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Origin of the Geometric Forces Accompanying Berry's Geometric Potentials

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We consider the dynamics of a particle carrying a magnetic moment in a strong magnetic field whose direction varies slowly in space. In particular, we discuss the geometric Lorentz-type and electric-type forces that were discovered in studies of Berry's phase. We show that the Lorentz-type force felt by the particle is caused by a small misalignment of the magnetic moment with respect to the magnetic field while the electric-type force is the time average of a strong oscillatory force induced by the precession of the magnetic moment.

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The investigation of the geometric phase [1], initiated by Berry in 1984 [2], has led to a series of studies of the appearance of gauge fields in adiabatic dynamics. The canonical example discussed in these studies is the problem of a neutral particle carrying a magnetic moment, moving in a magnetic field whose direction varies in space. These studies, to be reviewed below, have revealed surprising results regarding the dynamics of the particle [3]. In this work we discuss this problem through its equations of motion, with the goal of exploring the physical origins of these results.

The problem we analyze is that of a magnetic moment in a static, space-dependent, magnetic field. The Hamiltonian describing the system is

$$H=\mathbf{P}^2/2m-\gamma|\mathbf{B}(\mathbf{x})|\boldsymbol{l}\cdot\hat{\mathbf{b}}(\mathbf{x})\,, \tag{1}$$

where $\boldsymbol{l}$ is the internal angular momentum carried by the magnetic moment, $\boldsymbol{\mu}\equiv\gamma\boldsymbol{l}$ is the magnetic moment of the particle, $|\mathbf{B}(\mathbf{x})|$ is the magnitude of the magnetic field, and $\hat{\mathbf{b}}(\mathbf{x})$ is a unit vector in the direction of the magnetic field. For brevity, from now on we absorb the factor γ into the magnitude of the field $|\mathbf{B}(\mathbf{x})|$. We also omit the $\mathbf{x}$ dependence of the magnetic field wherever this does not lead to confusion. Particular examples of the Hamiltonian (1) were considered in various contexts by several authors [3–10], with the following similar results. If the magnetic field varies slowly enough in space, the adiabatic approximation can be employed. Employing that approximation, one finds that the dynamics of the internal angular momentum $\boldsymbol{l}$ (the fast degree of freedom of the problem) is characterized by the action $l_\parallel\equiv\boldsymbol{l}\cdot\hat{\mathbf{b}}$. This action, which is the component of $\boldsymbol{l}$ parallel to the magnetic field, is an adiabatic invariant. The effective Hamiltonian governing the dynamics of the particle (the slow degree of freedom of the problem) is found to include geometric vector and scalar potentials, induced by the spatial variation of $\hat{\mathbf{b}}$ [3]:

$$H_{\text{eff}}=\frac{[\mathbf{P}-\mathbf{A}(\hat{\mathbf{b}}(\mathbf{x}),l_\parallel)]^2}{2m}+\Phi(\hat{\mathbf{b}}(\mathbf{x}),l^2-l_\parallel^2)-l_\parallel|\mathbf{B}(\mathbf{x})|\,, \tag{2}$$

where $l\equiv|\boldsymbol{l}|$. The vector and scalar potentials depend on the direction of the magnetic field, but do not depend on the magnitude of the field $|\mathbf{B}|$ [3]. The dynamics of the particle is affected by the vector potential $\mathbf{A}(\hat{\mathbf{b}},l_\parallel)$ in a way similar to the effect of an electromagnetic vector potential on a charged particle. If the problem is solved quantum mechanically, one finds effects analogous to the Aharonov-Bohm effect, namely, geometric phases induced by the vector potential in interference phenomena. If the problem is treated classically, the vector potential affects the dynamics only through its curl, i.e., it induces a Lorentz-type force. The term "Lorentz type" refers to a force whose direction is perpendicular to that of the particle's velocity, and whose magnitude is proportional to that velocity. The scalar potential $\Phi(\hat{\mathbf{b}},l^2-l_\parallel^2)$ affects the classical dynamics of the particle through its gradient, that induces an "electric-type" force. The scalar potential and the electric-type force are proportional to $l^2-l_\parallel^2$ and are *inversely proportional to the mass of the particle.* We conclude this review of previous works by emphasizing that the induced Lorentz-type and electric-type forces are independent of the magnitude of the magnetic field $|\mathbf{B}|$ once that magnitude is strong enough to justify an adiabatic approximation.

The Lorentz-type force acting on a neutral particle due to the interaction of its magnetic moment with a space-dependent magnetic field is a surprising consequence of the spatial variation of $\hat{\mathbf{b}}$. It is surprising because the equations of motion derived from the full Hamiltonian (1) [see Eqs. (3) and (4) below] do not seem to include any velocity-dependent forces, and do seem to indicate that any force exerted on the particle should scale with the magnitude of the magnetic field $|\mathbf{B}|$. In this work we study these equations of motion. We first show that an approximation in which the magnetic moment is assumed to be aligned with the magnetic field at any point along the particle's path necessarily misses the two geometric forces. To reveal the origin of the Lorentz-type force, we then study the equation of motion for $\mathbf{l}$, expressed in a rotating frame of reference in which the magnetic field always aligns with the z axis. In that frame of reference the magnetic moment is subject to the influence of an effective magnetic field $\mathbf{B}'=\mathbf{B}+\dot{\hat{\mathbf{b}}}\times\hat{\mathbf{b}}$, where the time dependence of $\hat{\mathbf{b}}$ originates from the motion of the particle in a region of space where $\hat{\mathbf{b}}$ varies. The Lorentz-type force then results from approximating the magnetic moment as following the direction of $\mathbf{B}'$, rather than that of $\mathbf{B}$; i.e., it results from a slight misalignment of the magnetic moment relative to the direction of the magnetic field. This misalignment is of the order of $\dot{\mathbf{x}}/|\mathbf{B}|$, where $\dot{\mathbf{x}}$ is the velocity of the particle. We show that the force acting on the particle is proportional to the product of the magnitude of the magnetic field $|\mathbf{B}|$, with the misalignment of the magnetic moment, and that this product does indeed yield a $|\mathbf{B}|$-independent, velocity-dependent, Lorentz-type force. After that discussion, we turn to analyze the electric-type force, and show that this force is a time average of a strong oscillatory force, induced by the precession of the magnetic moment. We also explore the origin of the unique mass dependence of that force.

Our analysis of the problem starts from the equations of motion governing the dynamics of the particle and its internal angular momentum, as they are derived from the full Hamiltonian (1):

$$F_i=m\frac{d^2x_i}{dt^2}=l_\parallel\frac{\partial|\mathbf{B}(\mathbf{x})|}{\partial x_i}+|\mathbf{B}(\mathbf{x})|\mathbf{l}\cdot\frac{\partial\hat{\mathbf{b}}(\mathbf{x})}{\partial x_i}, \tag{3}$$

$$\frac{d\mathbf{l}}{dt}=\mathbf{l}\times\mathbf{B}(\mathbf{x}(t)), \tag{4}$$

where F_i is the ith component of the force exerted on the particle. The force F_i is composed of two components. The first component originates in the variation of the magnitude of the magnetic field $|\mathbf{B}|$. It is proportional to $l_\parallel$, and to the derivative of the magnitude of the magnetic field. The second component originates from spatial variations of the direction of the field, and is therefore the relevant term for the understanding of the Lorentz-type and electric-type forces. However, since $\partial\hat{\mathbf{b}}(\mathbf{x})/\partial x_i$ is perpendicular to $\hat{\mathbf{b}}(\mathbf{x})$, this second term is proportional to the part of $\mathbf{l}$ perpendicular to the magnetic field. *Therefore, the roots of the Lorentz-type and electric-type forces lie in the projection of the magnetic moment* $\mathbf{l}$ *on the plane perpendicular to the magnetic field.*

Following these observations, we now turn to an approximation scheme for the solutions to Eqs. (3) and (4). We start with Eq. (4), trying to obtain a solution for the angular momentum $\mathbf{l}(t)$ in terms of the path of the particle. Then, we substitute that solution in the equation for the position (3). As mentioned above, in the adiabatic limit, i.e., when $|\mathbf{B}|\to\infty$, the angle between $\mathbf{l}$ and the magnetic field $\mathbf{B}(\mathbf{x}(t))$ is an adiabatic invariant. We first focus on the case in which this angle is 0, namely, the case in which the magnetic moment approaches a direction parallel to the magnetic field. As shown below, for any finite value of the magnetic field, there is a component of the magnetic moment $\mathbf{l}$ perpendicular to the magnetic field $\mathbf{B}$, and this component is the origin of the Lorentz-type force.

We turn to Eq. (4), assume that the time-dependent magnetic field $\mathbf{B}(\mathbf{x}(t))$ is known, and try to solve for $\mathbf{l}$. The main difficulty in solving Eq. (4) is, of course, the time dependence of $\hat{\mathbf{b}}$. Attempting to "freeze" this time dependence, we express the problem in terms of a time-dependent coordinate frame, in which $\hat{\mathbf{b}}$ aligns with the z axis. We denote by $\mathbf{l}'$ the vector $\mathbf{l}$ described in terms of this time-dependent reference frame, and by $\boldsymbol{\omega}(t)$ the generator of the infinitesimal rotation that makes the z axis align with $\hat{\mathbf{b}}$. In other words, $\mathbf{r}'=\mathbf{r}+\mathbf{r}\times\boldsymbol{\omega}(t)dt$ is the infinitesimal rotation transforming a vector $\mathbf{r}$ in the coordinate frame where the z axis aligns with $\hat{\mathbf{b}}(t)$ to the vector $\mathbf{r}'$ in a coordinate frame where the z axis aligns with $\hat{\mathbf{b}}(t+dt)=\hat{\mathbf{b}}(t)+\dot{\hat{\mathbf{b}}}dt$. The alignment of $\hat{\mathbf{b}}$ with the z axis does not uniquely define the time-dependent frame, since it does not specify the directions of the time dependent x,y axes. Out of the infinitely many possible choices for these directions we choose the one that yields the smallest $\boldsymbol{\omega}(t)$, i.e., the one in which $\boldsymbol{\omega}(t)$ is perpendicular both to $\hat{\mathbf{b}}(t)$ and to $\hat{\mathbf{b}}(t+dt)$. Consequently,

$$\boldsymbol{\omega}(t)=\dot{\hat{\mathbf{z}}}\times\hat{\mathbf{z}}, \tag{5}$$

where $\hat{\mathbf{z}}$ is the unit vector $\hat{\mathbf{b}}$ as seen in the time-dependent frame, and $\dot{\hat{\mathbf{z}}}$ is the vector $\dot{\hat{\mathbf{b}}}$ as seen in the time-dependent frame. This choice of $\boldsymbol{\omega}(t)$ makes the coordinate frame follow a parallel transport trajectory [3,11]. The equation of motion for the vector $\mathbf{l}'$ is given by [12,13]

$$\frac{d\mathbf{l}'}{dt}=\mathbf{l}'\times[|\mathbf{B}(\mathbf{x}(t))|\hat{\mathbf{z}}+\dot{\hat{\mathbf{z}}}\times\hat{\mathbf{z}}]. \tag{6}$$

The equation of motion for $\mathbf{l}'$ is similar to that of $\mathbf{l}$ [Eq.

(4)], but with the magnetic field $\mathbf{B}$ replaced by $\mathbf{B}' = |\mathbf{B}(\mathbf{x}(t))|\hat{\mathbf{z}} + \dot{\hat{\mathbf{z}}} \times \hat{\mathbf{z}}$. Hence, the effective magnetic field felt by the vector $\boldsymbol{l}'$ is composed of two components: One is the adiabatic component, which is the original magnetic field, transformed to align with the z axis. The other is the nonadiabatic component, which arises from the time dependence of the reference frame, i.e., from the time dependence of $\hat{\mathbf{b}}$. The nonadiabatic component of $\mathbf{B}'$ is perpendicular to the adiabatic one. Approximating the solution to Eq. (6) by $\boldsymbol{l}' = l\hat{\mathbf{z}}$ amounts then to neglecting a component of magnitude $\dot{\hat{\mathbf{b}}}$ relative to a component of magnitude $|\mathbf{B}|$. When attempting to improve this approximation, it is important to note that although the direction of $\mathbf{B}'$ is generally time dependent, its time dependence is, in the limit $|\mathbf{B}| \rightarrow \infty$, slower than that of $\hat{\mathbf{b}}$. While the latter is independent of the magnitude of the magnetic field $|\mathbf{B}|$, the former is at least of the order of $|\mathbf{B}|^{-1}$. Thus, a refined approximation to the solution of Eq. (6) is

$$\boldsymbol{l}' = (l/|\mathbf{B}'|)(|\mathbf{B}|\hat{\mathbf{z}} + \dot{\hat{\mathbf{z}}} \times \hat{\mathbf{z}}), \tag{7}$$

i.e., the magnetic moment is approximated to follow the direction of $\mathbf{B}'$, rather than that of $\mathbf{B}$. As can be seen by direct substitution of Eq. (7) in Eq. (6), our refined approximation is indeed closer to the exact solution. The expression for the vector $\boldsymbol{l}$ in the "laboratory," static, frame is obtained from Eq. (7) by the replacements $\hat{\mathbf{z}} \rightarrow \hat{\mathbf{b}}$ and $\dot{\hat{\mathbf{z}}} \rightarrow \dot{\hat{\mathbf{b}}}$. Note that $\hat{\mathbf{b}}$ depends on time only through the time dependence of the particle's path, $\mathbf{x}(t)$. Thus, $\dot{\hat{\mathbf{b}}} = \dot{x}_j(t)\,\partial\hat{\mathbf{b}}/\partial x_j$ (a summation over repeated indices is understood). *Therefore, the magnetic moment has a component perpendicular both to the direction of the magnetic field $\hat{\mathbf{b}}$, and to the time derivative of this direction $\dot{\hat{\mathbf{b}}}$. The magnitude of this component is proportional to the velocity of the particle, and inversely proportional to the strength of the magnetic field.*

Although this conclusion appears, at first glance, to contradict the adiabatic invariance of the action, it actually does not. The adiabatic invariance states that $d(\boldsymbol{l}\cdot\hat{\mathbf{b}})/dt = 0$. However, using Eq. (4) we see that

$$\frac{d}{dt}(\boldsymbol{l}\cdot\hat{\mathbf{b}}) = \boldsymbol{l}\cdot\dot{\hat{\mathbf{b}}}\,; \tag{8}$$

i.e., adiabatic invariance requires that the magnetic moment $\boldsymbol{l}$ should not have a component along $\dot{\hat{\mathbf{b}}}$, but does not force its component along the direction $\dot{\hat{\mathbf{b}}} \times \hat{\mathbf{b}}$ to vanish. It is indeed this component that is responsible for the Lorentz-type force.

To calculate F_i, the ith component of the force acting on the particle [Eq. (3)], we have to take the scalar product of the magnetic moment $\boldsymbol{l}$ with the vector $\partial B/\partial x_i$. This scalar product can be taken in either the static or the time-dependent frame. We choose the static frame, in which

$$\frac{\partial B}{\partial x_i} = \frac{\partial|\mathbf{B}|}{\partial x_i}\hat{\mathbf{b}} + |\mathbf{B}|\frac{\partial\hat{\mathbf{b}}}{\partial x_i}\,; \tag{9}$$

i.e., it is composed of a component in the direction of the magnetic field, originating from variations in the magnitude of the magnetic field, and a component perpendicular to the field, originating from variations in the direction of the magnetic field. While the first component is proportional to the derivative of $|\mathbf{B}|$, the second is proportional to $|\mathbf{B}|$ itself. It is the product of that second component with the components of $\boldsymbol{l}$ perpendicular to the magnetic field which yields a force independent of $|\mathbf{B}|$. Calculating the force we get

$$F_i = \frac{l|\mathbf{B}|}{|\mathbf{B}'|}\left(\frac{\partial|\mathbf{B}|}{\partial x_i} + \frac{\partial\hat{\mathbf{b}}}{\partial x_i}\cdot(\dot{\hat{\mathbf{b}}}\times\hat{\mathbf{b}})\right). \tag{10}$$

In the adiabatic limit, when $|\mathbf{B}| \rightarrow \infty$, the magnitude of $\mathbf{B}'$ approaches that of $\mathbf{B}$. Taking that limit, and expressing $\dot{\hat{\mathbf{b}}}$ in terms of the particle's velocity, we finally arrive at

$$F_i = l\frac{\partial|\mathbf{B}|}{\partial x_i} + l\dot{x}_j\left(\frac{\partial\hat{\mathbf{b}}}{\partial x_i}\times\frac{\partial\hat{\mathbf{b}}}{\partial x_j}\right)\cdot\hat{\mathbf{b}}\,. \tag{11}$$

The first term of Eq. (11) is the force utilized in Stern-Gerlach-type experiments. The second term is the Lorentz-type force. It is indeed proportional and perpendicular to the velocity, and independent of $|\mathbf{B}|$. It is instructive, at this point, to express the unit vector $\hat{\mathbf{b}}$ as $\hat{\mathbf{b}} = (\sin\alpha\cos\beta, \sin\alpha\sin\beta, \cos\beta)$ where α,β are functions of $\mathbf{x}$. In terms of α and β, Eqs. (10),(11) are

$$\begin{aligned} F_i &= \frac{l|\mathbf{B}|}{|\mathbf{B}'|}\left(\frac{\partial|\mathbf{B}|}{\partial x_i} + (\dot{\alpha}\nabla\beta - \dot{\beta}\nabla\alpha)\sin\alpha\right) \\ &= l\frac{\partial|\mathbf{B}|}{\partial x_i} + l\dot{\mathbf{x}}\times(\nabla\alpha\times\nabla\beta)\sin\alpha\,. \end{aligned} \tag{12}$$

Note that the refinement of the adiabatic approximation to include a component of $\boldsymbol{l}$ perpendicular to $\hat{\mathbf{b}}$ does not affect the Stern-Gerlach-type force [the first term in Eqs. (10)-(12)]. The component of $\boldsymbol{l}$ perpendicular to $\hat{\mathbf{b}}$ is of the order of $|\mathbf{B}|^{-1}$, so that it yields a correction of the order of $|\mathbf{B}|^{-2}$ to the projection of $\boldsymbol{l}$ on $\hat{\mathbf{b}}$, and this correction does not affect the force in the limit $|\mathbf{B}| \rightarrow \infty$.

Up to this stage we have limited our attention to the case in which the magnetic moment approaches a complete alignment with the magnetic field. We now discuss the case in which $l_\parallel \neq l$, i.e., the case in which the Lorentz-type force is accompanied by an electric-type one. The important new point that should be discussed in this case is the precession of the magnetic moment around the effective magnetic field $\mathbf{B}'$. When a magnetic moment precesses around a magnetic field, it has a time-independent component parallel to the field, and an oscillatory time-dependent (precessing) component in the two directions perpendicular to the field. Now, as Eq. (3) shows, when the vector $\boldsymbol{l}$ has a component parallel to $\partial\hat{\mathbf{b}}/\partial x_i$, this component yields a force. Since $\partial\hat{\mathbf{b}}/\partial x_i$ is perpendicular to the original magnetic field $\mathbf{B}$, and hence nearly perpendicular to the effective field $\mathbf{B}'$, the component of $\boldsymbol{l}$ along $\partial\hat{\mathbf{b}}/\partial x_i$ is expected to oscillate in time too, and thus to yield an oscillatory force on the particle. This force can be expected to be large [proportional to $|\mathbf{B}|$—see Eq. (3)], but its rapid oscillations, of frequency $|\mathbf{B}|$, can be expected to result in a zero average. A careful analysis of that force, to be described below, shows that it does not average to zero, but rather yields the $|\mathbf{B}|$-independent electric-type force.

We start our analysis by writing an expression for $\boldsymbol{l}$ as a function of time. Following our refined approximation, $\boldsymbol{l}$ is expected to precess around the effective magnetic field $\mathbf{B}'$. Then, the internal angular momentum $\boldsymbol{l}$ is

$$\boldsymbol{l}(t)=l_\parallel\hat{\mathbf{b}}'+l_\perp[\cos(|\mathbf{B}|t)\hat{\mathbf{c}}+\sin(|\mathbf{B}|t)(\hat{\mathbf{b}}'\times\hat{\mathbf{c}})]\,, \quad (13)$$

where $l_\parallel^2+l_\perp^2=l^2$, $\hat{\mathbf{b}}'$ is a unit vector in the direction of $\mathbf{B}'$, and $\hat{\mathbf{c}}$ is an arbitrarily chosen unit vector perpendicular to $\hat{\mathbf{b}}'$. In the framework of our refined approximation $l_\parallel$ and $l_\perp$ are constants of motion. The $l_\parallel$ term of Eq. (13) yields the Lorentz-type force given by Eqs. (10)-(12), with l replaced by $l_\parallel$. Since this force was discussed above, we disregard it now and focus on the contribution of the $l_\perp$ term. For that term we neglect the difference between $\hat{\mathbf{b}}$ and $\hat{\mathbf{b}}'$. This difference induces a force that vanishes in the $|\mathbf{B}|\to\infty$ limit. We choose $\hat{\mathbf{c}}$ to be in the direction of $\partial\hat{\mathbf{b}}/\partial x_i$, and substitute Eq. (13) in the expression for the force, Eq. (3). We get

$$F_i^\perp=|\mathbf{B}|l_\perp\frac{\partial\hat{\mathbf{b}}}{\partial x_i}\cos|\mathbf{B}|t\,, \quad (14)$$

where $F_i^\perp$ denotes the $l_\perp$-dependent part of the force. As expected, the precession of the magnetic moment induces a big, though rapidly oscillating, force on the particle. As a result of that force, the smooth path of the particle, dominated by the Stern-Gerlach and Lorentz-type forces, is modulated onto rapid oscillations, dictated by the force (14). Altogether, the position of the particle is given by $\mathbf{x}(t)=\mathbf{x}_0(t)+\delta\mathbf{x}(t)$, where $\mathbf{x}_0(t)$ is a slowly varying smooth function of the time determined by the forces in Eq. (10) and the rapidly oscillating part $\delta\mathbf{x}(t)$ satisfies the following equation of motion:

$$m\frac{d^2\delta\mathbf{x}}{dt^2}=|\mathbf{B}|l_\perp\frac{\partial\hat{\mathbf{b}}}{\partial x_i}\cos|\mathbf{B}|t\,. \quad (15)$$

The solution of Eq. (15) is given by

$$\delta\mathbf{x}(t)=-\frac{l_\perp}{m|\mathbf{B}|}\frac{\partial\hat{\mathbf{b}}}{\partial x_i}\cos|\mathbf{B}|t\,.$$

Two points should be stressed regarding this expression for $\delta\mathbf{x}$. First, $\delta\mathbf{x}$ is inversely proportional to the mass. Second, although the force is proportional to $|\mathbf{B}|$, its rapid oscillations make the amplitude of $\delta\mathbf{x}$ *inversely* proportional to $|\mathbf{B}|$. Thus, in the adiabatic limit this amplitude becomes very small. However, as we now show, these oscillations are the source for the nonvanishing value of the time average of $F_i^\perp$. The period of precession is $2\pi/|\mathbf{B}|$, and thus the average of the force over one period is given by

$$\langle F_i\rangle(t)=\left(\frac{2\pi}{|\mathbf{B}|}\right)^{-1}\int_t^{t+2\pi/|\mathbf{B}|}l_\perp|\mathbf{B}|\frac{\partial\hat{\mathbf{b}}(\mathbf{x}(t'))}{\partial x_i}\cos|\mathbf{B}|t'dt'\,. \quad (16)$$

Substituting $\mathbf{x}(t)=\mathbf{x}_0(t)+\delta\mathbf{x}(t)$ and taking the adiabatic limit, the amplitude of the oscillations becomes small, and we can expand

$$\frac{\partial\hat{\mathbf{b}}(\mathbf{x}(t))}{\partial x_i}=\frac{\partial\hat{\mathbf{b}}_0}{\partial x_i}-\frac{\partial^2\hat{\mathbf{b}}_0}{\partial x_i\partial x_j}\frac{l_\perp}{m|\mathbf{B}|}\frac{\partial\hat{\mathbf{b}}_0}{\partial x_j}\cos|\mathbf{B}|t \quad (17)$$

[where $\hat{\mathbf{b}}_0\equiv\hat{\mathbf{b}}(\mathbf{x}_0(t))$] and substitute in Eq. (16). The first term in the expansion averages to zero but the second term yields

$$\langle F_i\rangle=-\frac{l_\perp^2}{2m}\frac{\partial\hat{\mathbf{b}}}{\partial x_j}\frac{\partial^2\hat{\mathbf{b}}}{\partial x_i\partial x_j}\,, \quad (18)$$

which is, of course, just the gradient of the scalar potential

$$\Phi=\frac{l_\perp^2}{4m}\sum_j\left(\frac{\partial\hat{\mathbf{b}}}{\partial x_j}\right)^2,$$

the scalar potential appearing in the effective Hamiltonian (2). The scalar potential is always positive, and it repels the particle from regions in which $\hat{\mathbf{b}}$ strongly varies.

The source of the electric-type force is then in the rapidly oscillating force resulting from the precession of the magnetic moment. If the mass of the particle is infinite, it does not respond to the oscillatory force, and the time average of the latter is zero. However, if the mass is finite, this force induces a small-amplitude oscillatory motion of the particle, and the combination of that motion with the precession makes the average of the oscillatory force nonzero.

Finally, we note that while the Lorentz-type force does not do work on the particle, the electric-type force does. It then transforms kinetic energy, $P^2/2m$, into potential energy, $-|\mathbf{B}|\boldsymbol{l}\cdot\hat{\mathbf{b}}(\mathbf{x})$, and vice versa. Consequently, it also changes the value of the action $\boldsymbol{l}\cdot\hat{\mathbf{b}}$. However, the work done by the electric-type force is independent of $|\mathbf{B}|$. Thus, the change in the action is of the order of $|\mathbf{B}|^{-1}$, i.e., it vanishes in the adiabatic limit.

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