PHYSICAL REVIEW A VOLUME 47, NUMBER 6 JUNE 1993

Meaning of the wave function

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(Received 18 December 1992)

So far, the wave function has been interpreted as a probability amplitude, which is given physical meaning by ensemble averages of a large number of identical systems at a given time. We give an alternative interpretation of the wave function for a *single system* by means of a measurement which lasts a long time. This is a measurement on a single quantum system which determines the expectation values of (not necessarily commuting) observables while the wave function is protected from collapsing because it undergoes another suitably chosen interaction. This type of measurement enables the distinction between states which are not orthogonal, but are protected by a suitable interaction with the states of their environment, even for a single system. It therefore gives a different ontological meaning to the wave function. Several experiments in which such a measurement is realized, which can in principle be performed using electrons, neutrons, or atoms, are studied.

PACS number(s): 03.65.Bz

I. INTRODUCTION

Since the discovery of quantum theory a very fundamental question has haunted physicists: What is the physical meaning of the wave function?

Schrödinger, when he introduced the wave function [1], regarded it as *ontological,* i.e., it exists as a real physical wave independent of our knowledge of it. But this view was abandoned and the probabilistic interpretation due to Born [2] was adopted. This gave the wave function an *epistemological* character in the sense that it became a device for making statistical predictions for future experiments on the basis of our present knowledge of the system, as explained below in Sec. II. The outcome of each experiment, according to this view, changes our knowledge and hence the wave function of the system.

This transition in the interpretation of the wave function took place for three reasons: (1) Even though a particle has a wave function that is in general extended over space, in all experiments performed so far it manifests as a localized point particle, e.g., in the interaction of the particle with a photographic plate, cloud chamber, counter, etc. (2) When a measurement is made on a quantum system, the wave function seems to collapse into one of the eigenstates of the observable being measured which cannot be described by quantum theory. However, the probability of collapse into any of the eigenstates can be predicted by quantum theory. (3) It is not possible to distinguish experimentally between two nonorthogonal wave functions of a given system with probability 1 because this would violate the unitarity of time evolution during the experiment.

But recently the concept of protective measurement was introduced [3]. This is a measurement of the wave function of a quantum system during which the wave function is prevented from changing appreciably by means of another interaction which it undergoes at the same time. Therefore this wave function does not collapse because of the measurement. This is explained in Sec. III. We also provide here a new physical meaning to the expectation value of an observable for a single quantum system as the outcome of a protective measurement on a *single system* as opposed to the usual statistical interpretation that can be made physically meaningful only by having an ensemble of identical systems.

The protective measurement of the spatial part of the wave function is described and illustrated by two specific examples in Sec. IV. In Sec. V protective measurement of a spin state by means of two possible experiments, using homogeneous and inhomogeneous magnetic fields, is described. Each of these proposed experiments show explicitly the transition from the usual measurement to the protective measurement. Finally, in Sec. VI we use this concept to show that (1), (2), and (3) above are not valid reasons for not giving reality to the wave function. This suggests that the wave function up to a phase may be ontological.

II. USUAL MEASUREMENT AND INTERPRETATION OF THE WAVE FUNCTION

Suppose a system Σ evolves, in the absence of any measurement, under the Hamiltonian H_0 and it is desired to measure the observable A. This can be done by letting the system interact with an apparatus so that the new

Hamiltonian is

$$H = H_0 + g(t) q A + H_a \ , \tag{2.1}$$

where q is an observable of the apparatus, which we take to be a coordinate of the apparatus, H_a is the Hamiltonian of the apparatus, and $g(t)$ represents the switching on and off of the interaction, i.e., $g(t)$ is nonzero only in some time interval, say $[0,T]$. Let $g(t) = g_0 f(t)$, where g_0 is a constant and $\int_0^T f(t) dt = 1$. The state $|\Phi\rangle$ of the combined system obeys Schrödinger's equation

$$i\hbar \frac{\partial}{\partial t} |\Phi\rangle = H |\Phi\rangle \ . \tag{2.2}$$

Let $|\Psi_i\rangle$ be the normalized eigenstates of A belonging to eigenvalues a_i in the Hilbert space of Σ. Suppose that the state of the combined system at $t=0$ was

$$|\Phi(0)\rangle = |\alpha\rangle |\Psi\rangle = |\alpha\rangle \sum_i c_i |\Psi_i\rangle \ , \tag{2.3}$$

where $|\alpha\rangle$ is a normalized state of the apparatus and at least two of the c_i's are nonzero. Then the effect of this interaction for small T is given by

$$|\Phi(T)\rangle = \exp\left[-\frac{i}{\hbar} \int_0^T H \, dt \right] |\Phi(0)\rangle$$

$$\simeq \sum_i c_i \exp\left[-\frac{i}{\hbar} g_0 q a_i \right] |\alpha\rangle |\Psi_i\rangle \ , \tag{2.4}$$

which is called the impulse approximation. Let π be the momentum of the apparatus conjugate to q, i.e.,

$$[\pi, q] = -i\hbar \ . \tag{2.5}$$

Then for the ith term in the summation (2.4), π has shifted by the value $g_0 a_i$. Therefore the momentum acquires different values for the different possible values of A. In the Heisenberg picture,

$$\frac{d}{dt}\pi = \frac{i}{\hbar}[H, \pi] = -g(t) A \ . \tag{2.6}$$

Again it is seen that π changes by different amounts for distinct eigenvalues a_i of A.

Hence by measuring the momentum π the value of A can be measured in principle. This is the first stage of the measurement process in which, returning to the Schrödinger picture, there is an *entanglement* between the eigenstates of A and the states of the apparatus. This is described by (2.4), which may be rewritten as

$$|\Phi(T)\rangle = \sum_i c_i |\alpha_i\rangle |\Psi_i\rangle \ , \tag{2.4'}$$

where $|\alpha_i\rangle \equiv \exp[-(i/\hbar) g_0 a_i q] |\alpha\rangle$ are states of the apparatus which, for sufficiently large g_0, are orthogonal for distinct a_i.

So far the wave function has undergone continuous, linear, unitary, reversible evolution governed by Schrödinger's equation. In the second stage of the measurement process an observation is made to determine in which stage $|\alpha_i\rangle$ the apparatus is in. Then, according to the usual Copenhagen interpretation, the wave function collapses into one of the states in an apparently discontinuous, irreversible manner:

$$\sum_i c_i |\alpha_i\rangle |\Psi_i\rangle \to |\alpha_k\rangle |\Psi_k\rangle \ . \tag{2.7}$$

The collapse (2.7) cannot be explained by Schrödinger's equation and can at present only be predicted statistically as having probability $|c_k|^2$.

A simple example of measurement is the determination of the z component of spin of a spin-$\frac{1}{2}$ particle, such as a neutron or a suitable atom, by means of an ideal Stern-Gerlach apparatus (Fig. 1) which has a "magnetic field" that is inhomogeneous, say, in the x direction [4]. Then, $H_a = \mathbf{p}^2/2m$ and $A = \sigma_x$, and q is the coordinate of the center of mass of the particle which then plays the role of the "apparatus." As the particle moves through this field in the time interval $[0,T]$ the entanglement (2.4′) takes place with the two states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ being eigenstates of σ_x with eigenvalues $a_1 = +1$ and $a_2 = -1$. Then, according to (2.6), the changes in momenta of the particle in these two states are opposite. Hence the wave function splits into two, which is the first stage of the measuring process. In the second stage, the wave function interacts with a macroscopic screen. Then, according to Schrödinger's equation, an entanglement of the form (2.4′) must again take place. But in fact only one spot appears on the screen for a given particle, which corresponds to the collapse (2.7). This spot is in one of two possible positions corresponding to the states $|\Psi_1\rangle$ and $|\Psi_2\rangle$ and this location can only be predicted probabilistically as having probabilities $|c_1|^2$ or $|c_2|^2$, respectively.

If the wave function is ontological then this collapse cannot be explained by present-day quantum mechanics. The epistemological interpretation of the wave function does not have this problem and is appealing for the following reasons. (a) If the wave function represents at least partially our knowledge of the system, then since the observation changes our knowledge of the system it is not surprising that we should subsequently change its

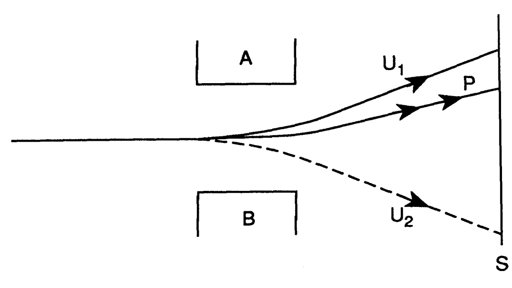

FIG. 1. Ideal Stern-Gerlach experiment for a spin-$\frac{1}{2}$ particle with and without protection. In general, as the particle with the magnetic moment enters the inhomogeneous magnetic field (between the pole pieces A and B of a magnet) its wave function splits into two whose approximate trajectories are U_1 and U_2. One spot appears on the screen S corresponding to the particle going along U_1 *or* U_2. But if a large magnetic field is present in the direction of the spin then its state is protected and the wave function does not split. So the particle has only one possible trajectory P. From two such experiments, the spin state of a single particle can be determined without collapsing it.

wave function. (b) If the wave function were a real objective representation of a single quantum system then we would expect this collapse (2.7) to occur deterministically. Moreover, a statistical prediction can be given physical meaning only by associating an ensemble of systems with the wave function, which suggests that Ψ cannot be a realistic description of a single system.

In spite of the above arguments in favor of an epistemological interpretation of Ψ, many physicists are dissatisfied with it because of the following reasons. (i) The projection postulate (2.7) lies outside of quantum mechanics as formulated by the wave equation (2.2). This is unlike other theories of physics in which the field equations and the equations of motion determine all the physics. (ii) The formulation of (2.7) requires the division between the quantum system from the final apparatus that collapses the wave function, which has not been clearly made in the orthodox interpretation, sometimes called the Copenhagen interpretation. Also, in the latter interpretation the apparatus is treated classically. But since the apparatus is made up of quantum-mechanical systems, such as protons, neutrons, electrons, and photons, it is reasonable to expect that it should also be treated quantum mechanically.

III. PROTECTIVE MEASUREMENT

A protective measurement differs from the usual measurement in the important respect that the wave function Ψ is protected from changing appreciably when the measurement is being made. Therefore the entanglement (2.4′) does not take place and instead

$$|\alpha(0)\rangle|\Psi(0)\rangle \to |\alpha(t)\rangle|\Psi(t)\rangle, \quad t>0 , \tag{3.1}$$

where $|\Psi(t)\rangle$ is approximately the same as $|\Psi(0)\rangle$ during the time interval $[0,T]$. Since the first stage of the measurement process, namely, the entanglement (2.4′), does not take place, the second stage, namely, the collapse (2.7), also does not take place. Also, on using (2.1) and (2.2),

$$\frac{d}{dt}\langle \alpha(t)\Psi(t)|\pi|\alpha(t)\Psi(t)\rangle = -g(t)\langle\Psi(t)|A|\Psi(t)\rangle , \tag{3.2}$$

which enables $\langle\Psi(t)|A|\Psi(t)\rangle$ to be determined from the change in momentum of the apparatus. Several such measurements of different observables A_i can be made without collapsing $|\Psi(t)\rangle$. Hence $\langle\Psi(t)|A_i|\Psi(t)\rangle$ can be determined for sufficiently large number of observables A_i from which $|\Psi(t)\rangle$ can be determined up to an overall phase, in principle. This enables the measurement of the wave function of a *single system* thereby giving a new ontological meaning to the wave function.

A protective measurement can be done in one of two ways.

(i) $|\Psi(t)\rangle$ is initially, at $t=0$, in a nondegenerate eigenstate of the Hamiltonian H. The interaction with the apparatus is such that H changes slowly during the measurement that takes place in the interval $[0,T]$ and $T \gg \hbar/\Delta E > 0$, where ΔE is the smallest of the energy differences between $|\Psi\rangle$ and other eigenstates of H. Also, the interaction is assumed to be sufficiently weak so that $|\Psi(t)\rangle$ is nearly equal to $|\Psi(0)\rangle$ up to a phase factor for $t\in[0,T]$. Then, by the adiabatic theorem [5], $|\Psi(t)\rangle$ remains as an eigenstate of $H(t)$ for all $t\in[0,T]$. Consequently, the wave function of the combined system changes as in (3.1), and the entanglement (2.4′) does not take place. Also, the interaction is assumed to be sufficiently weak so that $|\Psi(t)\rangle$ is nearly equal to $|\Psi(0)\rangle$ up to a phase factor for all $t\in[0,T]$. The latter condition is required in order that the state measured, as described in the previous paragraph, does not differ significantly from the original state because of the interaction.

We emphasize that we do not know what $|\Psi\rangle$ is before the measurement and need not know what H is. We need to know only that $|\Psi\rangle$ is protected in order to determine $|\Psi\rangle$. If, on the other hand, H is known and $|\Psi\rangle$ is known as an eigenstate of H belonging to a known simple eigenvalue E, it would be possible to *calculate* $|\Psi\rangle$. A new aspect of the protective measurement is that it enables $|\Psi\rangle$ to be directly *measured* and it is not necessary to know H or E.

(ii) For an arbitrary evolution according to Schrödinger's equation so that $|\Psi(t)\rangle$ is not necessarily an eigenstate of the Hamiltonian, it can be protected by measurements in the following way: Let $|\Psi_0(t)\rangle$ be the evolution of $|\Psi(0)\rangle$ determined by the unperturbed Hamiltonian H_0. Measure an observable $P(t)$, for which $|\Psi_0(t)\rangle$ is a nondegenerate eigenstate, a large number of times which are dense in $[0,T]$. Then $|\Psi(t)\rangle$ does not depart appreciably from $|\Psi_0(t)\rangle$ during the measurement and again we have (3.1) instead of (2.4′). The dense sequence of measurements prevent the wave function from collapsing while the observable A is being measured by an independent measurement described by the Hamiltonian (2.1).

To see this, consider the simple example of $P(t)$ being measured in the interval $[0,T]$ at times $t_n=(n/N)T$, $n=1,2,\ldots,N$, where N is an arbitrarily large number. Consider the branch of the evolution of the state of the combined system, consisting of the observed system and the apparatus, in which each measurement of $P(t_n)$ results in the state of the observed system being in $|\Psi_0(t_n)\rangle$:

$$|\Phi(T)\rangle_0 \equiv |\Psi_0(t_N)\rangle\langle\Psi_0(t_N)| \exp\left[-\frac{i}{\hbar}\frac{T}{N}H(t_N)\right]\cdots|\Psi_0(t_2)\rangle\langle\Psi_0(t_2)|$$
$$\times \exp\left[-\frac{i}{\hbar}\frac{T}{N}H(t_2)\right]|\Psi_0(t_1)\rangle\langle\Psi_0(t_1)|\exp\left[-\frac{i}{\hbar}\frac{T}{N}H(t_1)\right]|\Psi(0)\rangle|\alpha(0)\rangle$$

$$=|\Psi_0(t_N)\rangle\langle\Psi_0(t_N)|\exp\left[-\frac{i}{\hbar}\frac{T}{N}g(t_N)qA\right]\cdots|\Psi_0(t_3)\rangle\langle\Psi_0(t_2)|$$

$$\times\exp\left[-\frac{i}{\hbar}\frac{T}{N}g(t_2)qA\right]|\Psi_0(t_2)\rangle\langle\Psi_0(t_1)|\exp\left[-\frac{i}{\hbar}\frac{T}{N}g(t_1)qA\right]|\Psi_0(t_1)\rangle|\alpha_0(T)\rangle\ , \quad (3.3)$$

where $|\alpha_0(t)\rangle$ is the state of the apparatus when it evolves under the Hamiltonian H_a. We have assumed, for simplicity, that $[H_a,qA]$ is zero or its effect is negligible in the time interval $[0,T]$. The last expectation value in (3.3), to second order in $(1/N)$, is

$$\langle\Psi_0(t_1)|\exp\left[-\frac{i}{\hbar}\frac{T}{N}g(t_1)qA\right]|\Psi_0(t_1)\rangle=1-\frac{i}{\hbar}\frac{T}{N}g(t_1)q\langle A\rangle-\frac{1}{2\hbar^2}\frac{T^2}{N^2}g(t_1)^2q^2\langle A\rangle^2-\frac{1}{2\hbar^2}\frac{T^2}{N^2}g(t_1)^2q^2\Delta A^2$$

$$=\exp\left[-\frac{i}{\hbar}\frac{T}{N}g(t_1)q\langle A\rangle\right]\left[1-\frac{1}{2\hbar^2}\frac{T^2}{N^2}g(t_1)^2q^2\Delta A^2\right]\ ,$$

where $\langle A\rangle=\langle\Psi_0|A|\Psi_0\rangle$, and $\Delta A^2=\langle A^2\rangle-\langle A\rangle^2$ is the square of the uncertainty in A. This can be done with each exponential factor in (3.3). The product of the factors containing ΔA^2 tends to 1 as $N\to\infty$. Hence in the limit of $N\to\infty$, (3.3) reads

$$|\Psi(T)\rangle_0=|\Psi_0(T)\rangle\exp\left[-\frac{i}{\hbar}\int_0^T g(t)q\langle A\rangle dt\right]|\alpha_0(T)\rangle\ . \quad (3.4)$$

Thus in this limit the branch (3.3) undergoes a unitary evolution and therefore the contributions from the other branches corresponding to one or more of the measurements of $P(t)$ giving a state different from $|\Psi_0(t)\rangle$ tend to zero, i.e., the state of the combined system $|\Psi(T)\rangle=|\Psi(T)\rangle_0$. This is in the generalization of the "watched pot does not boil" effect [6] to the case when the "watched pot" is evolving under the Hamiltonian H_0. From the exponential operator in (3.4) [see also (2.6)] the momentum conjugate to q is shifted by

$$\Delta p=-\int_0^T\langle A\rangle g(t)dt\ . \quad (3.5)$$

Therefore by measuring Δp, $\langle A\rangle$ can be determined. By doing this experiment with different observables A, the wave function can be determined up to an overall phase for a single particle. Of course, if $P(t)$ is known it is possible to determine $|\Psi_0(t)\rangle$ by calculation. But the protective measurement enables $|\Psi_0(t)\rangle$ to be determined directly, by actual measurements, for a single particle.

Hitherto it was believed that for a single particle it is possible to experimentally distinguish between two orthogonal states but not two nonorthogonal states. This is because in the usual measurement the apparatus states $|\alpha_i\rangle$'s in (2.4′) must be orthogonal in order to be distinguishable. The unitarity of time evolution during $[0,T]$ then implies that the corresponding $|\Psi_i\rangle$'s in (2.3) must be orthogonal. The measurement collapses $|\Psi(T)\rangle$ into one of the orthogonal states, as indicated in (2.7), and the wave function of the collapsed state $|\Psi_k\rangle$ is then determined by calculation or by making measurements of other observables on a large number of systems in that state.

But since a protective measurement leads neither to entanglement nor collapse, it is possible to distinguish between two nonorthogonal states provided they are both protected. In method (i) this means that the states are eigenstates of two different Hamiltonians, so two separate experiments are performed; but each wave function can be determined (up to overall phase) for a single particle in that state. The states of the apparatus which distinguish between the two protected states are of course orthogonal after the measurement in order that they are distinguishable. But since the quantum-mechanical states of the environment which protects the states are orthogonal before the measurement, the unitarity of the time evolution of the combined system, including the apparatus and the environment, is not violated.

Also the protective measurement in addition to determining the state directly, shows the *manifestation of the entire wave function in its interaction with the probe.* The wave function is seen, in this type of measurement, as spread out over a region of space and not as the probability amplitude for finding the position of a point particle. This will become clear from the examples that follow.

As a simple example, consider the ideal Stern-Gerlach experiment discussed in Sec. II, which we now modify by protecting the spin state of the spin-$\frac{1}{2}$ particle by a large homogeneous magnetic field. Then the total magnetic field $\mathbf{B}(t)$ also has approximately a constant direction which we take to be the z direction for convenience for the time being. It is assumed that the expectation value of the coordinate q of the particle varies slowly during the interaction with $\mathbf{B}(t)$ so that the Hamiltonian (2.1) may be regarded as changing slowly. By the adiabatic theorem, if the spin was initially an eigenstate of $B\sigma_z$ then it remains as an eigenstate to a good approximation so that

$$\Psi(\mathbf{x},t)=\phi(\mathbf{x},t)\begin{bmatrix}1\\0\end{bmatrix}=\frac{\phi(\mathbf{x},t)}{2}\left\{\begin{bmatrix}1\\1\end{bmatrix}+\begin{bmatrix}1\\-1\end{bmatrix}\right\}\ . \quad (3.6)$$

Thus the adiabatic theorem ensures that the two eigenstates of σ_x on the right-hand side of (3.6) have the same space-time dependence and hence *the beam does not split* in spite of the inhomogeneity of the magnetic field in the x direction. If the beam strikes a screen, only one spot will appear halfway between the two spots which would have appeared if the state were not protected.

More generally, if the inhomogeneity of the ideal Stern-Gerlach apparatus is in the direction of an arbitrary unit vector $\mathbf{n}$, the beam clearly does not split provided the spin state is protected by a large homogeneous magnetic field in the unknown direction of the spin. Conversely, this nonsplitting for arbitrary $\mathbf{n}$ implies that the spin state is being protected. The Hamiltonian now is a special case of (2.1) corresponding to $A=\mu\sigma\cdot\mathbf{n}$ and q is the coordinate in the direction of $\mathbf{n}$. Schrödinger's equation implies that there must be a change in momentum of the entire beam according to

$$\frac{d}{dt}\langle\Psi|p|\Psi\rangle=-\mu g\langle\Psi|\sigma\cdot\mathbf{n}|\Psi\rangle \ , \tag{3.7}$$

where p is the momentum of the particle along $\mathbf{n}$. This is a special case of (3.2), with p now being the same as π. By measuring this momentum change from the change in velocity of the beam for three different directions of $\mathbf{n}$, in principle, the spin state can be determined for a single particle without collapsing this state. If the particle strikes the screen then it would do so at a spot between the two possible spots that it could strike if it were not protected. The precise location of the actual single spot it makes would depend on $\langle\Psi|\sigma\cdot\mathbf{n}|\Psi\rangle$. We emphasize again that we need not know along which direction the spin state was protected, but can determine this direction (which we earlier took to be in the z direction for simplicity) as described above.

Consider now the spin interacting with an actual inhomogeneous magnetic field $\mathbf{B}_1(\mathbf{x})$, satisfying Maxwell's equations [4], while its spin is in the direction of $\mathbf{m}$ and protected by a large homogeneous magnetic field $\mathbf{B}_0$ in the same direction. The Hamiltonian is $H=-\mu\mathbf{B}_1\cdot\sigma$. Then (3.7) is generalized to

$$\frac{d}{dt}\langle\Psi|\mathbf{p}|\Psi\rangle=\mu\langle\Psi|\nabla(\mathbf{B}_1\cdot\sigma)|\Psi\rangle=\mu\langle\Psi|\nabla(\mathbf{B}_1\cdot\mathbf{m})|\Psi\rangle \ . \tag{3.8}$$

In an actual Stern-Gerlach experiment there is a large magnetic field which is part of the apparatus so that the splitting takes place for the two spin states with quantization axis along its direction. Our treatment is different because $\mathbf{B}_0$ is *not* part of our apparatus and its direction, i.e., $\mathbf{m}$, is unknown to us. But by changing the orientation of our apparatus and therefore $\mathbf{B}_1(\mathbf{x})$ and measuring the acceleration determined by (3.8) the fixed direction of $\mathbf{m}$ and hence the spin state can be determined. This will be treated in detail in Sec. V B.

Returning now to the general discussion, simultaneous protective measurement of several observables A_α can be done by letting the observed system interact with these observables such that the Hamiltonian for the combined system is

$$H=H_0+\sum_\alpha\{g_\alpha(t)q_\alpha A_\alpha+H_{a\alpha}\} \ , \tag{3.9}$$

which generalizes (2.1). Each interaction is sufficiently weak and adiabatic as in (i) above, or dense measurements of the observable $P(t)$ are made as in (ii) above, in order that the measurement is protective. By measuring the changes in the momenta π_α conjugate to q_α, the values $\langle\Psi|A_\alpha|\Psi\rangle$ may be determined as described above. By determining sufficient number of these expectation values, $|\Psi\rangle$ can be determined up to an overall phase.

IV. PROTECTIVE MEASUREMENT OF THE SPATIAL WAVE FUNCTION

Suppose that the spatial part of the wave function $\Psi(\mathbf{x})$ is protected, for example, if the particle is in the ground state inside a box and the measurements made on it do not excite it. We do not know what Ψ is; we only know that it is protected. Then $\Psi(\mathbf{x})$ can be determined up to gauge transformations by protective measurements as follows: Make a protective measurement of the observable [7] $A=P_{\mathbf{x}}\equiv|\mathbf{x}\rangle\langle\mathbf{x}|$, which is the projection operator at position $\mathbf{x}$. This gives the value, via (3.2),

$$\rho(\mathbf{x})\equiv\langle\Psi|P_{\mathbf{x}}|\Psi\rangle=\Psi^*(\mathbf{x})\Psi(\mathbf{x}) \ . \tag{4.1}$$

Next, protectively measure $A=\frac{1}{2}(P_{\mathbf{x}}\boldsymbol{\pi}+\boldsymbol{\pi}P_{\mathbf{x}})$, which is the current density at $\mathbf{x}$, where $\boldsymbol{\pi}$ is the kinetic momentum operator. This gives the value

$$\begin{aligned}\mathbf{j}(\mathbf{x})&\equiv\langle\Psi|\tfrac{1}{2}(P_{\mathbf{x}}\boldsymbol{\pi}+\boldsymbol{\pi}P_{\mathbf{x}})|\Psi\rangle\\&=\frac{i\hbar}{2}\left[\left\{\left[\nabla+i\frac{e}{\hbar}A\right]\Psi^*(\mathbf{x})\right\}\Psi(\mathbf{x})-\Psi^*(\mathbf{x})\left[\nabla-i\frac{e}{\hbar}A\right]\Psi(\mathbf{x})\right]\\&=\rho(\mathbf{x})\{\hbar\nabla\theta(\mathbf{x})-eA(\mathbf{x})\} \ ,\end{aligned} \tag{4.2}$$

where $\theta(\mathbf{x})$ is the phase of $\Psi(\mathbf{x})$. As explained in Sec. III, the protective measurement does not change $\Psi(\mathbf{x})$, even when several observables are simultaneously measured protectively. Therefore $\rho(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$ can be determined for different points $\mathbf{x}$ for the same $\Psi(\mathbf{x})$. It can be shown that from $\rho(\mathbf{x})$ and $\mathbf{j}(\mathbf{x})$, for all points $\mathbf{x}$, $\Psi(\mathbf{x})$ can be determined up to gauge transformations. In this way $\Psi(\mathbf{x})$ can be determined up to gauge transformations, in principle, without collapsing the wave function for a single particle.

To bring out the essential idea behind this measure-

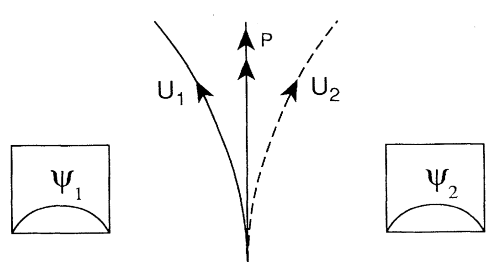

FIG. 2. A *single* particle with charge Q is in an equal superposition of normalized states Ψ_1 and Ψ_2 localized inside the boxes 1 and 2. If the state is unprotected then the trajectory of an electron moving midway between the two boxes would be either U_1 or U_2 corresponding to Q being in box 1 or 2. But for a protected state, the trajectory would be P as if half of the Q is in each box.

ment, we first consider a simple example which may be treated like a two-state system and then generalize to an infinite-dimensional Hilbert space. Consider the wave function of a single charged particle,

$$\Psi(\mathbf{x},t)=2^{-1/2}\{\Psi_1(\mathbf{x},t)+\Psi_2(\mathbf{x},t)\} \ , \tag{4.3}$$

where $\Psi_1(\mathbf{x},t)$ and $\Psi_2(\mathbf{x},t)$ are normalized wave functions respectively localized in their ground states in two small identical boxes 1 and 2 (Fig. 2). The particle has charge Q and will be denoted by Q. An electron which is prepared in the state of a small localized wave packet, whose spreading is assumed negligible, is shot so that if Q were not present it would go along a straight line through the midpoint perpendicular to the line of separation between the boxes. In the presence of Q, however, its trajectory would be influenced by the electric field of Q.

First suppose that Ψ is unprotected. Then the entanglement (2.4′) takes place and the combined system has the wave function

$$\Psi(\mathbf{x},\mathbf{x}',t)=2^{-1/2}\{\alpha_1(\mathbf{x}',t)\Psi_1(\mathbf{x},t)+\alpha_2(\mathbf{x}',t)\Psi_2(\mathbf{x},t)\} \ , \tag{4.4}$$

where $\alpha_1(\mathbf{x}',t)$ and $\alpha_2(\mathbf{x}',t)$ are the wave functions of the electron-taking trajectories influenced by the electric fields of Q in box 1 or 2, respectively. This implies that the electron will deviate either towards box 1 or box 2. Suppose that the electron subsequently hits a screen that is perpendicular to its original velocity. Then from the position where it hits the screen it can be determined if Q was in box 1 or 2. This means that the above wave function collapses to $\alpha_1(\mathbf{x}',t)\ \Psi_1(\mathbf{x},t)$ or $\alpha_2(\mathbf{x}',t)\Psi_2(\mathbf{x},t)$. This is a usual measurement of an observable A with two eigenstates corresponding to Q being in box 1 or 2. Suppose that the above measurement shows that Q is in box 1, which corresponds to the collapse

$$\Psi(\mathbf{x},\mathbf{x}',t)\rightarrow\alpha_1(\mathbf{x}',t)\Psi_1(\mathbf{x},t) \ . \tag{4.5}$$

Then every electron that is subsequently shot between the boxes in the same way will follow the same trajectory as the first electron corresponding to Q being in box 1.

But the above experiment would have a very different outcome if $\Psi(\mathbf{x},t)$ is protected. Now $\Psi(\mathbf{x},t)$ is orthogonal to

$$\Psi'(\mathbf{x},t)=2^{-1/2}\{\Psi_1(\mathbf{x},t)-\Psi_2(\mathbf{x},t)\} \ . \tag{4.6}$$

Also, if the Hamiltonian H_0 contains only local interactions and $\Psi(\mathbf{x},t)$ is an eigenfunction of H_0 then $\Psi'(\mathbf{x},t)$ must be also an eigenfunction of H_0 which is degenerate with $\Psi(\mathbf{x},t)$. Therefore it would then not be possible to protect $\Psi(\mathbf{x},t)$ using method (i) described in Sec. III. But it can be protected using only local interactions by allowing tunneling between the two boxes, say by joining them with a long tube whose diameter is small compared to the wavelength, which removes the degeneracy.

Suppose that Ψ is a nondegenerate energy eigenstate because of tunneling. Then in the above experiment assume that (a) the electron travels a distance between the boxes of the order of the separation distance between them in a time large compared to $\hbar/\Delta E$, where ΔE is the smallest of the energy differences between Ψ and the other energy eigenstates, and (b) at all times the potential energy of interaction between the electron and Q is small compared to ΔE. Then the adiabaticity condition and the weakly interacting condition required for method (i) in Sec. III are satisfied. Therefore, the measurement of A by means of the electron trajectory is a protective measurement. Then the trajectory of the electron would be a straight line as if each box contained half of the charged particle so that the electric field along the trajectory is zero (Fig. 2). The electric field experienced by the electron is like as if it is determined by the semiclassical Maxwell's equations

$$\partial_\nu F^{\nu\mu}(\mathbf{x},t)=e\langle\Psi|\hat{\mathbf{j}}^{\mu}(\mathbf{x},t)|\Psi\rangle \ , \tag{4.7}$$

where $\hat{\mathbf{j}}^{\mu}(\mathbf{x},t)$ is the four-current density operator, and $F^{\mu\nu}$ is the electromagnetic field.

But for the unprotected measurement considered earlier of course (4.7) is not valid and instead it is necessary to use the exact equation

$$\partial_\nu \hat{\mathbf{F}}^{\nu\mu}(\mathbf{x},t)=e\hat{\mathbf{j}}^{\mu}(\mathbf{x},t) \ , \tag{4.8}$$

where $\hat{\mathbf{F}}^{\mu\nu}$ is the quantized electromagnetic-field operator, in order to explain the outcome. Because (4.7) predicts that the electric field is the same as if half of the particle Q is in box 1 and the other half is in box 2, which implies that the electric field midway between the boxes is zero whereas as seen earlier the electron should deviate towards one of the boxes for an unprotected measurement. Indeed, if we open box 2 and find that Q is not there then it must be in box 1 leading to the collapse (4.5). Since looking into box 2 cannot suddenly change the electric field experienced by a far away particle, Eq. (4.7) cannot be valid. However, if we take the expectation value of (4.8) with respect to $|\Psi\rangle$, it is seen that the observed electric field is consistent with the state of Q producing it.

This example already illustrates how a protective measurement dramatically changes the manifestation of the wave function mentioned in Sec. III. For the usual unprotected measurements the wave function of Q interacts with the electron as if it is in box 1 or 2, making the electron move along the trajectory U_1 or U_2, as if Q is a localized point particle (Fig. 2). But for the protected measurement, the interaction is as if half of it is in box 1 and the other half is in box 2, giving rise to the very different trajectory P for the electron. It may be argued that because of the tunneling allowed between the two boxes, Q moves rapidly between the boxes and the time average of the corresponding electric field is being measured here. But we shall see below [Sec. VI, reasons (a) and (b)] that this interpretation is not valid.

More generally, suppose that Q is in the nondegenerate energy eigenstate

$$\Psi(\mathbf{x},t)=a\Psi_1(\mathbf{x},t)+b\Psi_2(\mathbf{x},t) \ , \tag{4.9}$$

where $|a|^2+|b|^2=1$. Then the electric field along the trajectory of the electron, in the above experiment performed protectively, would depend on the ratio $|b|/|a|$, and the trajectory would be curved accordingly. There-

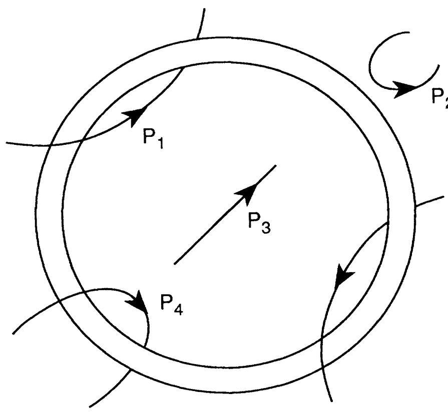

FIG. 3. A particle with charge Q is inside a circular tube in a nondegenerate eigenstate of energy. The tube is placed in a conducting cavity to prevent a transition of its state by the emission of a photon. Protective measurements by means of observation of suitable electron trajectories $P_1, P_2, P_3, \ldots$ near the tube can determine the wave function of Q up to gauge transformations, in principle, without collapsing the wave function.

fore, from the position where the electron hits the screen, $|a|$ and $|b|$ can be determined. To determine the phase difference between a and b it is necessary to do a measurement of an operator that connects Ψ_1 and Ψ_2 which therefore represents a nonlocal interaction. We therefore consider now a different gedanken experiment in which the entire wave function may be protected and measured, up to an overall phase factor, by means of only local measurements.

Consider the particle Q in a thin circular tube (Fig. 3) enclosing a magnetic flux Φ but with the magnetic field vanishing inside the tube. The Hamiltonian is

$$H_0 = \frac{1}{2m}(\mathbf{p} - Q\,\mathbf{A})^2 + V(\mathbf{x}) \ , \tag{4.10}$$

where $\mathbf{A}$ is the vector potential and $V(\mathbf{x})$ represents the effect of the walls of the tube. The eigenvalues of (4.10) are to a good approximation

$$E_n = \frac{(nh - Q\Phi)^2}{2mL^2} \ , \tag{4.11}$$

where n is an integer and L is the length of the tube. Assume now

$$Q\Phi \neq r\frac{h}{2} \ , \tag{4.12}$$

where r is any integer. Then the energies (4.11) are all distinct. Therefore a protective measurement of each eigenstate Ψ of H_0 can be made by shooting electrons near the tube so that conditions (a) and (b) above are satisfied and observing their trajectories. The electric and magnetic fields experienced by these electrons are given by (4.7) as if the charge and current densities are distributed over the entire tube corresponding to the wave function Ψ which has angular momentum nh. Therefore from the accelerations of the electrons the charge density $Q\rho$ and the current density $Q\mathbf{j}$ can be determined.

For this experiment the following conditions need to be satisfied. (a) Adiabaticity: The time of measurement should satisfy

$$T \gg \frac{h}{\Delta E} \ ,$$

where $\Delta E \sim nh^2/mL^2$ is the energy difference between the nth level and adjacent levels. (b) Weakness of the interaction: If E_I is the typical interaction energy between the electron and Q then $E_I/\Delta E \ll 1$. (c) Confinement: The energy of Q should be low enough so that it is confined to the tube. This would also imply that the velocity of Q is much less than the velocity of light which would justify the above nonrelativistic treatment. (d) Thermal fluctuations: The temperature Θ should be small enough so that there are no transitions due to exchange of thermal energy. Therefore $\Delta E \gg k\Theta$, where k is Boltzmann's constant.

In practice, Q interacts with the quantized electromagnetic field and therefore can undergo a transition from the given energy level to another by emitting a photon. The wave length λ of the emitted photon satisfies

$$\Delta E = \frac{hc}{\lambda} \ , \tag{4.13}$$

where ΔE is the difference between the energy levels. The emission of such a photon may be prevented by putting the tube inside a conducting cavity whose linear dimensions are much smaller than λ. Then the above-described measurement would be a protective measurement.

V. PROTECTIVE MEASUREMENT OF THE SPIN STATE

A. Protective measurement of the spin state by a uniform magnetic field

Consider a spin-$\frac{1}{2}$ particle with a magnetic moment, such as a neutron or a neutral atom, in a uniform magnetic field $\mathbf{B}_0$ in the z direction. Suppose it moves with constant momentum in the x direction for $x<0$, and passes through an additional small magnetic field $\boldsymbol{\beta}$ that is nonvanishing and uniform only in the region $0 \le x \le L$. As the neutron enters this region, in general the state splits into two because of the different momenta acquired by the two spin states for quantization axis along the direction of the total magnetic field, and correspondingly different phases are acquired by these two states. A similar splitting takes place when it emerges from this region. But when B_0 is large compared to β, we shall show that the state does not split into two, which corresponds to a protective measurement.

The Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} - \mu B_0 \sigma_z - \mu \boldsymbol{\beta}\cdot\boldsymbol{\sigma} \ . \tag{5.1}$$

The incoming beam is assumed to be the plane wave

$$\Psi(x,t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \exp(ikx - i\omega t) \ , \quad x<0 \ , \tag{5.2}$$

where

$$\hbar\omega=\frac{\hbar^2k^2}{2m}-\mu B_0 \; . \tag{5.3}$$

We suppose that the kinetic energy is much larger than μB_0 so that the reflected waves due to the step potentials at $x=0$ and $x=L$ are negligible. Let θ and ϕ be the polar angles of the total magnetic field $\mathbf{B}=\mathbf{B}_0+\boldsymbol{\beta}$ in the region $0\le x\le L$. Using Schrödinger's equation and matching boundary conditions, the transmitted waves are obtained to be

$$\Psi(x,t)=\Psi(x)\exp(-i\omega t) \; ,$$

$$\Psi(x)=\begin{cases}\cos\left[\frac{\theta}{2}\right]\exp(ik_+x)\Psi_+(x)+\sin\left[\frac{\theta}{2}\right]\exp(ik_-x)\Psi_-(x), & 0\le x\le L\\ \left[\exp\{i(k_+-k)L\}\cos^2\frac{\theta}{2}+\exp\{i(k_--k)L\}\sin^2\left[\frac{\theta}{2}\right]\right]\exp(ikx)\begin{bmatrix}1\\0\end{bmatrix}\end{cases} \tag{5.4}$$

$$+[\exp\{i(k_+-k')L\}-\exp\{i(k_--k')L\}]\exp(i\phi)\frac{\sin\theta}{2}\exp(ik'x)\begin{bmatrix}0\\1\end{bmatrix}, \quad L<x \; , \tag{5.5}$$

where

$$\Psi_+=\begin{bmatrix}\cos\frac{\theta}{2}\\ \exp(i\phi)\sin\frac{\theta}{2}\end{bmatrix} \qquad \Psi_-=\begin{bmatrix}\sin\frac{\theta}{2}\\ -\exp(i\phi)\cos\frac{\theta}{2}\end{bmatrix} \tag{5.6}$$

are orthonormal spin states with quantization axis along $\mathbf{B}$, and the new wave vectors satisfy

$$\hbar\omega=\frac{\hbar^2k_+^2}{2m}-\mu B=\frac{\hbar^2k_-^2}{2m}+\mu B=\frac{\hbar^2k'^2}{2m}+\mu B_0 \; . \tag{5.7}$$

Equations (5.4) and (5.5) show that the particle state splits into two states with different momenta in each of these regions, in general. The two spin states are thus entangled with the momentum states of the particle. If the incoming particle is a wave packet formed as a suitable superposition of the Ψ's corresponding to different values of k, then it splits into two wave packets in each of these regions.

But in the limiting case when B_0 is very large compared to β, θ is very small. Therefore in both regions the probability of finding the particle in the spin-down state is negligible, and hence

$$\Psi(x)=\begin{cases}\exp(ik_+x)\Psi_+(x), & 0\le x\le L \quad (5.8)\\ \exp\{i(k_+-k)L\}\exp(ikx)\begin{bmatrix}1\\0\end{bmatrix}, & L<x \; . \quad (5.9)\end{cases}$$

From (5.3) and (5.7),

$$k_+-k=\frac{m\mu\beta}{\hbar k}\cos\eta \; , \tag{5.10}$$

where η is the angle between $\boldsymbol{\beta}$ and the direction of the spin (z axis). Therefore there is a corresponding gain in momentum of the apparatus. As the wave packet passes through the region of $\boldsymbol{\beta}$, by measuring this gain in momentum, say, by the time of arrival, $\cos\eta$ can be determined. By repeating this experiment with two different orientations of $\boldsymbol{\beta}$ for the same particle the spin state, which is protected by $\mathbf{B}_0$, can be determined up to an overall phase for a single particle [8].

To see the transition from the unprotected to the protected experiment, the magnetic field $\mathbf{B}_0$ may be gradually increased. Then, according to (5.4), one of the two wave packets that the original wave packet splits into decreases in intensity while the other increases in intensity. Also, the difference between their velocities and therefore the distance of separation between the wave packets decreases. In the limit of the ratio of $B_0/\beta\rightarrow\infty$, so that $\theta\rightarrow 0$, the first wave packet disappears and only one wave packet emerges into the region $x>L$, which is also implied by (5.5).

B. Protective measurement of the spin state by an inhomogeneous magnetic field

Consider again a spin-$\frac{1}{2}$ particle with a magnetic moment, which while being protected by a large homogeneous magnetic field $\mathbf{B}_0$ in the z direction, goes through an apparatus containing an inhomogeneous magnetic field $g(t)\mathbf{B}_1(\mathbf{x})$. Suppose for simplicity that the particle has spin $\frac{1}{2}$. The Hamiltonian for the particle in its rest frame is

$$H=-\mu B_0\sigma_z-\mu g(t)\boldsymbol{\sigma}\cdot\mathbf{B}_1 \; . \tag{5.11}$$

Suppose that the polar coordinates of the total magnetic field

$$\mathbf{B}(\mathbf{x},t)=\mathbf{B}_0+g(t)\mathbf{B}_1(\mathbf{x}) \tag{5.12}$$

are $\theta(\mathbf{x},t)$ and $\phi(\mathbf{x},t)$ where θ is the angle between $\mathbf{B}$ and the z axis. For simplicity, choose $g(t)=g$ (constant) for $t\in[0,T]$ and $g(t)=0$ otherwise. If the initial state of the particle is given by

$$\Psi(\mathbf{x},t)=\begin{bmatrix}1\\0\end{bmatrix}\exp\left[i\frac{\mu B_0}{\hbar}t\right],\quad t<0\;, \tag{5.13}$$

then solving Schrödinger's equation with the Hamiltonian (5.11) gives

$$\Psi(\mathbf{x},t)=\cos\left[\frac{\theta}{2}\right]\exp\left[i\frac{\mu B(\mathbf{x})}{\hbar}t\right]\Psi_+(\mathbf{x},t)+\sin\left[\frac{\theta}{2}\right]\exp\left[-i\frac{\mu B(\mathbf{x})}{\hbar}t\right]\Psi_-(\mathbf{x},t)\;,\quad \text{for all } t\;, \tag{5.14}$$

where $\Psi_+(\mathbf{x},t)$ and $\Psi_-(\mathbf{x},t)$ are given by (5.6) but with the new θ and ϕ which are now functions of $\mathbf{x}$ and t.

We now consider two extreme limits: (i) $\mathbf{B}_0=\mathbf{0}$, i.e., the spin is not being protected. Then (5.14) yields

$$\Psi(\mathbf{x},t)=\cos\left[\frac{\theta}{2}\right]\exp\left[i\frac{\mu g T B_1(\mathbf{x})}{\hbar}\right]\Psi_+(\mathbf{x},t)+\sin\left[\frac{\theta}{2}\right]\exp\left[-i\frac{\mu g T B_1(\mathbf{x})}{\hbar}\right]\Psi_-(\mathbf{x},t)\;,\quad t>T\;. \tag{5.15}$$

Therefore as the particle comes out of the apparatus it is a superposition of the two wave packets (5.15) with different average momenta. Hence the beam splits into two.

(ii) $\mathbf{B}_0$ is very large compared to the Stern-Gerlach magnetic field. Then θ is very small and therefore (5.14) gives

$$\Psi(\mathbf{x},t)=\exp\left\{\frac{i\mu}{\hbar}(B_0 t+TgB_1(\mathbf{x})\cos\gamma)\right\}\begin{bmatrix}1\\0\end{bmatrix}\;,\quad t>T\;, \tag{5.16}$$

where γ is the angle between $\mathbf{B}_1$ and $\mathbf{B}_0$. Therefore the wave function does not split despite its interaction with the Stern-Gerlach apparatus. Hence the entanglement (2.4′) does not take place. However, according to (5.16), the particle state has acquired a momentum which is approximately $\mu Tg\langle\nabla\{B_1(\mathbf{x})\cos\gamma\}\rangle$, consistent with (3.8). By measuring this momentum, $\langle\nabla\{B_1(\mathbf{x})\cos\gamma\}\rangle$ can be determined without collapsing the wave function of the particle. By repeating an experiment several times with different inhomogeneous fields $\mathbf{B}_1(\mathbf{x})$, the direction of $\mathbf{B}_0$ and hence the direction of the spin may be determined. In this way the spin state may be determined for a single particle up to an overall phase.

In the special case of the ideal Stern-Gerlach experiment considered in Sec. III, $\mathbf{B}_1=-q\mathbf{n}$, where q is the coordinate in the direction of the unit vector $\mathbf{n}$. Then in the unprotected measurement in case (i), the beam splits into two which are in eigenstates of $\sigma\cdot\mathbf{n}$. In the protected experiment in case (ii), the beam does not split into two. But it deviates because of the momentum acquired which is $-\mu Tg\cos(\gamma)\mathbf{n}=-\mu Tg\langle\sigma\cdot\mathbf{n}\rangle\mathbf{n}$, consistent with (3.7). By repeating this experiment and measuring $\langle\sigma\cdot\mathbf{n}\rangle$ for different values of $\mathbf{n}$ the spin state may be determined for a single particle without collapsing the state. In practice, the position of the atom and therefore its trajectory may be determined by shining a laser beam of weak intensity on the atom.

To see the transition from cases (i) to (ii), we may gradually increase $\mathbf{B}_0$ from $\mathbf{0}$. Then, according to (5.14), the distance d between the two wave packets at a short distance from the Stern-Gerlach apparatus decreases to $d\cos\gamma$ as $B_0/gB_1\to\infty$, so that $\theta\to 0$. At the same time one of the wave packets decreases in intensity while the other increases in intensity so that in this limit there is only one wave packet corresponding to a protective measurement.

VI. DISCUSSION AND CONCLUSION

We have studied several possible experiments, which can be performed in principle, in which the wave function of a single quantum system can be determined up to gauge transformations. A particle manifests itself through its entire wave function during a protective measurement instead of manifesting like a point particle as in the usual measurement. The wave function neither becomes entangled nor collapses during a protective measurement. Also, we saw in Sec. III that two nonorthogonal states can be distinguished by two protective measurements. Therefore the reasons (1), (2), and (3) in Sec. I in favor of the epistemological interpretation do not appear to be valid.

But it may be argued that it was necessary for the wave function to be protected from collapse during the measurement. But this is not unlike in classical physics where also a system needs to be held in place in a given state when a measurement is being made, which nevertheless enables us to regard the description of the world by classical physics as ontological.

However, it takes a large amount of time to determine the wave function by method (i) in Sec. III. Although in method (ii) the measurement time T can be small, the Hamiltonian of interaction representing the large number of measurements being made is correspondingly large so that the time average would be meaningful even for small T. Therefore a protective measurement gives a new meaning to the wave function as a *time average* of a single system as opposed to the usual measurement at a given time which gives meaning to the wave function as an *ensemble average.*

But we emphasize that a protective measurement determines the manifestation of the wave function as an extended object unlike the usual measurement which gives $\Psi^*\Psi(\mathbf{x})$ the meaning of the probability density of finding the particle, regarded as real, at $\mathbf{x}$. It may be argued that, during a protective measurement, the particle moves to and fro and $\int_V\Psi^*\Psi(\mathbf{x})$ is the fraction of the time it spends in the volume V, which may be made as small as one wants. This interpretation would make the particle ontological and $\Psi^*\Psi(\mathbf{x})$ epistemological because it is a statistical average. However, we believe this argument to be invalid for the following reasons.

(a) If a particle is inside a one-dimensional box, it is well known that the energy eigenstates are standing waves with equally spaced nodes. For a given eigenstate, since $\Psi^*\Psi(\mathbf{x})$ vanishes at the nodes the above interpretation requires that the particle travels with infinite speed at each node in order that the time it spends in an infinitesimal neighborhood of each node is negligible. But there is no reason why the particle should speed up at each node because the potential is constant inside the box and there are no forces acting on the particle. Moreover, if the particle is charged then the sudden acceleration near each node would be expected to result in large radiation.

(b) The above interpretation requires hidden variables that describe the motion of the particle. This is not permitted by the Copenhagen interpretation, which cannot therefore be supported by the above argument. The only hidden variable theory which is not ruled out by Bell's theorem, as far as we know, is the one due to Bohm [9]. According to this interpretation, the velocity of the particle is proportional to the current density or the gradient of the phase of $\Psi(\mathbf{x})$. But if $\Psi(\mathbf{x})$ is a nondegenerate eigenstate of a real Hamiltonian, as in the above example of a particle in a box, then $\Psi(\mathbf{x})$ is proportional to a real function of $\mathbf{x}$ and therefore the "velocity" of the particle vanishes. Hence, the particle cannot move to and fro as required to the above interpretation.

In principle a protective measurement may be performed in atomic or nuclear scattering experiments, provided a large number of probes are scattered elastically from a *single* scattering center and the conditions for protective measurement stated in Sec. III are satisfied. In particular, the scattering interaction should be weak enough to preclude the possibility of the atom going into a state significantly different from the original ground state and falling back into the original ground state, which would be elastic scattering, but the probe would not be interacting with the original ground-state wave function, which we wish to measure. Such an experiment has not been performed so far, but may be feasible by using a trapped atom as the scattering center.

It is important to bear in mind, however, that in the examples considered above the protective measurement determines directly the current densities of conserved quantities such as charge, energy momentum, and angular momentum from their couplings to the electromagnetic and gravitational fields [10]. But this is done deterministically for a single system which gives reality to these conserved current densities. The wave function is then obtained from them up to gauge transformations, which are unitary transformations on the Hilbert space.

We have given an alternative, realistic interpretation of the wave function for a single system by means of the protective measurement. Since the universe consists of quantum-mechanical subsystems for which this interpretation can be applied, it may be reasonable to extrapolate this interpretation to the entire Universe. This would enable quantum theory to be applied to the entire Universe, which would legitimize quantum cosmology. In contrast, the Copenhagen interpretation cannot be applied to the wave function of the entire Universe because, first, there is no observer outside the Universe, which by definition is everything that exists, and, second, the probabilistic interpretation of the wave function can be given physical meaning only by having an ensemble of large number of identical systems whereas there is only one Universe.

But if the wave function is associated with a single system, then should it be regarded as "rigid", i.e., it never collapses, as in the Everett interpretation [11]? Or is it "fragile," i.e., it collapses when unprotected during measurement by a macroscopic apparatus due to some nonlinear modification of quantum mechanics [12]? Or should it be something intermediate, as in the casual interpretation [9], in which a dual ontology is given to the quantum system and the wave function that depends on the experimental setup? Or does the wave function during a measurement by a macroscopic apparatus evolve nonunitarily into a mixture which also must be regarded as ontological [13]? Or does something else happen during measurement? This question will be addressed in a future publication.

ACKNOWLEDGMENTS

We thank Abner Shimony for useful comments on presentation in an earlier version of this paper. This work was partially supported by NSF Grant No. PHY-8807812, and by Grant No. 424/91-1 of the Basic Research Foundation administered by the Israel Academy of Science and Humanities.

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[8] Another possible experiment is to interfere the state (5.9) with another coherent state which does not pass through $\boldsymbol{\beta}$, as in a neutron interferometric experiment. Then there would be a phase shift which is to first order, $\Delta\phi = (k_+ - k)L = m\mu\beta_3 L/\hbar k$ where β_3 is the component of β in the direction of $\mathbf{B}_0$. Again by measuring this phase shift for different orientations of $\boldsymbol{\beta}$, the spin state can be determined. But since in neutron interferometry, a large number of neutrons are needed to determine the phase shift, this cannot be regarded as a measurement of the state for a single particle.

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