that there is a correlation between the

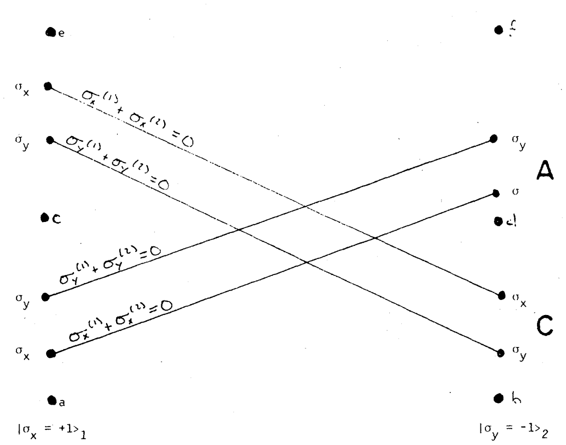

FIG. 11. Observers A and C conduct measurements of J_x and J_y in their respective frames of reference, each finding that $J_x=J_y=0$. A and C will agree on the state of particle 1 at points c and e and on the state of particle 2 at points d and f.

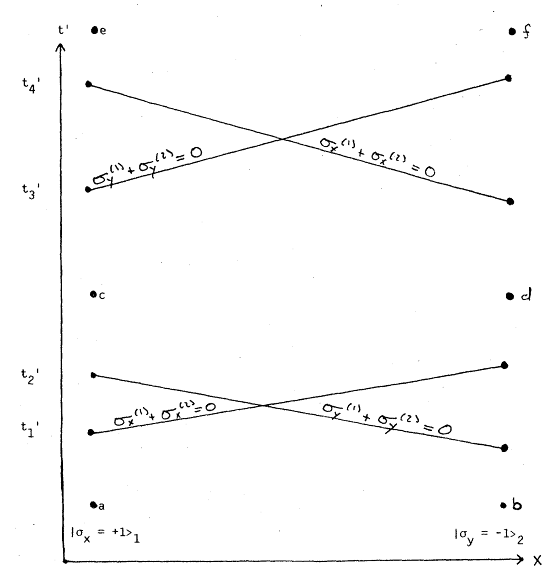

FIG. 12. Observer B's analysis of the single-time measurements performed by observers A and C in Fig. 11. B will agree with A and C on the state of particle 1 at points c and e and on the state of particle 2 at points d and f.

value of certain two-time observables associated with particle 1 and the values of certain two-time observables associated with particle 2. In this particular case, for example, B can say that

$$\sigma_x^{(1)}(t_1')\pm\sigma_y^{(1)}(t_2')=-[\sigma_x^{(2)}(t_4')\pm\sigma_y^{(2)}(t_5')] ,$$

since A's experiments (as seen by B) show that $\sigma_x^{(1)}(t_1')+\sigma_x^{(2)}(t_4')=\sigma_y^{(1)}(t_2')+\sigma_y^{(2)}(t_5')=0$. We conclude, therefore, that there are two equivalent descriptions of the system which are related to one another by a Lorentz transformation: one description in which the single-time states of the particles are correlated to one another, and one description in which the two-time states of the particles are correlated to one another.

The equivalence between the varying (i.e., single-time versus two-time) descriptions of a quantum system given by two or more Lorentz observers can be illustrated by means of the following example. Consider a case in which observers A (who moves with uniform velocity in the $+x$ direction with respect to B) and C (who moves with uniform velocity in the $-x$ direction with respect to B) conduct a "singlet test" (i.e., single-time measurements of J_x and J_y) in their respective frame of reference, with positive results.[8] What description will A, B, and C give of particles 1 and 2 at the spacetime points c, d, e, and f shown in Fig. 11?

Let us assume that the initial state of the two-particle system—that is, the state of the system in the common past of all three observers—is $|\sigma_x=+1\rangle_1|\sigma_y=-1\rangle_2$ (although the argument that follows is completely general). Let us also assume that all three observers are aware that the outcome of A's and C's measurements is that $J_x=J_y=0$. Then observers A and C will *retrodict* from the outcome of A's singlet test that the state of particle 2 at point d must be $|\sigma_x=-1\rangle_2$, since that state must satisfy the relation $\sigma_x^{(2)}(d)=-\sigma_x^{(1)}(a)$ (where a is a point in the observers' common past). In a similar fashion, they will *retrodict* from C's singlet test that the state of particle 1 at point c is $|\sigma_y=+1\rangle_1$. A and C will then *predict* from the outcome of A's and C's measurements, respectively, that the final state of particle 2 (that is, its state at point f) is $|\sigma_y=-1\rangle_2$ and that the final state of particle 1 is $|\sigma_x=+1\rangle_1$. A and C will agree, then, that the net effect of their four measurements is to change each particle's spin twice, in such a way as to return each particle to its initial state.

What will be B's description of the two-particle system subjected to A's and C's measurements? First, we note that B sees the following *two-time* measurements carried out on the particles:

$$\sigma_x^{(1)}(t_1')+\sigma_x^{(2)}(t_3')=0 ,\quad \sigma_y^{(1)}(t_2')+\sigma_y^{(2)}(t_4')=0 ,$$
$$\sigma_y^{(1)}(t_3')+\sigma_y^{(2)}(t_1')=0 ,\quad \sigma_x^{(1)}(t_4')+\sigma_x^{(2)}(t_2')=0 .$$

Next, we note that, from B's viewpoint, the two consecutive measurements of $\sigma_y^{(1)}$ at t_2' and t_3' must yield the same results [i.e., $\sigma_y^{(1)}(t_2')=\sigma_y^{(1)}(t_3')$], as must the two consecutive measurements of $\sigma_x^{(2)}$ at t_2' and t_3' $[\sigma_x^{(2)}(t_2')=\sigma_x^{(2)}(t_3')]$. Accordingly, B can restate the outcome of the four measurements as

$$\sigma_x^{(1)}(t_1')+\sigma_x^{(2)}(t_2')=0 ,\quad \sigma_y^{(1)}(t_2')+\sigma_y^{(2)}(t_1')=0 ,$$
$$\sigma_y^{(1)}(t_3')+\sigma_y^{(2)}(t_4')=0 ,\quad \sigma_x^{(1)}(t_4')+\sigma_x^{(2)}(t_3')=0 .$$

B therefore sees two consecutive sets of crossed measurements of $\sigma_x(t_j',t_k')$ and $\sigma_y(t_k',t_j')$ carried out on the two-particle system. (The fact that the order of the measurements differs in the two pairs of measurements is unimportant, as was shown earlier.) Based on this analysis (which is illustrated in Fig. 12), B will conclude that the state of the two particles at points c and d (which are at equal times in his frame) is $|\sigma_y=+1\rangle_1|\sigma_x=-1\rangle_2$ and that the final state of the system (its state at points e and f) is $|\sigma_x=+1\rangle_1|\sigma_y=-1\rangle_2$. B's *multiple-time* or *many-time* description of the two-particle system will therefore be completely equivalent to the single-time descriptions of the system given by A and C.

Lorentz transformations, then, can generate many-time properties out of single-time properties; what amounts, in some particular frame, to the measurement of a single-time property, amounts to the measurement of a many-time property in another frame. Many-time descriptions of nonrelativistic systems might be considered luxuries—after all, we can covariantly avoid such descriptions; we can, if we wish, covariantly prohibit all but single-time experiments). Such descriptions of relativistic systems, on the other hand, simply cannot be done without.

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[3]We are not concerned at this point with the rather complicated question of how conditions giving rise to this interaction Hamiltonian might be achieved in the laboratory.

[4]Y. Aharonov and D. Z. Albert, Phys. Rev. D **21**, 3316 (1980).

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[6]S. S. D'Amato, in *Proceedings of the Conference on Fundamental Questions in Quantum Mechanics, Albany, New York, 1984,* edited by L. M. Roth and A. Inomata (Gordon and Breach, New York, 1985).

[7]The rationale for defining the two-time eigenstate of $\sigma_x(t_1,t_2)$ and $\sigma_y(t_2,t_1)$ in this fashion is given by S. S. D'Amato, Ph.D., University of South Carolina, 1984.

[8]As is shown in Ref. 5, when two Lorentz observers attempt to monitor the history of a two-particle system by performing measurements in overlapping regions of spacetime, their measurements disrupt one another. As a result, the outcome of A's and C's measurements in this example will both be uncertain. However, it *may* occur that both A and C find that $J_x=J_y=0$.