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# Measurement process in relativistic quantum theory

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An instantaneous nondemolition measurement technique for composite quantum-mechanical systems (those which consist of several separate subsystems) is described, and that technique is applied to the design of verification experiments for some nonlocal variables and states. Relativistic restrictions on the measurability of nonlocal properties are derived, and other, less familiar, varieties of measurement are discussed.

## I. INTRODUCTION

There are a number of well-known difficulties in the interpretation of the measurement process in nonrelativistic quantum theory; and the theory of relativity produces new restrictions on the measurement process: The measurement of certain nonlocal variables is known to contradict the principle of causality. It was once thought that in relativistic quantum theory only local variables are measurable; but lately it has emerged[1] that this is not so: certain nonlocal states *can* be verified by experiment. On the other hand *not* all of them are measurable, and indeed there are states whose measurement would contradict the principle of causality. In this paper we will generalize the results of Ref. 1 and will give a description of the measurable operators and the measurable states in a composite system.

In this work we start with the *assumption* that we can measure any *local* operator, and we *investigate* which *nonlocal* operators and states are measurable. We are interested in the following question. Does it have physical meaning to speak about nonlocal variables at a particular time? The measuring procedures we seek, in response to this question, will consequently be instantaneous.

The organization of the paper is as follows. In Sec. II we describe the method for verifying that a nonlocal operator has a given value. This will also include a description of the measurement of modular variables. We will use those operator measurements in Sec. III for verification of nonlocal states. We will demonstrate the general measurement procedure in an explicit example: the verification of a nonlocal state of a system consisting of three subsystems separated in space. The restrictions on our method which follow from the causality principle will be discussed in Sec. IV. We will see there what nonlocal operators and states cannot be measured by the methods of the previous sections. Then in Sec. V we will propose a new kind of verification for quantum states using local "exchange" interactions that is suitable for those "unmeasurable" states. We will also describe an example of a nonlocal measurable operator with nondegenerate eigenstates (measurable, that is, in the familiar, von Neumann,[2] sense).

## II. MEASUREMENT OF NONLOCAL VARIABLES

First we shall need to explain what we *signify* by "measurement" in this particular section. It is different from the usual definition of measurement in quantum mechanics. We define a measurement here as the nondemolition verification that a certain variable $A$ has a given value $a$. If before the measurement the observed variable has the value $a$ then the experiment will produce the result "yes," and the state of the system will *not* change. If our initial state is a linear combination of eigenstates of the observed operator with different eigenvalues of $A$ then the experiment will produce the result "yes" or "no" with appropriate probability. In case the answer is "yes," the final state will be the projection of the initial state on the degenerate space of eigenstates of $A$ with eigenvalue $a$. If the answer is "no," then the final state will be orthogonal to that space. The difference between this and the usual definition of measurement in quantum mechanics is that we do *not* require that the other eigenstates of the observed variable (with *other* eigenvalues than $a$) be unaltered during the measuring process. The only requirement is that if we start with a state wherein the $A \neq a$, then $A \neq a$ at the *end* of the measurement as well.

Our first nonlocal variable will be the sum of the local variables $A_1$ and $A_2$ that are related to spatially separate parts of the system. We are interested in nonlocal measurement; that is, after the measurement we would like to know the value of the sum $A_1+A_2$ *without* knowing the values of $A_1$ and $A_2$ separately.

We have to verify that $A_1+A_2=a$. By redefining $A_2 \rightarrow A_2 - a$ we see that our problem is the verification that

$$A_1+A_2=0 . \tag{1}$$

Our measuring device consists of two separate parts which have canonical coordinates $q_1$ and $q_2$. We prepare this composite device in the nonlocal state

$$q_1-q_2=0, \quad \pi_1+\pi_2=0 , \tag{2}$$

where $\pi_i$ is the momentum conjugate to $q_i$. We can do this by local interaction when the two parts of the measuring device are initially brought together. Then we separate those parts and position them at the appropriate parts of the observed system.

The next stage of our measurement procedure is the local interaction between appropriate components of the measuring device and those of the observed system. The interactions are short and simultaneous. The time of the measurement is defined by the time of this interaction. The Hamiltonian of the interaction is

$$H_{\rm int}=g(t)(q_1A_1+q_2A_2)\ , \tag{3}$$

where $g(t)$ is nonzero only during a short interval of time $[t_0,t_0+\epsilon]$ and it satisfies the normalization condition

$$\int_{t_0}^{t_0+\epsilon} g(t)dt=1\ . \tag{4}$$

Then in the Heisenberg picture

$$\dot{\pi}_1=-g(t)A_1,\ \ \dot{\pi}_2=-g(t)A_2\ . \tag{5}$$

$A_1$ and $A_2$ are not changed by interaction (3), and we can take $\epsilon$ small enough so that $A_1$ and $A_2$ will not be changed (as a result of their *own* dynamics) during the time of the interaction. Then, using initial condition (2), and normalization of $g(t)$ (4) we find from (5) that

$$(\pi_1+\pi_2)_{t>t_0+\epsilon}=-(A_1+A_2)_{t=t_0}\ . \tag{6}$$

The last step of our measuring procedure consists of local measurements of $\pi_1$ and $\pi_2$. We will perform those measurements immediately after the local interactions at time $t=t_0+\epsilon$. This completes the measurement of $A_1+A_2$.

Indeed, we see from (6) that knowing $\pi_1$ and $\pi_2$ after the interaction gives us the value of $A_1+A_2$. At time $t=t_0+\epsilon$ there is no local observer that knows the value of $A_1+A_2$. Such knowledge would require bringing the results of the local measurements of $\pi_1$ and $\pi_2$ together, and that would require some additional finite period of time. But since the values $\pi_1$ and $\pi_2$ have been indelibly recorded at time $t=t_0+\epsilon$ (by means of those final local measurements), then the measurement of $A_1+A_2$ (given that the "measurement" of $A_1+A_2$ is taken to mean the indelible recording, in some macroscopic form, of the value $A_1+A_2$ at time $t_0+\epsilon$) is unambiguously completed at that time.

The measurement is nonlocal. After the measurement we know neither $A_1$ nor $A_2$. It is nondemolition. If in the initial state $A_1+A_2=0$ then the interaction Hamiltonian acting on the initial state gives us zero:

$$H_{\rm int}\,|\,\psi_{\rm in}\rangle=g(t)q_1(A_1+A_2)=0\ . \tag{7}$$

We have shown that our measurement procedure is the nondemolition verification that $A_1+A_2=0$. Now, suppose that we start with the initial state

$$\alpha\,|\,A_1+A_2=0\rangle+\beta\,|\,A_1+A_2=b\,\rangle,\ \ b\neq0\ . \tag{8}$$

Then the measurement will induce the transformation

$$\begin{aligned}(\alpha\,|\,A_1+A_2=0\rangle+\beta\,|\,A_1+A_2=b\,\rangle)\,|\,\pi_1+\pi_2=0\rangle&\to\alpha\,|\,A_1+A_2=0\rangle\,|\,\pi_1+\pi_2=0\rangle+\beta\,|\,A_1+A_2=b\,\rangle\,|\,\pi_1+\pi_2=-b\,\rangle\\ &\to\begin{cases}\text{yes, probability }|\alpha|^2, & |\,A_1+A_2=0\rangle\\ \text{no, probability }|\beta|^2, & |\,A_1+A_2=b\,\rangle\ .\end{cases}\end{aligned} \tag{9}$$

We see that our procedure will satisfy all the requirements of our definition of measurement. More than this, it is a measurement of $A_1+A_2$ in the usual sense of quantum mechanics. Indeed if we start with the state $|\,A_1+A_2=b\,\rangle$ then

$$H_{\rm int}\,|\,\psi_{\rm in}\rangle=g(t)q_1b\,|\,\psi_{\rm in}\rangle\ . \tag{10}$$

This Hamiltonian is not equal to zero as in (7), but it acts only on the measuring device, and therefore all eigenstates of the operator $A_1+A_2$ are unchanged during the interaction.

We can generalize this method to the measurement of the sum of $N$ local operators $\sum_{i=1}^N A_i$ related to $N$ separate parts of the composite system. The measuring device will consist of $N$ parts and it will be prepared in the initial state

$$q_i-q_j=0,\ \ i,j=1,2,\ldots,N\ ,\qquad \sum_{i=1}^N \pi_i=0\ . \tag{11}$$

The interaction Hamiltonian is

$$H_{\rm int}=g(t)\sum_{i=1}^N q_iA_i\ . \tag{12}$$

Then we find $\sum_{i=1}^N A_i$ by local measurements of $\pi_i$

$$\left[\sum_{i=1}^N A_i\right]_{t=t_0}=-\left[\sum_{i=1}^N \pi_i\right]_{t=t_0+\epsilon}\ . \tag{13}$$

We can also measure any linear combination $\sum_{i=1}^N \alpha_iA_i$. We define $A_i'=\alpha_iA_i$ and then measurement of $\sum_{i=1}^N\alpha_iA_i$ will be the measurement of $\sum_{i=1}^N A_i'$. We can also measure $\prod_{i=1}^N A_i$ (in case the eigenvalues of the operators $A_i$ are positive): This is equivalent to measuring $\sum_{i=1}^N A_i''$, where $A_i''=\ln A_i$; but we cannot measure *all* possible functions of $A_i$.

A more general class of nonlocal variables that we *can* measure are the *modular* sum of local variables: $(\sum_{i=1}^N A_i)\,{\rm mod}\,a$. In case the system consists of two parts, the measurement of the modular sum is equivalent to measurement of the sum

$$(A_1+A_2)\,\mathrm{mod}a=b \Longleftrightarrow B_1+B_2=b\ ,$$

where

$$B_1=\begin{cases}A_1\,\mathrm{mod}a, & A_1\,\mathrm{mod}a\le b\ ,\\ A_1\,\mathrm{mod}a-b, & A_1\,\mathrm{mod}a>b\ ,\end{cases}$$
$$B_2=A_2\,\mathrm{mod}a\ . \tag{14}$$

But if the system has more than two parts then measurement of the modular sum is a new problem.

For performing this measurement we will use the same Hamiltonian and the same measuring device. The difference will be in the initial state of the measuring device:

$$q_i-q_j=0,\quad i,j=1,2,\dots,N\ ,$$
$$\left[\sum_{i=1}^{N}\pi_i\right]\mathrm{mod}a=0,\quad q_i\,\mathrm{mod}\frac{2\pi\hbar}{a}=0\ . \tag{15}$$

We have the same Hamiltonian; therefore, again $\sum_{i=1}^{N}A_i$ will be equal to the change of $\sum_{i=1}^{N}\pi_i$ during the interaction. But now $\sum_{i=1}^{N}\pi_i$ in the initial state is defined only modulo $a$. So the measurement of $\sum_{i=1}^{N}\pi_i$ after the interaction will give us the value of $(\sum_{i=1}^{N}A_i)\,\mathrm{mod}a$:

$$\left[\sum_{i=1}^{N}A_i\right]\mathrm{mod}a=\left[-\sum_{i=1}^{N}\pi_i\right]_{t>t_0+\epsilon}\mathrm{mod}a\ . \tag{16}$$

We now show that this measurement procedure is the nondemolition verification that $(\sum_{i=1}^{N}A_i)\,\mathrm{mod}a=0$. We will see that the time-translation operator during the measurement, which acts on the initial state of the system and the measuring device, does not change that initial state:

$$\exp\left[-\frac{i}{\hbar}\int_{t_0}^{t_0+\epsilon}H_{\mathrm{int}}dt\right]|\psi_{\mathrm{in}}\rangle=|\psi_{\mathrm{in}}\rangle\ . \tag{17}$$

This is equivalent to

$$\left[\frac{1}{\hbar}\int_{t_0}^{t_0+\epsilon}H_{\mathrm{int}}dt\right]\mathrm{mod}2\pi\,|\psi_{\mathrm{in}}\rangle=0\ . \tag{18}$$

$A_i$ and $q_i$ do not change during the interaction; therefore, using (4) we get

$$\frac{1}{\hbar}\int_{t_0}^{t_0+\epsilon}H_{\mathrm{int}}dt\,|\psi_{\mathrm{in}}\rangle=\frac{1}{\hbar}q_1\sum_{i=1}^{N}A_i\,|\psi_{\mathrm{in}}\rangle\ , \tag{19}$$

and taking into account the initial state of the system and the measuring device we deduce that Eq. (18) is satisfied. Therefore our measurement is a nondemolition measurement.

If we start with a state that is a linear combination of eigenstates of the operator $\sum_{i=1}^{N}A_i$, then the measurement will be described by a transformation that is similar to (9); it is easy to see that it satisfies all other requirements of our definition of measurement.

## III. MEASUREMENT OF NONLOCAL STATES

Now we are going to describe the method for measuring nonlocal states using sets of measurements of the type considered in Sec. II. What we mean by measurement of state $|\phi\rangle$ is the nondemolition verification that the state of the system is $|\phi\rangle$. If we start with the state $|\psi\rangle=\alpha|\phi\rangle+\beta|\phi_\perp\rangle$ where $|\phi_\perp\rangle$ is orthogonal to $|\phi\rangle$, then the measurement will produce the result "yes" with probability $|\alpha|^2$, and the final state will in those cases be $|\phi\rangle$; it will produce the result "no" with probability $|\beta|^2$ and the final state in those cases will be orthogonal to $|\phi\rangle$ (but will *not* necessarily be $|\phi_\perp\rangle$). To begin with we study a system that consists of two separate parts with $K$ orthogonal states in each. We will designate local bases in each part as $|i\rangle_1$ and $|j\rangle_2$, $i,j=1,2,\dots,K$. The general state of the system can be written as

$$|\psi\rangle=\sum_{i,j=1}^{K}\beta_{ij}\,|i\rangle_1|j\rangle_2\ . \tag{20}$$

We can always find new bases in the separate local parts such that the state $|\psi\rangle$ will have the form (we will call it *canonical)*

$$|\psi\rangle=\sum_{i=1}^{K}\alpha_i\,|i\rangle_1|i\rangle_2\ . \tag{21}$$

More than this, we can choose local bases such that all $\alpha_i$ will be real and non-negative.

We will prove this statement in matrix language. $U^1$ and $U^2$ will be unitary transformations that relate the new and the old bases in local parts of our system:

$$|i\rangle_1=\sum_t U^1_{it}\,|t\rangle_1,\quad |j\rangle_2=\sum_s U^2_{js}\,|s\rangle_2 \tag{22}$$

($|t\rangle_1$ and $|s\rangle_2$ are new bases).

Then the statement we wish to prove will be equivalent to

$$\sum_{ij}U^1_{it}\beta_{ij}U^2_{js}=\alpha_t\delta_{ts} \tag{23}$$

or in matrix notation ($\alpha\equiv\alpha_t\delta_{ts}$)

$$U^{1T}\beta U^2=\alpha\ . \tag{23a}$$

We have to show that for any matrix $\beta$ there are unitary matrixes $U^1$ and $U^2$ such that the matrix $U^{1T}\beta U^2$ will be diagonal. Any matrix $\beta$ can be decomposed into the product of a Hermitian semipositive matrix $H$ and unitary matrix $U$:

$$\beta=HU\ . \tag{24}$$

The matrix $H$ can be diagonalized by a similarity transformation with a unitary matrix $U'$. The diagonalized matrix will be non-negative and we will call it $\alpha$:

$$U'^{-1}HU'=\alpha\ . \tag{25}$$

Now we define

$$U^1\equiv(U'^{-1})^T,\quad U^2\equiv U^{-1}U'\ , \tag{26}$$

then

$$U^{1T}\beta U^2=U'^{-1}HUU^{-1}U'=\alpha\ .$$

Therefore any state $|\psi\rangle$ can be brought to the canonical form (21). Now we will give the measuring procedure that verifies the state $|\phi\rangle$, the canonical form of which is

$$|\phi\rangle=\frac{1}{\sqrt{K}}\sum_{i=1}^{K}|i\rangle_1|i\rangle_2 . \quad (27)$$

The measurement of the state will include measurements of two nonlocal operators. The first is verification that $A_1+A_2=0$ where

$$A_1|i\rangle_1=-i|i\rangle_1,\quad A_2|i\rangle_2=i|i\rangle_2 . \quad (28)$$

This measurement is a nondemolition verification that our state has a canonical form in given local bases without defining the coefficients $\alpha_i$. The next measurement has to specify $\alpha_i$; in our case, it has to verify that all $\alpha_i$ are equal.

We define unitary local operators that will act in every local part of the system:

$$\begin{aligned}&U_1|i\rangle_1=|i+1\rangle_1,\quad U_1|K\rangle_1=|1\rangle_1 ,\\ &U_2|i\rangle_2=|i+1\rangle_2,\quad U_2|K\rangle_2=|1\rangle_2 .\end{aligned} \quad (29)$$

It is easy to see that among the states that have canonical form in our basis only the state $|\phi\rangle$ (27) will not change under the transformation $U_1U_2$:

$$U_1U_2|\phi\rangle=|\phi\rangle . \quad (30)$$

Now we define $B_1$ and $B_2$:

$$e^{iB_1}=U_1,\quad e^{iB_2}=U_2 . \quad (31)$$

Then, taking in account that $B_1$ and $B_2$ commute we have $U_1U_2=e^{i(B_1+B_2)}$ and therefore Eq. (30) is equivalent to

$$(B_1+B_2)\,\mathrm{mod}2\pi=0 . \quad (32)$$

So the second measurement that will complete the verification of our state is the measurement of the modular sum of $B_1$ and $B_2$. $B_1$ and $B_2$ are Hermitian local operators; therefore we can relate them to physical variables.

In a similar way we can verify the state that has the following canonical form:

$$|\phi\rangle=\frac{1}{\sqrt{N}}\sum_{n=1}^{N}|i_n\rangle_1|i_n\rangle_2 , \quad (33)$$

where $\{i_n\}$ is any subset of the set of indices $\{1,\ldots,K\}$.

We start with nondemolition local measurements (it is enough to perform them in one part only) which verify that the local state is not $|i_t\rangle_1$, where $\{i_t\}=\sim\{i_n\}$. After measurement of the sum of $A_1+A_2$ [see (28)], then, rather than using (29), we will define

$$\begin{aligned}&U_1|i_n\rangle_1=|i_{n+1}\rangle_1,\quad U_1|i_N\rangle_1=|i_1\rangle_1 ,\\ &U_2|i_n\rangle_2=|i_{n+1}\rangle_2,\quad U_2|i_N\rangle_2=|i_1\rangle_2 ,\end{aligned} \quad (34)$$

and the rest of the procedure is exactly the same as above [see (30), (31), and (32)]. As we will see in the next section, states for which the $\alpha_i$'s are *not* of the same magnitude cannot be verified by this procedure.

Now we will generalize this method to the composite system that has $M>2$ separated parts. We will measure the state

$$|\phi\rangle=\frac{1}{\sqrt{K}}\sum_{i=1}^{K}|i\rangle_1|i\rangle_2\cdots|i\rangle_M . \quad (35)$$

Again, the measurement procedure has two stages. The first is a verification that the state has only terms such as $\prod_{m=1}^{M}|i\rangle_m$ and the second stage is a verification that all these terms have equal coefficients $1/\sqrt{K}$.

For completion of the first stage we have to perform $K-1$ measurements of the sum of two local operators. The operators are

$$\begin{aligned}&A_1|i\rangle_1=-i|i\rangle_1 ,\\ &A_l|i\rangle_l=i|i\rangle_l,\quad l=2,3,\ldots,M\end{aligned} \quad (36)$$

and the measurements are verifications that

$$A_1+A_l=0,\quad l=2,3,\ldots,M . \quad (37)$$

The second stage is exactly the same as in the case of the two-part system. We define $U_m$ and $B_m$ as in (29) and (31) and our state will be specified by the equation

$$\prod_{m=1}^{M}U_m|\phi\rangle=|\phi\rangle \quad (38)$$

that is equivalent to

$$\left[\sum B_m\right]\mathrm{mod}2\pi=0 , \quad (39)$$

and we know how to verify this last proposition.

We will demonstrate our methods on a simple example of the measurement of the nonlocal state of the system that consists of three parts with two orthogonal states in each part. The state that we are going to verify is

$$|\phi\rangle=\frac{1}{\sqrt{2}}(|1\rangle_1|1\rangle_2|1\rangle_3+|2\rangle_1|2\rangle_2|2\rangle_3) . \quad (40)$$

Our measurement procedure will have two stages. The first is a verification that the state has the form (in a given basis)

$$|\phi\rangle=\alpha_1|1\rangle_1|1\rangle_2|1\rangle_3+\alpha_2|2\rangle_1|2\rangle_2|2\rangle_3 \quad (41)$$

and this will be done by two measurements. One will be a verification that the state has the form

$$|\phi\rangle=\alpha_1|1\rangle_1|1\rangle_2|\cdots\rangle_3+\alpha_2|2\rangle_1|2\rangle_2|\cdots\rangle_3 , \quad (42)$$

and the other will show that the state is of the form

$$|\phi\rangle=\alpha_1|1\rangle_1|\cdots\rangle_2|1\rangle_3+\alpha_2|2\rangle_1|\cdots\rangle_2|2\rangle_3 . \quad (43)$$

The last stage is a verification that $\alpha_1=\alpha_2$.

We will use a measuring device that consists of three separate parts with two orthogonal states in each part. It is similar to our system. The local states of the measuring device shall be designated as $|\underline{i}\rangle_m$, $i=1,2$; $m=1,2,3$. The local interaction between parts of the measuring device and parts of the system will be

$$
\begin{aligned}
&|1\rangle_m |\underline{1}\rangle_m \to |1\rangle_m |\underline{1}\rangle_m ,\\
&|1\rangle_m |\underline{2}\rangle_m \to |1\rangle_m |\underline{2}\rangle_m ,\\
&|2\rangle_m |\underline{1}\rangle_m \to |2\rangle_m |\underline{2}\rangle_m ,\\
&|2\rangle_m |\underline{2}\rangle_m \to |2\rangle_m |\underline{1}\rangle_m .
\end{aligned} \tag{44}
$$

The local measurements will be the measurements of operators $\underline{A}_m$:

$$\underline{A}_m |\underline{i}\rangle_m = i |\underline{i}\rangle_m . \tag{45}$$

For performing the first measurement of the first stage we will prepare two parts of the measuring device (MD) in the state:

$$|\psi_{\rm MD}\rangle = \frac{1}{\sqrt{2}}( |\underline{1}\rangle_1 |\underline{1}\rangle_2 + |\underline{2}\rangle_1 |\underline{2}\rangle_2 ) . \tag{46}$$

Then we perform the local interaction (44). This will not change our state $|\phi\rangle$ (40). After the interaction we will perform local measurements of $\underline{A}_1$ and $\underline{A}_2$. We see that only if the results are $\underline{A}_1=1$, $\underline{A}_2=1$, or $\underline{A}_1=2$, $\underline{A}_2=2$ does the state have the form (42). We will repeat the procedure for parts 1 and 3 and we will verify that the state has the form (43) also. Therefore it has the form (41). now we have to verify that $\alpha_1=\alpha_2$.

Our method tells us to define the unitary operator $U_m$ (30):

$$U_m |1\rangle_m = |2\rangle_m, \quad U_m |2\rangle_m = |1\rangle_m , \tag{47}$$

or in matrix representation

$$U_m = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

The operators $B_m$ are defined by

$$e^{iB_m} = U_m ;$$

Therefore,

$$B_m = \frac{\pi}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} .$$

Now we will change the local bases in such a way that $B_m$ will be diagonal. The required unitary transformation is

$$T = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \tag{48}$$

then

$$B_m \to \pi \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

and we have to measure

$$\left[ \sum B_m \right] \bmod 2\pi = 0 . \tag{49}$$

In the new basis our state $|\phi\rangle$ is

$$|\phi\rangle = \frac{1}{\sqrt{4}}( |1\rangle_1 |1\rangle_2 |1\rangle_3 + |1\rangle_1 |2\rangle_2 |2\rangle_3 + |2\rangle_1 |1\rangle_2 |2\rangle_3 + |2\rangle_1 |2\rangle_2 |1\rangle_3 ) \tag{50}$$

and if the state has the form (41) but $\alpha_1 \neq \alpha_2$ then it will have other terms such as $|1\rangle_1 |2\rangle_2 |1\rangle_3$, for example, that do not satisfy Eq. (49).

To verify that $(\sum B_m) \bmod 2\pi = 0$ we will prepare the measuring device in the initial state:

$$|\psi_{\rm MD}\rangle = \frac{1}{\sqrt{4}}( |\underline{1}\rangle_1 |\underline{1}\rangle_2 |\underline{1}\rangle_3 + |\underline{1}\rangle_1 |\underline{2}\rangle_2 |\underline{2}\rangle_3 + |\underline{2}\rangle_1 |\underline{1}\rangle_2 |\underline{2}\rangle_3 + |\underline{2}\rangle_1 |\underline{2}\rangle_2 |\underline{1}\rangle_3 ) . \tag{51}$$

We will perform the local interaction (44) (in the new local bases). We see that state $|\phi\rangle$ (50) will not be changed by this interaction:

$$|\phi\rangle |\psi_{\rm MD}\rangle \to |\phi\rangle \psi_{\rm MD}\rangle . \tag{52}$$

Now we will perform local measurements of $\underline{A}_m$. If and only if the results satisfy the equation

$$\left[ \sum \underline{A}_m \right] \bmod 2 = 1 , \tag{53}$$

then we will have verified that the state is $|\phi\rangle$.

In our example the system and the measuring device may have physical realizations. They may consist of three separated spin-$\frac{1}{2}$ particles. The local measurement of $\underline{A}_m$ will in that case be a Stern-Gerlach experiment. There is no theoretical problem with the realization of the interaction Hamiltonian related to the local transformation (44). This is the reason we chose this way for verifying our state rather than using our general method [see (15)].

## IV. WHAT WE CANNOT MEASURE

The measurement of nonlocal states which we have considered so far all consist of sequences of measurements of nonlocal operators. These separate nonlocal operator measurements all consist of a number of couplings of the measuring apparatus to *local* operators of the measured system (the operators $A_m$ of Sec. III), and consequently those nonlocal operators all invariably possess complete sets of eigenstates which are also eigenstates of all of the local operators (the $A_m$) in question.

We have been unable to imagine, and we suspect that there do not exist, any measuring procedures for nonlocal states which satisfy the requirements of relativistic causality, and which satisfy the definition of state measurement given in Sec. III, which are not of the general type just described.

We shall prove, in this section, that any nonlocal state of any composite system which is measurable by such procedures as these (which, as we just said, we believe to be the only procedures available) must necessarily have the following property: If the system in question is divided in any way whatever into two nonempty parts, and bases of the state spaces of these two separate parts are chosen so that the state of the composite system (written in these bases) has the canonical form (21) (which, as we have shown above, can always be done) then the absolute values of the nonzero coefficients $\alpha_i$ must all be equal.

It should be noted that such states as these (states which have the property just described) are precisely those which were shown in Sec. III to be measurable, and consequently, the results of Sec. III, when combined with those of the present section, imply that states are measurable (by means of such procedures as we have here described) if and only if they have the property just described.

Our proof will proceed as follows. We shall first describe the restrictions on measurable operators which arise as a consequence of the requirement of causality, and then we shall show that there exist quantum states that cannot be specified by specifying the values of any combination of measurable operators.

We shall designate as $|i\rangle_m$ the eigenstates of the local operators that we will use in our measuring procedure [$A_m$ of the formula (12)]. Then the eigenstates of the nonlocal operator will be $\prod_{m=1}^{M}|i_m\rangle_m$, where $i_m$ may have all possible values. Some of these states will be degenerate. (If not, what we do amounts to the verification of a *local* state.) We shall start with considering the case of a two-part system. For example, the measurement previously described, that which verifies that the system has the canonical form in a given basis, has degenerate eigenstates $|i\rangle_1|i\rangle_2$, while other eigenstates $|i\rangle_1|j\rangle_2$, $i\neq j$ have different eigenvalues.

The causality principle gives us the following restriction on the degenerate eigenstates $|i\rangle_1|j\rangle_2$ of a measurable nonlocal operator. If $|i_1\rangle_1|j_1\rangle_2$, $|i_1\rangle_1|j_2\rangle_2$, $|i_2\rangle_1|j_1\rangle_2$ are degenerate then $|i_2\rangle_1|j_2\rangle_2$ is also degenerate with the above states. We shall prove it by showing that if $|i_2\rangle_1|j_2\rangle_2$ is the eigenstate of the nonlocal operator with a distinct eigenvalue then we can send information from one part of the system to another faster than light. The proof runs as follows.

We shall prepare the measuring device that will perform the nonlocal measurement of the operator at time $t=t_0$. Before the measurement we prepare the state $|\psi\rangle_2$ in the second part

$$|\psi\rangle_2=\alpha_1|j_1\rangle_2+\alpha_2|j_2\rangle_2,\quad \alpha_1,\alpha_2\neq 0\ . \tag{54}$$

In part one we decide to prepare state $|i_1\rangle_1$ or state $|i_2\rangle_1$ at the time $t=t_0-\epsilon$. In part two we perform a local verification measurement of state $|\psi\rangle_2$ at the time $t=t_0+\epsilon$. If at the time $t=t_0-\epsilon$ we prepare state $|i_1\rangle_1$ then the nonlocal measurement will not change the state $|\psi\rangle_2$ and the result of the last measurement in part two will be positive with probability 1. But if the initial state of part one was $|i\rangle_2$ then the state of part two after the nonlocal measurement will be either $|j_1\rangle_2$ or $|j_2\rangle_2$ with appropriate probabilities, and the probability of a positive result in the verification experiment of $|\psi\rangle_2$ will be less than 1. Therefore, *local* actions in part one can change the probability of results of local measurements in part *two* that take place a time $2\epsilon$ later. Since $\epsilon$ may be arbitrarily small, this completes the proof of the statement.

Nondemolition verification that the operator has a given value is defined as the nondemolition verification that the state can be any linear combination of degenerate eigenstates with that eigenvalue. In our case the operator measurement will tell us that in the basis of the eigenstates of the local operators our nonlocal state has the form

$$|\phi\rangle=\sum\beta_{ij}|i\rangle_1|j\rangle_2\ , \tag{55}$$

where the matrix elements $\beta_{ij}$ are nonzero only if $|i\rangle_1|j\rangle_2$ is one of the degenerate eigenstates of the observed operator. The meaning of what we have just proven about degenerate states of the operator is that we can bring the matrix $\beta_{ij}$ to the block composed form by appropriate reshuffling of the eigenstates $|i\rangle_1$ and $|j\rangle_2$. Thus all the information about the state $|\phi\rangle$ that we can get from one measurement of such an operator is that the state has the form (55), wherein the $\beta_{ij}$ are known to be zero outside of the given block and their values within that block are unknown. Our method of state measurement is a verification that the matrix $\beta_{ij}$ has the given block forms in different bases.

For any state, we can verify that it has canonical form in a given basis: $\sum\alpha_i|i\rangle_1|i\rangle_2$. We can assume that all $\alpha_i$ are nonzero. (If some of them are zero our method simply and automatically ignores the appropriate states $|i\rangle_1$ and $|i\rangle_2$.) The first stage of the measurement of the state is the verification that in one basis (the basis of canonical form) the matrix $\beta$ is diagonal. It is the matrix $\alpha$ and it is nonsingular. If we change the local bases by transformations $U^1$ and $U^2$ then the matrix will be

$$\beta=U^{1T}\alpha U^2 \tag{56}$$

and it will be nonsingular again. Our measurement verifies that $\beta$ has block form. But a matrix in block form may be nonsingular only if it is block diagonal. So all that any single measurement can do is to verify that in some basis $\beta$ has block-diagonal form.

By mixing the eigenstates of the basis inside the blocks we can bring the matrix $\beta$ into the diagonal form (but not necessarily into the matrix $\alpha$). We can perform the measurement verifying that it is diagonal in the basis and clearly this will give us no less information than the original verification of block-diagonal form. Therefore what we have to investigate is to ascertain which are the states that cannot be specified by knowing that they have canonical form in various different bases. We shall prove that there are such states, and that for each state, its canonical form has at least two nonzero distinct $|\alpha_i|$.

We can see that the density matrices in every local part of the measured system have the same block-diagonal form as the matrix $\beta$. Indeed

$$\hat{\rho}^1_{ij}=\sum_t\beta_{it}\bar{\beta}_{tj}|i\rangle_1{}_1\langle j|\ ,$$
$$\hat{\rho}^2_{ij}=\sum_t\beta_{ti}\bar{\beta}_{jt}|i\rangle_2{}_2\langle j|\ , \tag{57}$$

or in matrix representation

$$\rho^1=\beta\beta^\dagger,\quad \rho^2=\beta^\dagger\beta\ ; \tag{57a}$$

therefore if $\beta$ is diagonal then $\rho^1$ and $\rho^2$ are also diagonal and $\rho^1_{ij}=\rho^2_{ij}=|\gamma_i|^2\delta_{ij}$. The set of characteristic values of any matrix is basis independent and therefore the set $\{|\gamma_i|\}$ is equivalent to the set $\{|\alpha_i|\}$. If all the $|\alpha_i|$ are distinct, therefore there is a one-to-one correspondence between characteristic values $|\alpha_i|^2$ and the local eigenvectors or, in operator language, between the eigenvalues $|\alpha_i|^2$ and eigenstates $|i\rangle_1$ (or $|i\rangle_2$). In this case the state has canonical form only for one set of basis eigenstates. We can verify that the state has canonical form in one basis and this is the maximal information about the state that we can get by means of nondemolition experiments.

If some $|\alpha_i|$ are equal we can mix between appropriate local eigenstates of the basis ($|i\rangle_1$, in one part and $|i\rangle_2$ in the second, wherein all $|i\rangle$ correspond to equal eigenvalues $|\alpha_i|^2$), and then we can verify again that the state has canonical form in the new basis. By this procedure we can specify relative phases between appropriate $\alpha_i$ (for all sets of equal $|\alpha_i|$). If *all* $|\alpha_i|$ are equal then we can specify phases of all $\alpha_i$, we find the value $|\alpha_i|$ from the normalization, and consequently we verify the state. We demonstrated this procedure explicitly in Sec. III. However, if there are distinct $|\alpha_i|$ we cannot mix the states corresponding to these eigenvalues $|\alpha_i|$, thereby getting canonical form in the new basis. Therefore we can verify neither relative phases between these $\alpha_i$ nor relative absolute values and consequently we cannot verify those states.

The conjecture that there is no other causal way to perform nondemolition verification of a nonlocal state, except for by a set of nondemolition operator measurements using only local interactions, gives us the following result. The only nonlocal measurable states of a system with two separate parts are the states that have canonical form with all $|\alpha_i|$ equal. We can generalize this statement for composite systems with many parts. We divide any system into two subsystems and then the measurable states have to be of the form

$$|\phi\rangle=\frac{1}{\sqrt{K}}\sum_{i=1}^{K}|\psi_i\rangle_1|\psi_i\rangle_2 , \tag{58}$$

where $|\psi_i\rangle_\alpha$, $i=1,2,\ldots,k$ are orthonormal states in part $\alpha$, $\alpha=1,2$. From this follows that for any measurable state of a composite system the density matrices in each separate part of the system must be similar to the diagonal matrix, where all its nonvanishing values must be equal.

## V. OTHER VARIETIES OF MEASUREMENT

Let us discuss again our nonlocal measurement. We verify by a nondemolition experiment some nonlocal property of the system. In Sec. II that property was that the nonlocal operator has a certain given value and in Sec. III that property was that the system is in some given nonlocal state. The measurement is instantaneous in the sense that was described in Sec. II. It is nondemolition only for states that have the property for which we are looking. If we start with some other state then the measurement will give the answer "yes" or "no" with appropriate probability. If it says "yes" then the final state has the property, while if it says "no" then the final state definitely does not have the property.

If we use our definition of measurement then the state

$$|\phi\rangle\equiv\alpha_1|1\rangle_1|1\rangle_2+\alpha_2|2\rangle_1|2\rangle_2 , \quad |\alpha_1|\neq|\alpha_2|\neq 0 \tag{59}$$

is unmeasurable. We can prove that the measurability of $|\phi\rangle$ contradicts the principle of causality. But there are other kinds of measurement for which $|\phi\rangle$ is measurable (and it is this which gives us possibility of speaking about the state $|\phi\rangle$).

First, we can *prepare* state $|\phi\rangle$. We prepare locally the states

$$\alpha_1|1\rangle_1+\alpha_2|2\rangle_1 \text{ and } \frac{1}{\sqrt{2}}(|1\rangle_2+|2\rangle_2) . \tag{60}$$

Then the initial state of the system will be

$$\begin{aligned}\frac{1}{\sqrt{2}}&(\alpha_1|1\rangle_1+\alpha_2|2\rangle_1)(|1\rangle_2+|2\rangle_2)\\ &=\frac{1}{\sqrt{2}}(\alpha_1|1\rangle_1|1\rangle_2+\alpha_2|2\rangle_1|2\rangle_2)\\ &\quad+\frac{1}{\sqrt{2}}(\alpha_1|1\rangle_1|2\rangle_2+\alpha_2|2\rangle_1|1\rangle_2) .\end{aligned} \tag{61}$$

Now we verify by the measurement procedure of Sec. II that the state has canonical form. This will give the answer "yes" with probability $\frac{1}{2}$ and in these cases the final states will be $|\phi\rangle$. The experiment, of course, may or may not be successful.

Another variety of measurement is a particular kind of nondemolition verification that the state is $|\phi\rangle$. We know how to accomplish this for *every* nonlocal state $|\phi\rangle$. This verification measurement does not satisfy all the requirements of our definition of state measurement. It is nondemolition for the state $|\phi\rangle$ but this time the final state will be $|\phi\rangle$ in *any* case, *without* dependence on the initial state.

In this measurement we will use a measuring device which has a Hilbert space isomorphic to our system and we will prepare it in a state $|\underline{\phi}\rangle$ that corresponds under the isomorphism to $|\phi\rangle$. Then we will switch on some local simultaneous interactions that will produce an "exchange" between the state of the system and the state of the measuring device. The interactions that will do that are interactions between every separable part $t$ of the system and the corresponding part of the measuring device. These will be described by the transformation

$$|i\rangle_t|\underline{j}\rangle_t\to|j\rangle_t|\underline{i}\rangle_t , \tag{62}$$

where $|\underline{i}\rangle_t$ is a set of orthogonal states in one separate component $t$ of the measuring device.

We see, indeed, that this transformation leads to exchanging of the states

$$|\psi\rangle|\underline{\phi}\rangle\to|\phi\rangle|\underline{\psi}\rangle .$$

Therefore if we prepare the measuring device in a given state $|\underline{\phi}\rangle$, then the final state of the system will always be $|\phi\rangle$. This measurement procedure brings, instantaneously, all information about the state of the system (indeed the state itself) into the measuring device. Now, albeit the

Hilbert spaces of the measuring device and the measured system will be isomorphic, their *Hamiltonians* may be very different. Indeed, the free Hamiltonian of the device may be effectively zero, so that the effect of the exchange will be to freeze the state of the measured system at the instant of the exchange in the measuring device. But there is another sense in which it is difficult to claim that the measuring process is complete after the exchange. It is not only that there is no local observer that can immediately know the result of the measurement (that we have encountered before), but also that we cannot perform local measurements on the measurement device and thereby obtain a set of results (in separated places) that, after being brought together, will give us the answer. We need to bring the parts of the measuring device itself together in one place. Then the state of the measuring device will be local and we have assumed that we can measure any local state.

This "exchange" measurement has another limitation as well. It may be used only as a state measurement. We cannot produce an "exchange" measurement of an operator. This is true not only for the usual definition of an operator measurement but, also, for verification of the given value of the operator in the case that there are degenerate eigenstates with this value. We can perform this verification using the methods described in Sec. II for a quantum system that is correlated to another system. We can do this without destroying the correlation. However, this is something that clearly cannot be accomplished by an "exchange" measurement without touching the "other" system. Let us come back, now, to the measurement that was discussed above.

The usual definition of measurement in quantum mechanics is different from the one we have outlined here. We want to inquire whether our measurement procedure satisfies the requirements of the usual definition of measurement. If it does, then there is a complete set of orthogonal states that are unaffected by the measurement procedure. These are the eigenstates of the measured operator. We can see that the measurement procedure of Sec. II *is*, indeed, measurement in the usual sense. It is, in particular, the measurement of the nonlocal operator $\sum A_i$. The eigenstates of that operator will not be changed (except in overall phase) by the interaction Hamiltonian (12) when the initial state of the mechanical device is given by (11).

So we know how to do a measurement of certain nonlocal operators in the usual sense. All the measurable nonlocal operators that were considered in Sec. II have degenerate eigenstates. Those measurements do not specify the state of the system completely. This was done in Sec. III. But the measurement procedure of Sec. III is not a measurement of an operator. It has no complete set of eigenstates. So we cannot generalize our earlier statement: We cannot say that any complete orthogonal set of states that are measurable in the sense of Sec. III defines a measurable operator.

Let us give an example of a nonlocal operator the measurability of which would violate causality. The operator will have the following set of nondegenerate eigenstates:

$$
\begin{aligned}
|\psi_1\rangle &= \frac{1}{\sqrt{2}}(|1\rangle_1|1\rangle_2 + |2\rangle_1|2\rangle_2)\,,\\
|\psi_2\rangle &= \frac{1}{\sqrt{2}}(|1\rangle_1|1\rangle_2 - |2\rangle_1|2\rangle_2)\,,\\
|\psi_3\rangle &= |1\rangle_1|2\rangle_2, \quad |\psi_4\rangle = |2\rangle_1|1\rangle_2\,.
\end{aligned} \tag{63}
$$

We will contradict the principle of causality in the following way: (i) preparing state $|2\rangle_2$ in part two at time $t \ll t_0$; (ii) preparing state $|1\rangle_1$ or $|2\rangle_1$ at time $t = t_0 - \epsilon$; (iii) measurement of the operator at time $t = t_0$; (iv) local verification of the state $|2\rangle_2$ at time $t = t_0 + \epsilon$.

The probability of the result of the local measurement (iv) in part two at time $t_0 + \epsilon$ *will depend on our choice at time* $t_0 - \epsilon$ *in part one*, albeit part one is separated by an arbitrary distance from part two.

Therefore not every nonlocal operator with measurable eigenstates is itself measurable. Thus the following question arises: Does there exist a nonlocal measurable operator with nondegenerate eigenstates? The answer is yes. The nonlocal operator that we will take will have the following nondegenerate eigenstates:

$$
\begin{aligned}
|\phi_1\rangle &= \frac{1}{\sqrt{2}}(|1\rangle_1|1\rangle_2 + |2\rangle_1|2\rangle_2)\,,\\
|\phi_2\rangle &= \frac{1}{\sqrt{2}}(|1\rangle_1|1\rangle_2 - |2\rangle_1|2\rangle_2)\,,\\
|\phi_3\rangle &= \frac{1}{\sqrt{2}}(|1\rangle_1|2\rangle_2 + |2\rangle_1|1\rangle_2)\,,\\
|\phi_4\rangle &= \frac{1}{\sqrt{2}}(|1\rangle_1|2\rangle_2 - |2\rangle_1|2\rangle_2)\,.
\end{aligned} \tag{64}
$$

We will take the local operators $A_i$ (28); then $|\phi_1\rangle$ and $|\phi_2\rangle$, as well as $|\phi_3\rangle$ and $|\phi_4\rangle$ will be degenerate eigenstates of the operator $(A_1 + A_2)\text{mod}2$ that we know how to measure. Next we will perform the appropriate local unitary transformations (48) and we will measure the operator $(A_1 + A_2)\text{mod}2$ as it is defined in the new bases. Now the degenerate eigenstates will be $|\phi_1\rangle$ and $|\phi_3\rangle$ as well as $|\phi_2\rangle$ and $|\phi_4\rangle$. For the measurement that consists of these two measurements, the states $|\phi_i\rangle$ are eigenstates and they are nondegenerate.

There is no contradiction between the measurability of this operator and causality. This happens because the probability of any given result for any local measurement in all separate parts of the system is the same for all four eigenstates.

## VI. CONCLUSION

In this work we have presented a method for the measurement of nonlocal states in composite systems that have $N$ separate parts with $K$ orthogonal states in every part. The general form of the measurable states is

$$
|\phi\rangle = \frac{1}{\sqrt{K}} \sum_{i=1}^{K} \prod_{j=1}^{N} |i\rangle_j\,. \tag{65}
$$

Some of those states are familiar ones. If $K=N=2$, (65) is the Einstein-Podolsky-Rosen (EPR) -Bohm state that was used later by Bell in his original paper about the Bell inequality. If $N=2$ and $K\rightarrow\infty$, then the state is similar to the original EPR state. We proved that at least for $N=2$ these are the only measurable nonlocal states, all of which have the following local property: any local measurement in any separate part has the same probability to produce *any* given result. In other words the density matrix in all separate parts is proportional to the unit matrix. This explains why these measurements do not contradict causality. Finally we saw that there are measurable nonlocal operators. The eigenstates of those operators have a form (65).

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