Charge Superselection Rule*

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Questions concerning superselection rules are considered. Two experiments are discussed. In the first, coherent superpositions of different angular-momentum states are constructed. In the second, coherent superpositions of states with different charge are constructed in complete analogy with the angular-momentum case. We suggest that, contrary to a widespread belief, interference may be possible between states with different charges.

I. INTRODUCTION

THE idea of a superselection rule as a rule preventing certain mathematical state vectors from being realized in nature was introduced by Wick, Wightman, and Wigner.[1,2]

A selection rule for a quantity A asserts the conservation of A. Selection rules exist for energy, momentum, angular momentum, electric charge, lepton number, and baryon number, and the number of fermions modulo 2. According to Wick, Wightman, and Wigner, some of these selection rules can be elevated to the status of superselection rules. A superselection rule asserts the impossibility of preparing a coherent superposition of two eigenstates of A with different eigenvalues. More generally, no physically realizable density matrix contains off-diagonal elements connecting states of different A.

It was conjectured by Wick, Wightman, and Wigner that superselection rules exist for fermion number modulo 2 and for the total electric charge.

This is the first of a series of articles in which we discuss the validity and meaning of such rules. In this article we suggest a method of exhibiting interference between states of different charge.

Imagine an isolated laboratory which is neither interacting nor correlated to any outside system.

We consider experiments done entirely inside the laboratory. We may, therefore, assume that the total charge of the laboratory is definite.

We note that if the laboratory is divided into two parts—a subsystem upon which our interest is focused, and the rest of the laboratory—then the state of the laboratory can be written

$$|\psi_{\text{lab}}\rangle=\sum_q |q\rangle|Q-q\}f(q)\,,$$

where $|q\rangle$ is a state of the subsystem with eigenvalue q, $|Q-q\}$ is a state of the rest of the laboratory with eigenvalue $Q-q$, and Q is the total charge of the laboratory. Since each value q will be correlated with different orthogonal states of the rest of the laboratory, the density matrix for the subsystem will not have elements connecting different q.

It would seem then that unless the entire laboratory has an indefinite value of Q, coherence between different q for a subsystem is not possible, as emphasized by Wigner. By analogy, it would seem that establishing interference between different particle momenta (localizing a particle) is impossible unless the laboratory is localized. This result applies to an observer outside the laboratory who indeed is correct in concluding that if the laboratory has definite momentum with respect to him, no part of the laboratory is localized in his frame of reference. Nevertheless, one may ask for the description as seen by the experimenter inside the laboratory. His question is whether or not the subsystem is localized with respect to the laboratory frame of reference.

The equivalent question for charge has not been formulated before. In this article, we investigate this problem and show that interference between different charge states has analogous meaning.

We precede this discussion with a detailed description of an arrangement in which interference in the above sense is established between different angular-momentum states, to indicate more fully the relevance of such interference in a familiar context.

II. ANGULAR MOMENTUM

In this section we consider an experiment to prepare a coherent superposition of $\sigma_z=+1$ and $\sigma_z=-1$ for an electron initially in the state $|\sigma_z=+1\rangle$. This can be done with the aid of a magnetic field in the x direction. The electron passes through the magnetic field, precessing in the usual way around the x axis. The system is arranged in such a way as to leave the electron spin in the y direction when the electron leaves the field. Such a final state is a coherent superposition of $\sigma_z=\pm1$. That it is coherent can be seen by measuring σ_y. An incoherent superposition would give the value $\frac{1}{2}$ for the probability that $\sigma_y=1$ or -1, while the present state has probabilities 1 and 0. The measurement of σ_y can be done in various ways by the use of a second magnetic field.[3]

The phase θ in the superposition $|\sigma=+1\rangle+re^{i\theta}|\sigma=-1\rangle$ has an interpretation in terms of the direction

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[1] G. C. Wick, A. S. Wightman, and E. P. Wigner, Phys. Rev. **88**, 101 (1952).

[2] R. F. Streater and A. S. Wightman, *TCP, Spin and Statistics and All That* (W. A. Benjamin, Inc., New York, 1964).

[3] See, however, E. P. Wigner, Z. Physik **131**, 101 (1962).

of polarization of the electron. By changing θ, we rotate the polarization of the electron about the z axis.

Since angles are always measured with respect to a frame of reference, with the zero of angle chosen arbitrarily, the phase in the $\sigma=\pm 1$ superposition is defined relative to an angular frame of reference, say the second magnetic field. We analyze the system in terms of a quantum-mechanical system of three parts, each free to rotate about z. There are the electron e, the preparation magnet H_1, and the second magnet H_2 used to measure the electron spin. The magnets will be thought of as a system of spin-$\frac{1}{2}$ particles with magnetic moments. If the system of two magnets has a very large moment of inertia, then the experiment cannot be sensitive to the exact total angular momentum of H_1 and H_2, since, in any case, the angular velocity of the system will be approximately zero. We therefore do not lose generality if we choose the total angular momentum $L_{z1}+L_{z2}$ of H_1 and H_2 to be zero.

The electron enters with $\sigma_z=+1$. The initial state is then

$$\sum_{L_1,L_2,\alpha,\beta} \Psi(L_1,L_2,\alpha,\beta)\delta(L_1+L_2)|L_1\alpha,L_2\beta\rangle\sigma_z=+1\rangle, \qquad (1)$$

where L_i equals the z component of angular momentum of the ith magnet and α and β describe all other degrees of freedom of the magnets. We shall suppress the dependence on α and β in what follows.

At any later time the state will be

$$\varphi(L_1L_2)\delta(L_1+L_2)|\sigma=+1\rangle|L_1,L_2\rangle + \varphi_2(L_1L_2)\delta(L_1+L_2-\hbar)|\sigma=-1\rangle|L_1,L_2\rangle. \qquad (2)$$

The density matrix for the electron alone can be seen to be diagonal in the z component of σ, since different σ_z will be correlated with different L_1+L_2.

More generally, if the entire universe is in a state of definite L_z, then the density matrix for any subsystem must be diagonal in L_z, since different L_z for the subsystem will be correlated with different L_z for the rest of the universe.

Are we to assume that the diagonal nature of the density matrix means that the originally proposed experiment is impossible, or that a superselection rule for L_z exists even for subsystem of the universe?

The answer for the simple system of electron and magnets is that since we prepared the entire system in an eigenstate of L_z, the orientation of the xy plane is completely uncertain with respect to an external observer. Hence, in particular, the orientation of the electron in the xy plane remains totally ambiguous. However, in the experiment described above, the reference frame for the electron is the system of magnetic fields. Therefore, questions concerning the over-all orientation of the electron with respect to magnetic fields are of no importance, i.e., *the coherence of states of different angular momentum is measured relative to a frame of reference.* In other words, the statement that the electron emerges from the first magnet in an eigenstate of σ_x has a meaning only relative to the second magnet and not relative to an external observer.

In order to insure consistency of the experiment, it is important that the frames of reference provided by H_1 and H_2 be consistent. That is, the orientation of H_1 with respect to H_2 must not be ambiguous. This requires the wave function $\Psi(L_1,L_2)$ to have a form

$$\Psi(L_1L_2)\sim e^{i(L1-L2)\theta}, \qquad (3)$$

so that the magnetic fields will have a definite, nonfluctuating angle θ between them.

We summarize by saying that the relative phase between the states $|\sigma=\pm 1\rangle$ is always measured with respect to a zero angle defined by the orientation of one of the magnets, which by definition provides the xy frame of reference.

Furthermore, if consistency is to be checked in some other frame, the dynamical degrees of freedom of the two frames must be correlated so as to give the two frames a definite relative orientation.

III. CHARGELIKE QUANTITIES

In this section, we discuss an experiment which can be performed in principle, in which charge plays an analogous role to that of the angular momentum in the previous section.

Imagine two cavities, C_1 and C_2, each having a single mode which may be populated by an arbitrary number of negatively charged mesons. (We neglect electromagnetic interactions in our discussion.) A state is prepared in which the charges Q_1 in C_1 and Q_2 in C_2 are uncertain, while the total charge Q_1+Q_2 is definite. Such a state may be prepared, for example, by feeding the cavities with mesons, each being "split" before entering the cavities by a suitable "half-reflecting" mirror. Each meson will be prepared with a wave function $\Psi_1+e^{i\varphi}\Psi_2$, where Ψ is nonvanishing in C_1 and Ψ_2 is nonvanishing in C_2.

If the total number of mesons is large enough, the state thus prepared will be, to a good approximation, a state of definite phase $\Phi_1-\Phi_2$, where Φ_i is the phase conjugate to Q_i.

This preparation is analogous to the preparation of the two magnets in the last section which were prepared in a state of well-defined relative orientation and definite total angular momentum.

After the cavities have been so prepared, a proton is sent into C_1. It emerges from C_1 as either a proton or a neutron. We claim that relative to the frame of reference provided by C_1 the emerging nucleon is in a coherent superposition of zero and plus charge with a definite phase between neutron and proton components. To check this, we send the nucleon through C_2 and observe, as we shall show, that the probability of the nucleon emerging from C_2 as either proton or neutron exhibits interference between proton and neutron.

In fact we can construct a "laboratory" including an arbitrary number of cavities that serve as frames of reference. One cavity may be arbitrarily chosen as the zero of phase. Relative to that origin, we can prepare superpositions of charge states which can be checked at any other cavity in an interference experiment of the kind we shall describe. Consistent results will be obtained in that the phase of the superposition will be independent of the cavity at which it is checked.

We now discuss in detail the setup described. Let $|n\rangle$ be a state with n negative mesons. Consider a state $|Q,\theta\rangle$ for cavity 2 which has an approximately well defined charge Q and phase θ:[4]

$$|Q,\theta\rangle=\sum_n \frac{Q^{n/2}}{(n!)^{1/2}}e^{in\theta}|n\rangle. \tag{4}$$

We show how such a state can be used to measure the phase of a proton-neutron superposition if Q is chosen very large.

We assume that protons and neutrons interact with mesons through a Hamiltonian

$$H=(\sigma^+a^-+\sigma^-a^+)g(t), \tag{5}$$

where σ^+ takes a proton into a neutron, σ^- takes a neutron into a proton, and a^- and a^+ are meson destruction and creation operators. The function $g(t)$ is a function which equals g from $t=0$ to $t=T$ and equals zero for all other times. The interval O to T represents the time during which the nucleon is in contact with the mesons. The initial state is $|Q,\theta\rangle$ for the cavity C_2 and $\alpha|P\rangle+\beta|N\rangle$ for the nucleon.

Using the Hamiltonian of Eq. (5) and assuming large Q, we find that at time T the state is

$$[\alpha\cos(gTQ^{1/2})+\beta i\sin(gTQ^{1/2})e^{-i\theta}]|P\rangle \\ +[\alpha i\sin(gTQ^{1/2})e^{i\theta}+\beta\cos(gTQ^{1/2})]|N\rangle. \tag{6}$$

The proton probability is

$$\alpha^*\alpha\cos^2(gTQ^{1/2})+\beta^*\beta\sin^2(gTQ^{1/2}) \\ -i\alpha\beta^*e^{i\theta}\cos(gTQ^{1/2})\sin(gTQ^{1/2}) \\ +i\alpha^*\beta e^{-i\theta}\cos(gTQ^{1/2})\sin(gTQ^{1/2}). \tag{7}$$

Hence if the relative phase between α and β is φ, then the probabilities of a proton and a neutron after passing through C_2 depend on $(\theta-\varphi)$.

In fact, we may say that with respect to the reference frame provided by C_2, the neutron-proton relative phase is $(\theta-\varphi)$ and the initial nucleon state is

$$|\alpha|\ |P\rangle+|\beta|\ e^{i(\theta-\varphi)}\ |N\rangle. \tag{8}$$

Suppose now that we prepare a nucleon state by passing a proton through C_1 with C_1 in the state $|Q\theta\rangle$. Upon emerging, the nucleon has the probability $\cos^2(gTQ^{1/2})$ of being a proton and $\sin^2(gTQ^{1/2})$ of being a neutron. We wish to determine the relative phase between the proton and neutron components by passing the emerging beam through C_2, which is in the state $|Q'\theta'\rangle$. The probability that the nucleon emerging from C_2 will be a proton is obtained by multiplying the transition operator for passing through C_1 by that for passing through C_2. The result gives

$$\cos^2(gTQ^{1/2})\cos^2(gT\sqrt{Q'})+\sin^2(gTQ^{1/2})\sin^2(gT\sqrt{Q'}) \\ -2[\cos(gTQ^{1/2})\sin(gTQ^{1/2})\cos(gT\sqrt{Q'}) \\ \sin(gT\sqrt{Q'})][\cos(\theta-\theta')] \tag{9}$$

for the proton probability.

Comparing Eq. (9) with Eq. (6) gives a value $(\theta-\theta')$ for the relative phase between neutron and proton when referred to the frame provided by C_2.[5]

In the above experiment, the two states $|Q\theta\rangle$ and $|Q'\theta'\rangle$ for C_1 and C_2 are, of course, not eigenstates of charge. Actually they are minimum uncertainty packets in charge and phase. Hence we have not yet been able to set up the required relative reference frames with a system of definite charge. However, if we superpose states $|Q\theta_1\rangle$ and $|Q'\theta_2\rangle$, keeping the phase difference $(\theta_1-\theta_2)$ fixed at $(\theta-\theta')$ according to

$$|i\rangle=\int|Q\theta_1\rangle|Q'\theta_2\rangle\delta(\theta_1-\theta_2-\theta+\theta')e^{i(Q+Q')\theta_1}, \tag{10}$$

then we obtain a state of definite charge $Q+Q'$. Furthermore, all probabilities for proton or neutron states are precisely the same as for the initial state $|Q\theta\rangle\ |Q'\theta'\rangle$. Hence the apparatus described by Eq. (10) constructs a state which, when referred to the phase frame of C_2, is a coherent superposition of proton and neutron with phase $(\theta-\theta')$.

IV. CONCLUSION

Our conclusion is that coherence between different values of an additive conserved quantity Q is really a way of speaking about the special interference effects which occur when the source and detectors of particles are correlated and coherently share a fixed amount of Q between them.

In order to establish the existence or nonexistence of a superselection rule in the present sense, it is necessary to examine the nature of the available interactions to see whether the procedures outlined above can actually be carried out without loss of coherence.

Our preliminary investigations, which will be described in a separate article, indicate the following points for electron charge.

1. If the electromagnetic coupling between charge and photons is turned off, then there exists no limitation on the coherence of superpositions of charge.

2. Because of the actual coupling, the electromagnetic energy in the meson cavities will cause the phase

[4] R. J. Glauber, Phys. Rev. **131**, 2766 (1963).

[5] We refer to the state of the nucleon after having been prepared at 1 and before being detected at 2.

relations between cavities to become uncertain in an amount of time which depends on the fine-structure constant and the geometry.

3. The potential energy seen by a charge when passing between cavity 1 and cavity 2 is uncertain because of the uncertain value of charge in each cavity. Hence a charged particle will develop an uncertain phase relative to an uncharged particle in the motion from 1 to 2.

4. It follows from the above points that as long as the fine-structure constant is small compared to 1, our experiments can be carried out. For a given fine-structure constant, an upper bound on the distance separating the two cavities exists beyond which the interference will be washed out. This upper bound becomes infinite as the fine-structure constant becomes zero.

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Effect of a Coordinate Measurement on the Statistical Ensemble of a Quantum-Mechanical System*

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An analysis is given of the change in the statistical ensemble when a measurement of a generalized coordinate is made on a quantum-mechanical system. The results are naturally described in terms of the changes in the Wigner distribution of the system. Some of these changes are the same as those in classical theory.

I. INTRODUCTION

We develop in this paper a method for discussing the measurement of quantum-mechanical observables with continuous eigenvalue spectra. In this method the Wigner distribution function[1] has a role similar to that of the Gibbs phase-space density in the corresponding classical problem. The analysis applies to a wide class of physical situations, such as a field or a material particle.

In classical theory, the ensemble describing the statistical properties of a system is specified by the Gibbs phase density ρ_G, such that $\rho_G(q,p)dqdp$ is the probability of finding the system in the phase-space element $dqdp$. When a measurement is made, the Gibbs density must be changed to incorporate the information obtained from the measurement. For a measurement of a system with one degree of freedom, which establishes that the coordinate q lies in a range $q_1-\frac{1}{2}\Delta q\leq q\leq q_1+\frac{1}{2}\Delta q$, the Gibbs density just after the measurement is

$$\begin{aligned}\rho_G^{(1)}(q,p)&=w_G^{-1}\rho_G^{(0)}(q,p)\,, & |q-q_1|&<\tfrac{1}{2}\Delta q\\ &=0\,, & |q-q_1|&>\tfrac{1}{2}\Delta q\,,\end{aligned}\qquad(1.1)$$

where $\rho_G^{(0)}$ is the Gibbs density just before the measurement, and

$$w_G=\int_{q_1-\frac{1}{2}\Delta q}^{q_1+\frac{1}{2}\Delta q}dq\int_{-\infty}^{+\infty}dp\;\rho^{(0)}(q,p)\,.\qquad(1.2)$$

Equation (1.1) is an expression of the "reduction" of the classical ensemble by the measurement. The members of the original ensemble having coordinates inconsistent with the measurement are discarded; those which are consistent form the new ensemble. The factor w_G, which normalizes the new ensemble, is the probability of finding the coordinate q in the range Δq in the original ensemble. Some information about the original ensemble is preserved by those members retained in the new ensemble.

In quantum theory, the statistical state of a system is described by the density matrix ρ. When a measurement is made, the density matrix must be changed to incorporate the information obtained from the measurement. The density matrices just before and just after the measurement, ρ_0 and ρ_1 respectively, are related by the equation

$$\rho_1=w^{-1}P\rho_0P\,,\qquad(1.3)$$

$$w=\mathrm{Tr}[\rho_0P]\,,\qquad(1.4)$$

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[1] E. P. Wigner, Phys. Rev. 40, 749 (1932).