Remote operations and interactions for systems of arbitrary-dimensional Hilbert space: State-operator approach

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We present a systematic simple method for constructing deterministic remote operations on single and multiple systems of arbitrary discrete dimensionality. These operations include remote rotations, remote interactions, and measurements. The resources needed for an operation on a two-level system are one ebit and a bidirectional communication of two cbits, and for an *n*-level system, a pair of entangled *n*-level particles and two classical "nits." In the latter case, there are $n-1$ possible distinct operations per *n*-level entangled pair. Similar results apply for generating interaction between a pair of remote systems, while for remote measurements only one-directional classical communication is needed. We further consider remote operations on *N* spatially distributed systems, and show that the number of possible distinct operations increases here exponentially, with the available number of entangled pairs that are initially distributed between the systems. Our results follow from the properties of a hybrid state-operator object (stator), which describes quantum correlations between states and operations.

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I. INTRODUCTION

Over the recent years entanglement has been examined as a resource that allows new types of communication tasks, such as teleportation, dense coding, and other local manipulations of entanglement [1]. These studies exploit the relation between quantum nonlocality and the structure of the Hilbert space. A more recent avenue of research examines the relation between entanglement and the dynamical evolution of several systems. Here, two basic questions have been examined: First, what is the entanglement creation capability of a given Hamiltonian that acts on a pair of systems [2]. The second question deals with the reverse problem: what types of nonlocal operations on two or more remote systems can be generated, using a given resource of entangled states, by applying local operations and performing classical communication (LOCC).

In this paper we will be interested in the second question. Previous work has demonstrated that certain operations like a remote-controlled NOT (CNOT), may consume less entanglement than what is needed when applying teleportation techniques [3,4]. For probabilistic nonlocal operations, an isomorphism between the physical operations and the required entanglement has been discovered, which for certain operations necessitates less than one ebit per operation [5]. A closely related question, raised by Huelga *et al.* [6], concerns the possibility of implementing a unitary transformation on a remote system.

The purpose of this paper is to present a systematic approach for constructing a class of deterministic remote unitary transformation, and remote interactions between several distributed systems. We assume that the parties share entangled states and are allowed to perform only local operations and bidirectional classical communication. For remote measurements, one-directional classical communication is sufficient.

A special characteristic of our method is that the generators that give rise to the transformation are controlled locally by the two parties. The structure of the complete operation is in a sense "split" and determined by the local observers that posses the distributed parts of the system. Therefore, in the special case that the generators are known only locally, one cannot perform the operation using ordinary teleportation techniques.

To clarify this, consider the remote unitary operation

$$U_B = \exp[i\alpha\sigma_{n_B}] \tag{1}$$

that Alice and Bob wish to apply on a state $|\Psi_B\rangle$ of Bob. The axis n_B, which defines $\sigma_{n_B} = \overline{n_B}\cdot\overline{\sigma_B}$, is determined by Bob, while the angle of rotation α is determined by Alice.

Similarly, if Alice and Bob wish to apply a remote interaction

$$U_{AB} = \exp[i\alpha\sigma_{n_A}\sigma_{n_B}] \tag{2}$$

on a pair of spins in some arbitrary state $|\Psi_{AB}\rangle$, with one spin at the hands of Alice and the other with Bob, then, the axes n_A and n_B, which fix the local generators σ_{n_A} and σ_{n_B}, are controlled locally by Alice and Bob, respectively.

Our approach relies on the properties of a new hybrid object, which we introduce in Sec. II. This object describes quantum correlations between states of one party, say Alice, and operations acting on an arbitrary state of Bob. It turns out that certain remote operations can be translated to certain properties of this hybrid state-operator object, which we will refer to as a "stator." The possible remote operations are hence associated with properties of the stator alone and are independent of the nature of the state(s) upon we intend to act remotely. By identifying the appropriate stator we are able to apply a remote operation on an arbitrary state(s).

In Sec. III, we describe the physical context in which stators can be prepared by applying LOCC on shared entanglement and the system. In Sec. IV, we show how to use stators to construct remote rotations for a two-level (spin-half) system. Then, in Sec. V, we consider the general problem of operating on an *n*-level system. In Sec. VI, we study the case of *N* multiple systems, and in Sec. VII we show how to promote remote unitary operations into remote interactions and measurements.

II. THE STATOR

We begin by introducing an object, which we shall refer to as a ''stator.'' A stator is a hybrid linear construction of states in Alice's Hilbert space and operators acting on Bob's system. The purpose of introducing this object is twofold. First, stators simplify considerably the construction of remote unitary operations and interactions via entanglement by providing us with a systematic general approach that can be easily generalized to an arbitrary number of *n*-level systems. Second, we found that these objects, which describe quantum correlations between states on one side and operators on the other side, assist us to develop an intuition regarding remote operations that may turn out helpful in other problems.

Let us then begin by defining what is a stator. We denote the Hilbert spaces of two remote observers, Alice and Bob, by $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. Instead of describing quantum correlations between states of Alice and Bob, we wish now to describe quantum correlations between *states* in $\mathcal{H}_A$ and resulting *actions* described by operators $O(\mathcal{H}_B)$ acting on an *arbitrary* state in $\mathcal{H}_B$. Hence we now construct a hybrid state operator or shortly a ''stator,'' $\mathcal{S}$, that lives in the space

$$\mathcal{S}\in\{\mathcal{H}_A\times O(\mathcal{H}_B)\}. \tag{3}$$

In close analogy to an entangled state, a stator has the general form

$$\mathcal{S}=\sum_{i=1}^{N_A}\sum_{j=1}^{N_B^2} c_{ij}|i\rangle\otimes B_j, \tag{4}$$

where $|i\rangle\in\mathcal{H}_A$, B_i act on states in $\mathcal{H}_B$ and c_{ij} are c numbers. The sum runs up to $N_A=\dim(\mathcal{H}_A)$, and $(N_B)^2$ where $N_B=\dim(\mathcal{H}_B)$. The operators B_i may be regarded as vectors in a N_B^2 dimensional Hilbert space with an inner product defined as $\langle B_i,B_j\rangle=\mathrm{tr}(B_i^\dagger B_j)$. An inner product between stators can hence be defined as $\langle\mathcal{S}_1,\mathcal{S}_2\rangle=\mathrm{tr}(\mathcal{S}_1^\dagger\mathcal{S}_2)$. We can now apply a unitary transformation on the states and operators and rewrite the above stator as a Schmidt decomposition

$$\mathcal{S}=\sum_{i=1}^{N} c_i|i\rangle\otimes B_i, \tag{5}$$

where $N=\min(N_A,N_B^2)$. (For simplicity we use the same notation for $|i\rangle$ and B_i in the new basis.) The above decomposition is not unique when $\dim\mathcal{H}_A\neq\dim\mathcal{H}_B$. The above states and operators satisfy the orthogonality relations

$$\langle B_i,B_j\rangle=\delta_{ij},$$

$$\langle i|j\rangle=\delta_{ij}. \tag{6}$$

Although this structure resembles the form of an entangled state it does not describe a fixed amount of entanglement because it applies to any general state of Bob. When we act with a stator on a general state $|\Psi_B\rangle\in\mathcal{H}_B$ we get

$$\mathcal{S}|\Psi_B\rangle\in\mathcal{H}_A\otimes\mathcal{H}_B. \tag{7}$$

Therefore, even if the stator has a maximal entanglementlike structure [as in Eq. (16) below], the measure of entanglement, $E(\mathcal{S}|\Psi_B\rangle)$, depends on the nature of $|\Psi_B\rangle$. For a general state of Bob, we may get any value of $E(\mathcal{S}|\Psi_B\rangle)$, from zero to one even for a ''maximal'' stator.

Most important to us will be the following property. For every stator we can construct an *eigenoperator equation*

$$O_A\mathcal{S}=\lambda_B\mathcal{S}. \tag{8}$$

Thus, by operating on the stator with an operator $O_A\in\mathcal{H}_A$ in Alice's Hilbert space, we get back the same stator multiplied by an *eigenoperator* now acting in Bob's Hilbert space $\mathcal{H}_B$. In general, the operators O_A and eigenoperators λ_B need not be Hermitian. In the present work we require that both O_A and λ_B are Hermitian operators.

Let us consider the eigenoperator equation in some detail:

$$\sum_i c_iO_A|i\rangle\otimes B_i=\sum_i c_i|i\rangle\otimes\lambda_BB_i. \tag{9}$$

Take the inner product with B_j and $|k\rangle$ and use the orthogonality relations. If $N_B^2>N_A$ we obtain that $\langle B_j,\lambda_BB_k\rangle=0$ for $j>N_A$. Similarly if $N_A>N_B^2$, $\langle k|O_A|j\rangle=0$ for $k>N_B$. The other nontrivial relation becomes

$$c_j\langle k|O_A|j\rangle=c_k\langle B_j,\lambda_BB_k\rangle. \tag{10}$$

Or denoting by $O_{ij}=\langle i|O_A|j\rangle$ and $\lambda_{ij}=\langle B_i,\lambda_BB_j\rangle$

$$c_jO_{kj}=c_k\lambda_{jk}. \tag{11}$$

For a given c_i we must hence satisfy N^2 equations. We may use the above equation to obtain the relations

$$\frac{O_{kj}}{O_{jk}}\frac{\lambda_{kj}}{\lambda_{jk}}=\frac{c_k^2}{c_j^2} \tag{12}$$

and

$$O_{kj}O_{jk}=\lambda_{kj}\lambda_{jk}. \tag{13}$$

If both O_A and λ_B are Hermitian operators, Eq. (12) yields

$$c_i=e^{i\theta}c_j, \tag{14}$$

where θ is some real number. Since the Hermiticity of O_A and λ_B is essential for our method of performing deterministic remote operations, the coefficients c_i in the stator must all be equal up to a phase. As we show in Sec. III this implies that the resources needed to prepare a stator, satisfying an eigenoperator equation with Hermitian operators, are *maximally* entangled states.

For Hermitian operators Eq. (13) further implies a relation between the matrix elements

$$|O_{jk}|^2 = |\lambda_{jk}|^2. \tag{15}$$

As a first example, let $\dim \mathcal{H}_A = 2$ be spanned by the eigenstates $|0_A\rangle$ and $|1_A\rangle$ of σ_{z_A}, and consider the operator $\sigma_{n_B} \in O(\mathcal{H}_B)$ such that $\sigma_{n_B}^2 = I_B$. Consider now the stator

$$\mathcal{S} = |0_A\rangle \otimes I_B + |1_A\rangle \otimes \sigma_{n_B}, \tag{16}$$

which we shall refer to in the sequel as a two-level stator. $\mathcal{S}$ satisfies the eigenoperator equation

$$\sigma_{x_A}\mathcal{S} = \sigma_{n_B}\mathcal{S}. \tag{17}$$

As straightforward, but useful consequence, any analytic function f also satisfies

$$f(\sigma_{x_A})\mathcal{S} = f(\sigma_{n_B})\mathcal{S} \tag{18}$$

and particularly

$$e^{i\alpha\sigma_{x_A}}\mathcal{S} = e^{i\alpha\sigma_{n_B}}\mathcal{S}, \tag{19}$$

where α is any real number or Hermitian operator in $\mathcal{H}_A$. The above relation already indicates why stators can be useful for generating remote operations. We note that a unitary operation of Alice gives rise to a similar unitary operation acting on Bob's side.

The above construction can be generalized to the case $\dim \mathcal{H}_A = n$, which becomes relevant if Bob owns an n-level system. Let $|i_A\rangle$, $i = 0,1,...,n-1$, be an orthogonal basis of $\mathcal{H}_A$, and choose $U_B \in O(\mathcal{H}_B)$ be the nth root of the unity: $U_B^n = I_B$. We can then construct the n-level stator

$$\mathcal{S} = |0_A\rangle \otimes I_B + |1_A\rangle \otimes U_B + \cdots |n-1_A\rangle \otimes U_B^{n-1}. \tag{20}$$

The relevant eigenoperator equation than becomes

$$V_A\mathcal{S} = U_B\mathcal{S}, \tag{21}$$

where V_A is a shift operator defined by $V_A|m_A\rangle = |(m-1)_A\rangle$, $m = 1,...,n-1$, and $V_A|0_A\rangle = |(n-1)_A\rangle$. By operating with $V_A + V_A^\dagger$ we then obtain the Hermitian eigenoperator $U_B + U_B^\dagger$, and acting with $i(V_A - V_A^\dagger)$ yields the eigenoperator $i(U_B - U_B^\dagger)$. Similarly we can construct any powers of $U_B + U_B^\dagger$ and $i(U_B - U_B^\dagger)$. We can further generalize our construction to the case that Bob has at hand several systems (which may be removed from each other) of arbitrary dimension. We discuss this case in Sec. VI.

In passing we comment that while being in appearance very similar, a stator, $\mathcal{S} = |0_A\rangle \otimes I_B + |1_A\rangle \otimes \sigma_{n_B}$ and a CNOT operator, $U_{\text{CNOT}} = |0_A\rangle\langle 0_A| \otimes I_B + |1_A\rangle\langle 1_A| \otimes \sigma_{n_B}$, are fundamentally different things. A stator is a mathematical object that captures the correlations between states on Alice's side and operations acting on Bob's system. U_{CNOT} is a unitary operation that acts on both Alice's and Bob's systems. We cannot prepare U_{CNOT} and keep it for later use. On the contrary, a stator $\mathcal{S}$ may be prepared and kept, until we decide

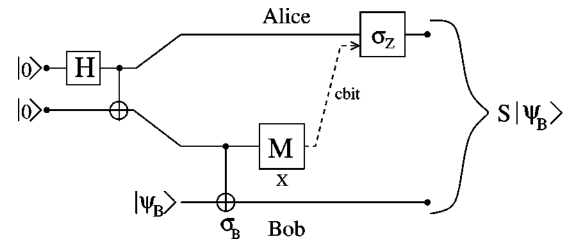

FIG. 1. Preparation of a stator acting on Bob's state.

what remote operations we wish to perform. The stator does not act on Alice's system, instead Alice applies on it a unitary transformation that generates (up to trivial rotations) a remote unitary operation, as in Eq. (19). The CNOT is more restricted, as it can induce only controlled 1_B and σ_{n_B} operations. A remote CNOT can be generated as a special case using the stator approach (see Sec. VII).

III. PREPARATION OF STATORS

We have seen that stators allow us to obtain state-independent relations between Alice's actions and their result on Bob's state. We proceed then to describe the process that will be referred to as "preparation" of a stator. Hence, given by an unknown state, $|\Psi_B\rangle \in \mathcal{H}_B$, and some shared entangled state $|\text{ent}\rangle$, our aim is to transform this initial state by performing some LOCC operation into

$$|\text{ent}\rangle \otimes |\Psi_B\rangle \rightarrow \mathcal{S}|\Psi_B\rangle. \tag{22}$$

We first describe in details the simplest case in which Alice and Bob use one ebit of shared entanglement to prepare a two-level stator as depicted in Fig. 1. The initial state at the hands of Alice and Bob is in this case

$$\frac{1}{\sqrt{2}}(|0_a0_b\rangle + |1_a1_b\rangle) \otimes |\Psi_B\rangle. \tag{23}$$

For practical purposes, in the following we will denote by the small letters, a and b, the shared ancillary entangled systems of Alice and Bob, respectively.

Bob starts by performing a controlled-NOT interaction (with respect to σ_{n_B}) between the qubit (b) and his state $|\Psi_B\rangle$, described by the unitary transformation

$$U_{bB} = |0_b\rangle\langle 0_b| \otimes I_B + |1_b\rangle\langle 1_b| \otimes \sigma_{n_B}. \tag{24}$$

Here σ_{n_B} is an operator acting in $\mathcal{H}_B$ satisfying $\sigma_{n_B}^2 = I_B$. ($\mathcal{H}_B$ need not be two dimensional; for instance, Bob's system may contain several spins, in which case $\sigma_{n_B} = \sigma_{n_B}^1\sigma_{n_B}^2\cdots$.)

This yields the state

$$\frac{1}{\sqrt{2}}(|0_a0_b\rangle \otimes I_B + |1_a1_b\rangle \otimes \sigma_{n_B})|\Psi_B\rangle. \tag{25}$$

Next he performs a measurement of σ_x of the entangled qubit to project out a certain value. The resulting state is now

$$\frac{1}{2}(|0_b\rangle \pm |1_b\rangle)\otimes(|0_a\rangle\otimes I_B \pm |1_a\rangle\otimes\sigma_{n_B})|\Psi_B\rangle. \tag{26}$$

Finally Bob informs Alice what was the result of his measurement by sending Alice one classical bit of information. For the case that $\sigma_x = -1$ Alice performs a trivial π rotation around the $\hat{z}$ axis and flips the $-$ sign to a $+$ sign. The resulting state of the system is now given by

$$\frac{1}{2}(|0_b\rangle \pm |1_b\rangle)\otimes(|0_a\rangle\otimes I_B + |1_a\rangle\otimes\sigma_{n_B})|\Psi_B\rangle. \tag{27}$$

Since Bob's previously entangled qubit factors out, the final state of Alice's qubit and Bob's system can be obtained by letting the stator

$$\mathcal{S} = |0_a\rangle\otimes I_B + |1_a\rangle\otimes\sigma_{n_B} \tag{28}$$

act on $|\Psi_B\rangle$. This completes the preparation of a two-level stator $\mathcal{S}$, which now operates on Bob's system. We further discuss preparation of n-level stators in connection with remote operations on an n-level system in Sec. V.

In passing we recall that the coefficients c_i in a general stator (5) are determined by the nature of the entangled state, $\Sigma c_i|i_a\rangle|i_b\rangle$, used to prepare the stator. Since the requirement that O_A and λ_B are Hermitian forces us to construct a stator with equal (up to a phase) coefficients, c_i, the resources needed to construct such a stators have to be maximally entangled states.

IV. REMOTE UNITARY TRANSFORMATIONS

Suppose that Bob has a system in the unknown state $|\Psi_B\rangle$ on which Alice and Bob wish to act on with a unitary transformation described by a rotation

$$U_B = e^{i\alpha\sigma_{n_B}} \tag{29}$$

with $\sigma_{n_B}^2 = I_B$ and α a real arbitrary number.

We will now show that the transformation (29) can be performed, provided that the generator σ_{n_B} is known to Bob, and the parameter α of rotation is known to Alice. To this end, they start by using a shared ebit to prepare, as described in the preceding section, the stator

$$\mathcal{S} = |0_a\rangle\otimes I_B + |1_a\rangle\otimes\sigma_{n_B}, \tag{30}$$

which operates on Bob's state. σ_{n_B} enters here as a result Bob's choice to perform a CNOT with respect to σ_{n_B} as in Eq. (24).

Next, Alice performs on her qubit a unitary transformation

$$U_a = e^{i\alpha\sigma_{x_a}}, \tag{31}$$

where $\sigma_{x_a}|0_a\rangle = |1_a\rangle$ and $\sigma_{x_a}|1_a\rangle = |0_a\rangle$. Using the fact that when acted with σ_{x_a} the stator satisfies an eigenoperator equation with an eigenoperator σ_{n_B}, we have

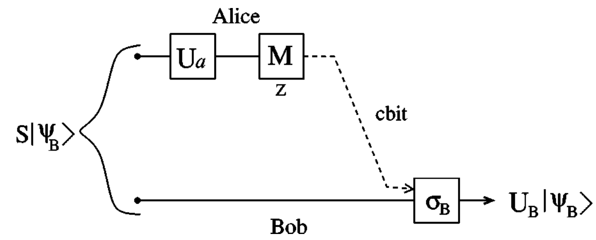

FIG. 2. Usage of a stator to operate a remote rotation.

$$e^{i\alpha\sigma_{x_a}}\mathcal{S} = e^{i\alpha\sigma_{n_B}}\mathcal{S}. \tag{32}$$

Hence after the rotation the state is

$$(|0_a\rangle\otimes I_B + |1_a\rangle\otimes\sigma_{n_B})e^{i\alpha\sigma_{n_B}}|\Psi_B\rangle. \tag{33}$$

Depending upon the final state of Alice's qubit, they managed to produce the required rotation, modulo possible extra trivial rotations. To eliminate these rotations, Alice measures the state of her qubit. If it is $|0_a\rangle$, we have produced the required transformation. If it turns out to be in the state $|1_a\rangle$ she needs to inform Bob to perform a trivial π rotation, $U_\pi = \exp(i\pi\sigma_{n_B}/2)$, which corrects for the extra σ_{n_B} above. This completes the process (see Fig. 2).

The resources that Alice and Bob require for remote rotation applied on a two-level system are hence, one ebit of shared entanglement and two cbits. They communicate one cbit first from Bob to Alice to prepare the stator, and one cbit from Alice back to Bob to complete the required rotation with probability 1. For both cbits we have that $p(1) = p(0) = 1/2$, i.e., they are unbiased. Therefore, the exchanged classical communication contains no information on the state of Bob or the angle of rotation.

The role of the exchanged cbits is as follows: the first cbit is needed in order to obtain the correct stator [fix the sign in Eq. (26)]. Without this one would have obtained with probability 1/2 the correct rotation U and with probability 1/2 the rotation $U^\dagger$. (For the case of remote measurements discussed in Sec. VII, this uncertainty in the sign may be irrelevant initially and may be corrected at a later stage of the process.) The second cbit sent from Alice to Bob is clearly needed from the causality requirement. A process that uses less than one cbit of communication from Alice to Bob clearly violates the causality.

V. REMOTE OPERATIONS ON *n*-LEVEL SYSTEMS

We now apply our method for the case of an n-level system. First we identify the n-level stator with the appropriate generator of rotations as an eigenoperator. To prepare this stator, Alice and Bob apply LOCC on their shared entangled state and Bob's n-level system. Next Alice performs a unitary transformation on half of the entangled pair on her side, followed by a measurement, and informs Bob via a classical channel how to correct his system to complete the rotation.

As we shall shortly see, the required resources in this case are two maximally entangled n-level systems, and two classical "nits" (each containing n possible values), one sent from Bob to Alice to complete the preparation, and the second from Alice back to Bob to complete the remote operation. However, unlike the two-level case, the number of possible unitary operations per given entangled n-level pair is here larger and given by any general linear combinations of $n-1$ generators. The rotation around a given axis is one of the possible operations.

To illustrate this let us first demonstrate the process for the case $n=3$ of a spin-one particle. For a rotation around the z axis (with the axis of rotation being chosen as before by Bob) we need to identify a stator that satisfies the eigenoperator equation

$$A\mathcal{S}=L_Z\mathcal{S}, \tag{34}$$

where L_Z is the appropriate generator of rotation. When applying A^2 on the stator, we get $A^2\mathcal{S}=L_Z^2\mathcal{S}$. Therefore, L_Z^2 is another eigenoperator of $\mathcal{S}$. Since $L_Z^3=L_Z$, these are the only eigenoperators.

Since for $n=3$ we have two distinct eigenoperators, the most general remote transformation that we are able to construct, using two maximally entangled three-level systems (qutrit), has the form

$$U_B=\exp[i(\alpha L_Z+\beta L_Z^2)], \tag{35}$$

where α and β are chosen by Alice.

Recalling the discussion in Sec. II, the appropriate $\mathcal{S}$ for this case is a three-level stator of the form

$$\mathcal{S}=|0_a\rangle\otimes I_{\Psi_B}+|1_a\rangle\otimes U_{\Psi_B}+|2_a\rangle\otimes U_{\Psi_B}^2, \tag{36}$$

where the requirement $U_{\psi_B}^3=I_{\Psi_B}$ dictates the form

$$U_{\Psi_B}=e^{(2\pi i/3)L_z}. \tag{37}$$

(Here we used the subscript Ψ_B in U_{Ψ_B} in order to distinguish between the full remote operation U_B applied by Alice and Bob and local transformations U_{Ψ_B} applied by Bob.) Since for a spin-one particle we have $e^{i\theta L_z}=1+i\sin\theta L_z+L_z^2(\cos\theta-1)$, we identify the operator A in Eq. (34) as

$$A=\frac{(V-V^\dagger)}{2i\sin\frac{2\pi}{3}}, \tag{38}$$

where V and $V^\dagger$ are the raising and lowering operators defined in Sec. II.

Having identified the required stator and the operators A, we next describe the preparation and rotation processes. We begin with a shared pair of maximally entangled qutrits and Bob's state $|\Psi_B\rangle$

$$(|0_a0_b\rangle+|1_a1_b\rangle+|2_a2_b\rangle)|\Psi_B\rangle. \tag{39}$$

Bob applies the unitary operation U_{bB} on his state and his half (b) of the entangled pair

$$U_{bB}=|0_b\rangle\langle 0_b|\otimes I_{\Psi_B}+|1_b\rangle\langle 1_b|\otimes U_{\Psi_B}+|2_b\rangle\langle 2_b|\otimes U_{\Psi_B}^2. \tag{40}$$

This results with the state

$$|\Psi_{\text{tot}}\rangle=[|0_a0_b\rangle\otimes I_{\Psi_B}+|1_a1_b\rangle\otimes U_{\Psi_B}+|2_a2_b\rangle\otimes U_{\Psi_B}^2]|\Psi_B\rangle. \tag{41}$$

Now to generate the stator (36), we need to eliminate Bob's entangled particle. Hence Bob measures his particle b in the following basis:

$$|0_b'\rangle=\frac{1}{\sqrt{3}}[|0_b\rangle+|1_b\rangle+|2_b\rangle]$$

$$|1_b'\rangle=\frac{1}{\sqrt{3}}[|0_b\rangle+e^{(2\pi i/3)}|1_b\rangle+e^{(4\pi i/3)}|2_b\rangle]$$

$$|2_b'\rangle=\frac{1}{\sqrt{3}}[|0_b\rangle+e^{(4\pi i/3)}|1_b\rangle+e^{(2\pi i/3)}|2_b\rangle]. \tag{42}$$

Let us rewrite the state (41) in the terms of the new basis vectors:

$$\begin{aligned}|\Psi_{\text{tot}}\rangle=\{&[|0_a\rangle\otimes I_{\Psi_B}+|1_a\rangle\otimes U_{\Psi_B}+|2_a\rangle\otimes U_{\Psi_B}^2]|0_b'\rangle\\&+[|0_a\rangle\otimes I_{\Psi_B}+e^{(2\pi i/3)}|1_a\rangle\otimes U_{\Psi_B}\\&+e^{(4\pi i/3)}|2_a\rangle\otimes U_{\Psi_B}^2]|1_b'\rangle\\&+[|0_a\rangle\otimes I_{\Psi_B}+e^{(4\pi i/3)}|1_a\rangle\otimes U_{\Psi_B}\\&+e^{(2\pi i/3)}|2_a\rangle\otimes U_{\Psi_B}^2]|2_b'\rangle\}|\Psi_B\rangle.\end{aligned} \tag{43}$$

According to one of the three particular outcomes of Bob's measurement of the particle b the state of Alice's particle a and Bob's particle Ψ_B evolves to

$$[|0_a\rangle\otimes I_{\Psi_B}+|1_a\rangle\otimes U_{\Psi_B}+|2_a\rangle\otimes U_{\Psi_B}^2]|\Psi_B\rangle \tag{44}$$

or to the states

$$[|0_a\rangle\otimes I_{\Psi_B}+e^{(2\pi i/3)}|1_a\rangle\otimes U_{\Psi_B}+e^{(4\pi i/3)}|2_a\rangle\otimes U_{\Psi_B}^2]|\Psi_B\rangle$$

and

$$[|0_a\rangle\otimes I_{\Psi_B}+e^{(4\pi i/3)}|1_a\rangle\otimes U_{\Psi_B}+e^{(2\pi i/3)}|2_a\rangle\otimes U_{\Psi_B}^2]|\Psi_B\rangle.$$

Bob transmits this classical outcome (classical ''trit'') to Alice. Notice that the three results appear with equal probability of 1/3 hence the classical trit is unbiased. In the last two cases Alice performs the following transformations on her particle a in order to correct the state to the form (44): $C_1=|0_a\rangle\langle 0_a|+\exp(4\pi i/3)|1_a\rangle\langle 1_a|+\exp(2\pi i/3)|2_a\rangle\langle 2_a|$ and $C_2=|0_a\rangle\langle 0_a|+\exp(2\pi i/3)|1_a\rangle\langle 1_a|+\exp(4\pi i/3)|2_a\rangle\langle 2_a|$, respectively. We can interpret Eq. (44) as the stator (36) operating on the state $|\Psi_B\rangle$. This completes the preparation process.

In order to generate a general rotation, Alice acts on her particle with the unitary operator

$$U_a = e^{i(\alpha A + \beta A^2)} \tag{45}$$

and performs a measurement to collapse the state into one of the states $|n_a\rangle$. Notice that as in the preparation process the results are again unbiased. She then sends a classical trit to Bob and informs him of the result of her measurement. For the cases that Alice obtained $|1_a\rangle$ or $|2_a\rangle$, Bob then performs the rotations $U^2_{\Psi_B}$ and U_{Ψ_B}, respectively. This completes the procedure of generating a remote rotation.

The above procedure can be applied to an arbitrary n-level system. The maximally entangled state of two qutrits is then replaced by a maximally entangled pair of n-level systems. After applying the interaction U_{bB} the total state becomes

$$|\Psi_{\rm tot}\rangle = \left[\sum_{m=0}^{n-1} |n_a n_b\rangle \otimes U^m_{\Psi_B}\right] |\Psi_B\rangle \tag{46}$$

with

$$U_{\Psi_B} = e^{(2\pi i/n)L_Z} \tag{47}$$

and L_Z the appropriate rotation generator for the n-level system.

Bob then performs a measurement of his half of the entangled pair b in the following basis:

$$|m'_b\rangle = \frac{1}{\sqrt{n}} \sum_{m_b=0}^{n-1} \exp\left[\frac{2\pi i}{n} m'_b m_b\right] |m_b\rangle, \tag{48}$$

where $m'_b = 0,\ldots,n-1$. He sends to Alice one "nit" to inform her which of the n-possible outcomes was obtained. Alice on her side operates the relevant unitary operation. It can be shown that for $n>3$ the relevant operator A in Eq. (34) becomes a linear combination of powers of $V - V^\dagger$ for n odd, and of $V + V^\dagger$ for even n. The total number of independent combinations is $n-1$. To complete the process Alice then performs a measurement and sends Bob one classical nit. This enables him to perform one of the operations $U^m_{\Psi_B}$, $m = 0,\ldots,n-1$, which complete the process.

To summarize, for an n-level system, we use the resources of one pair of maximally entangled n-level systems and a two-way classical communication of one nit in each direction. This enables to apply a general remote transformation of the form

$$U_B = \exp[i(\alpha_1 L_Z + \alpha_2 L_Z^2 + \cdots + \alpha_{n-1} L_Z^{n-1})], \tag{49}$$

where Bob determines the axis Z, and Alice determines the $n-1$ angles α_i.

VI. OPERATIONS ON MULTIPLE SYSTEMS

Consider next the case that on Bob's side, we have N distinguishable separate systems in some arbitrary state $|\Psi_{B_{1,\ldots,N}}\rangle$

$$|\Psi_{B_{1,\ldots,N}}\rangle \in \mathcal{H}_{B_1} \otimes \cdots \otimes \mathcal{H}_{B_N} \tag{50}$$

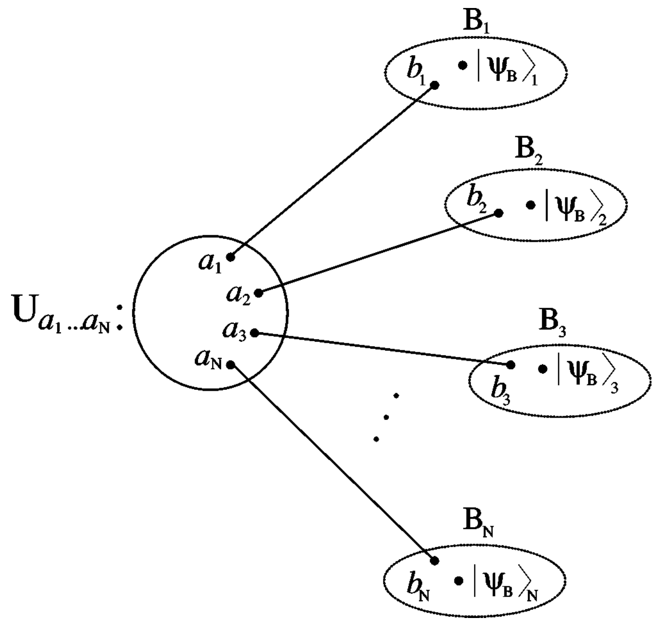

FIG. 3. Remote operation on N distributed systems.

with $\dim \mathcal{H}_{B_i} = n_i$. The N systems may be distributed to N different remote spatially separated locations denoted by B_i.

To examine the operations possible with our method we further assume that we distribute between Alice and B_i N maximally entangled pairs as depicted in Fig. 3. For a given system of dimensionality n_i, we match a maximally entangled n_i-level pair shared between Alice and B_i.

Clearly we now can repeat our method and generate $n_i - 1$ operations on the ith system by using the shared entangled pairs to prepare N stators, each one connecting between Alice and the system B_i. However, it now turns out that with N stators at hand, we can generate an exponentially larger class of operations, most of them corresponding to interactions between several remote subsystems.

To exemplify this, consider first the simplest case of N two-level (spin-half) systems. In this case the resources needed are N shared ebits between Alice and B_i and classical bidirectional communication of $2N$ classical bits: two cbits between Alice and a given B_i. As before, each B_i has the choice of fixing the local axis of rotation which fixes N generators $\sigma_{n_{B_i}}$, $i = 1,\ldots,N$.

We can repeat the preparation of a stator $\mathcal{S}_i$ for each spin separately as described in Sec. III. The total stator is then

$$\mathcal{S}_{\rm tot} = \otimes_{i=1}^N (|0_{a_i}\rangle \otimes I_{\Psi_{B_i}} + |1_{a_i}\rangle \otimes \sigma_{n_{B_i}}). \tag{51}$$

The above stator satisfies an eigenoperator equations

$$\sigma_{x_i} \mathcal{S}_{\rm tot} = \sigma_{B_i} \mathcal{S}_{\rm tot}. \tag{52}$$

However, since the different N generators commute, we also have that any *product* of separate eigenoperators is also an eigenoperator. The total number of eigenoperators is then

$$\sum_{m=1}^{N} C_N^m = 2^N - 1. \tag{53}$$

It follows then that Alice has the freedom of selecting the 2^N-1 angles that generate rotations and interactions between the spins.

For example, the most general remote operation for the case $N=3$ becomes

$$U_B = \exp\left[i\sum_{m=1}^{3} \alpha_m \sigma_{B_m} + i\frac{1}{2}\sum_{m\neq n} \beta_{mn}\sigma_{B_m}\sigma_{B_n} + i\gamma\sigma_{B_1}\sigma_{B_2}\sigma_{B_3}\right]. \tag{54}$$

We can easily apply our method for any configuration of N separated n_i-levels systems. (In general n_i may not be equal.) Let us consider the case with $n_i = n$ for all i. Then the total number of operators is easily computed to be

$$\sum_{m=1}^{N} (n-1)^m C_N^m = n^N - 1. \tag{55}$$

Therefore, with the aid of N pairs of n-level maximally entangled pairs and bidirectional classical communication of $2N$ nits, we can apply n^N-1 remote operations.

Finally, we note that the N separated subsystems can be viewed as a single system of dimensionality $D=\Pi_{i=1}^{N} n_i$. Hence by the results of the preceding section, we can use one D-level stator to act on the system as a whole. The number of distinct operations will then be given by $D-1$, in agreement with the results obtained in Eqs. (53) and (55).

VII. GENERATING REMOTE INTERACTIONS AND MEASUREMENTS

In the preceding section we have already seen examples where Alice can act remotely on several spatially separated systems and effectively generate an interaction between remote subsystems. For instance, for two remote spins systems Alice can use two ebits and four cbits to generate the interaction

$$U_{B_1,B_2} = e^{i\alpha\sigma_{B_1}\sigma_{B_2}}. \tag{56}$$

Here the local axes of rotation, $\overline{n_i}$ ($\sigma_{B_i} = \overline{n_i}\cdot\overline{\sigma}_{B_i}$), are determined locally by the local observers B_i, and the coupling strength α is controlled by Alice.

There is yet another simple method to generate remote interaction between Bob's system and a system A located with Alice. Inspecting Eq. (19), we note that in fact the angle α can be promoted to an operator acting on a system A of Alice. Hence in the case of a two-level stator, with an eigenoperator σ_{n_B} we have also the relation

$$e^{i\lambda O_A \sigma_{x_A}}\mathcal{S} = e^{i\lambda O_A \sigma_{n_B}}\mathcal{S}, \tag{57}$$

where the stator $\mathcal{S}$ is defined as in Eq. (16), and O_A is an Hermitian operator acting on an arbitrary dimensional system A of Alice.

A simple generalization of the procedures in Secs. III and IV now allows performing remote interaction between separate systems. The only modification needed is to replace the unitary rotation performed by Alice to her half of the entangled pair (a) with the unitary operation

$$U_{Aa} = e^{i\lambda O_A \sigma_{x_a}} \tag{58}$$

acting on (a) and on her system A. [For the n-level case σ_{x_a} needs to be replaced by an appropriate operator, e.g., the operator A defined in Eq. (38).]

For example suppose Alice's system is another spin-half particle, and we wish to apply remotely a controlled-NOT operation [3,4] between Alice's and Bob's spins, taking Alice's system as a control and Bob's system as a target. To this end we notice that

$$U_{\text{CNOT}} = |\uparrow_A\rangle\langle\uparrow_A|\otimes I_B + |\downarrow_A\rangle\langle\downarrow_A|\otimes\sigma_{x_B} = \exp\left[-i\frac{\pi}{4}(1-\sigma_{z_A})(1-\sigma_{x_B})\right]. \tag{59}$$

To apply this transformation we prepare the stator $\mathcal{S} = |0_a\rangle\otimes I_B + |1_a\rangle\otimes\sigma_{x_B}$, which satisfies

$$\exp\left[-i\frac{\pi}{4}(1-\sigma_{z_A})(1-\sigma_{x_a})\right]\mathcal{S} = \exp\left[-i\frac{\pi}{4}(1-\sigma_{z_A})(1-\sigma_{x_B})\right]\mathcal{S}. \tag{60}$$

Therefore, we can in a straightforward manner use our procedure to construct a remote CNOT.

As a special case of remote interactions we can further consider remote measurements. Hence Alice's system A will be considered as a measuring device. We can use another spin as a measuring device (pointer) or let us introduce a continuous measuring device M with conjugate coordinates P and Q, where P plays the role of the "pointer."

Let us describe a remote "Stern-Gerlach" measurement of Bob's spin system along a certain direction. Alice informs Bob to fix the axis n_B according to the direction she wishes to perform the measurement. After completing the preparation of the stator she applies the unitary operation

$$U_{Ma} = e^{iQ\sigma_{x_a}} \tag{61}$$

that yields the state

$$\mathcal{S}e^{iQ\sigma_{n_B}}|\Psi_B\rangle|M\rangle. \tag{62}$$

She can now observe the variable P of the measuring device and read the outcome of the measurement. Bob's state will reduce to the corresponding outcome. Therefore, in a remote measurement we need only a one-directional communication of *only one* bit of information from Bob to Alice.

We conclude with several comments. The remote measurement process can be in fact completed instantaneously on a spacelike surface. Alice does not need to wait to obtain a classical bit to perform her measurement. In this case Alice generates the operation $\exp(\pm iQ\sigma_{n_B})$ with probability 1/2 for each $\pm$ possibility. Hence, in accordance with the causality, the result of the measurement can be interpreted only *after* she obtains the classical bit from Bob. This approach can be easily generalized for general systems as well as for performing measurements nonlocal observables.

Finally, it is interesting to note that the present method consumes less entanglement resources (one ebit instead of two, and two cbits instead of four) compared to methods using teleportations.

VIII. CONCLUSION

We presented a systematic method for constructing deterministic remote operations on single and multiple systems of arbitrary dimensions. Our approach requires bidirectional classical communication of unbiased bits between the parties and leaves the control over the generators that act on each system at the hands of the local observers. In this way the control over the full structure of the unitary operation is split among several remote observers. It is also worth mention that when the local information is kept secret, the operations cannot be achieved using teleportationlike schemes. These properties may be helpful for constructing new cryptographic tools.

To facilitate the construction of remote operations we have introduced an object—the stator—which describes correlations between states of one system and operations acting on an arbitrary state of another remote system. We hope that stators may turn out useful for other problems regarding the relation between entanglement and remote interactions.

Note added. Recently we have learned of other results obtained independently by Huelga, Plenio, and Vaccaro [7] and by Yang and Gea-Banacloche [8].

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