Superoscillations and tunneling times

Yakir Aharonov,[1,2] Noam Erez,[1] and Benni Reznik[1]
[1]*School of Physics and Astronomy, Tel Aviv University, Tel Aviv 69978, Israel*
[2]*Department of Physics, University of South Carolina, Columbia, South Carolina 29208*
(Received 17 October 2001; published 17 May 2002)

It is proposed that superoscillations play an important role in the interferences that give rise to superluminal effects. To exemplify that, we consider a toy model that a wave packet to travel in zero time and negligible distortion, a distance arbitrarily larger than the width of the wave packet. The peak is shown to result from a superoscillatory superposition at the tail. Similar reasoning applies to the dwell time.

DOI: 10.1103/PhysRevA.65.052124 PACS number(s): 03.65.Xp, 03.65.Ta

I. INTRODUCTION

Superluminal effects have been predicted in conjunction with various quantum systems propagating in a forbidden zone. In these regimes, the (semiclassical) kinetic energy is negative, making the semiclassical tunneling time ill defined, and various operational definitions of the velocity of a wave packet have been proposed, in many examples, giving differing values. In recent years, a number of experiments with superluminal photons have been performed, reviving interest in the problem, as well as controversy. The theoretical predictions have been verified—in fact, different tunneling times have been measured in accordance with the different operational definitions appropriate for the various experimental setups. Although there is no longer much controversy over the facts (former claims of breakdown of causality have been cleared up), there is still some disagreement on the merits of various operational definitions of the time delay. For extensive reviews that cover the experimental situation, as well as the theoretical background and bibliography see Refs. [1–3]. In the present paper, we largely follow the notations of Chiao and Steinberg [3].

Causality is not violated. The signal velocity (the velocity of propagation of an *abrupt* disturbance) is always subluminal. Other "velocities" may well be superluminal, for example, the group velocity of a wave packet. In the latter case, the (local) peak of the packet appears at a point where constructive interference builds up (this effect is often termed as "pulse reshaping"), much earlier than the arrival of a freely propagating wave function. Thus the information stored in the tail may be traveling way ahead of the peak and can possibly be used to anticipate it. It is important to note, however, that the signal velocity is never measured—to measure it one needs a detector with infinite sensitivity. All the operationally defined velocities can, in principle, become superluminal. In the example dicussed in this paper, both the Wigner time and a clock time (to be defined below) turn out to give superluminal velocities.

The purpose of this paper is to investigate further the nature of the interferences that give rise to superluminality [4–10]. It has been noted by Steinberg [11,12] that the superluminality phenomenon is associated with postselection: for instance, from a sample of particles that scatter off a barrier we examine only the subsample that tunnel through. When a preselected and postselected systems [19,21,22] is subjected to a nondisturbing, "weak" measurement, the outcome of the measurement, known as the "weak value," can attain values that lie outside the spectrum of eigenvalues of the measured observable [13,20]. Weak values may hence be naturally related to the superluminal phenomenon, as indeed, Steinberg has already argued that the dwell time is a weak value of a projector to the tunneling domain. The appearance of unusual weak values has been associated with a unique interference structure [14], for which Berry [15] coined the term "*superoscillations*." As an instructive example of a superoscillatory function $F(k)$ consider

$$F(k;N,L)=\left[\left(\frac{1-L/x_0}{2}\right)e^{ikx_0/N}+\left(\frac{1+L/x_0}{2}\right)e^{-ikx_0/N}\right]^N. \tag{1}$$

Here, $N>1$ is an integer, and L and x_0 being the super and reference shifts. For small k we expand $\exp(ikx_0/N)$ and find

$$F(k;N,L)=e^{-ikL}\left[1+\frac{(L^2-x_0^2)k^2}{2N}+O(N^{-2})\right]\cong e^{-ikL}. \tag{2}$$

Although this function is a superposition of waves e^{ikx} with $|x|\leqslant x_0$, in the interval $|k|\ll\sqrt{N}/\sqrt{L^2-x_0^2}\equiv\Delta k$, $F(k)$ behaves nearly as a pure wave e^{ikL} with L arbitrarily larger than x_0. In the regime $|k|<\Delta k$ the function oscillates rapidly. The number of these "superoscillations" is $\sim\sqrt{N}$. This remarkable feature is derived at the expense of having the function grow exponentially in other regions. In the example above, for $|k|>\Delta k$, we get $F\sim e^N$.

In this paper we will suggest that at least for certain cases, the constructive interference giving rise to superluminal effects, originates from a similar structure of superoscillations. To exemplify that, we consider a toy model, (which extends on a previous proposal of Olkhovsky Recami, and Salesi [16]), which allows for a wave packet to traverse, in a vanishing time and negligible distortion (the transmitted wave packet is the first derivative of the incoming packet), a distance *arbitrarily longer than the original size of the wave packet*. Hence the peak is here reconstructed from the exponentially small tail of the wave function. As far as we know, in the examples discussed to date, the superluminal shift of the wave-packet is restricted. It is comparable to or much smaller than the initial wave packet size. We then show that the resulting group-delay and dwell times vanish [17]. Finally, coming back to the role of postselection we provide a rigorous proof for Steinberg's claim [11,12] that the delay time is a weak value of a projector operator.

The paper proceeds as follows. In Sec. II we calculate the dwell time and the group-delay time for tunneling through n-δ-function barriers. In the low-energy limit, both turn out to be zero. We also derive the condition for the calculations to apply for a wave packet. Using this condition, we see that for this system, the (negative) delay can be larger than the uncertainty associated with the length of the wave packet. Section III deals with the relation between superluminality and interference effects in the tail of the wave function, and superoscillations. The applicability to the example of Sec. II, of the explanations of superluminality given in other cases, is discussed. Finally, in Sec. IV we elaborate on Steinberg's claim that the dwell time is a weak value.

II. THE GROUP-DELAY AND DWELL TIME FOR A PARTICLE TUNNELING THROUGH AN n-δ-FUNCTION POTENTIAL

Olkhovsky Recami and Salesi [16] showed that a Schrödinger particle tunneling through a double rectangular barrier traversed the distance between the bumps instantaneously in the limit that its kinetic energy was much smaller than the height of the barrier. Unlike previous examples of superluminal tunneling, the length of the region of superluminality consists of an arbitrarily long portion with zero potential, between the bumps. Replacing the rectangular barriers in the example discussed in Ref. [16] by δ-function potentials, the calculations can be made somewhat simpler, and are easily generalized to n arbitrary δ-function bumps (still using the approximation of low kinetic energy).

In this section we make a direct calculation of the transmission coefficient for the stationary scattering of a scalar particle obeying the Schrödinger equation, off a multiple δ-function potential. The Schrödinger equation is of course nonrelativistic and displays an unphysical superluminal dispersion. However, the time-independent equation is the same as for the scalar relativistic wave equation, and we focus on the Schrödinger equation merely for a simple concrete interpretation. The origin of the superluminality in this case is a reshaping of the tail of the wave function. For a further discussion of the justification of using the Schrödinger equation for investigating superluminal tunneling times see the review by Chiao and Steinberg [3].

A. Transmission through a multiple δ-function potential

Consider the Schrödinger equation with the following potential:

$$V(x)=\sum \alpha_i \delta(x-L_i), \quad L_0=0. \tag{3}$$

The energy eigenfunctions have the form (for $x<0$ and $x>L_n$)

$$\psi(x)=\begin{cases} Ae^{ikx}+Be^{-ikx}, & x<0 \\ Ce^{ikx}+De^{-ikx}, & x>L_n . \end{cases} \tag{4}$$

The coefficients satisfy

$$\begin{pmatrix} A \\ B \end{pmatrix} = M \begin{pmatrix} C \\ D \end{pmatrix}, \tag{5}$$

$$M=\prod_{i=1}^{n}\left[\beta_i\begin{pmatrix} 1 & e^{-2ikL_i} \\ -e^{2ikL_i} & -1 \end{pmatrix}+I\right], \quad \beta_j=\frac{m\alpha_j}{ik}. \tag{6}$$

In the limit of small kinetic energy ($|\beta_i|\gg 1$), we can drop the I matrices, as long as $n<\beta_i$. It is then straightforward to prove by induction on n that

$$M=\prod_{1}^{n}\beta_i\begin{pmatrix} \prod_{i=2}^{n}(1-z_i) & \prod_{i=2}^{n}(z_i^{-1}-1) \\ -\prod_{i=2}^{n}(1-z_i) & -\prod_{i=2}^{n}(z_i^{-1}-1) \end{pmatrix}+O(1), \tag{7}$$

where $z_1=1$, $z_i=\exp[2ik(L_i-L_{i-1})]$ $(i=2\cdots n)$.

As usual, we examine the case of "stationary scattering." To get the (amplitude) transmission coefficient for probability current flowing from the left, t, we put $A=1$, $B=r$, $C=t$, $D=0$ in Eq. (4)

$$\psi(x)=\begin{cases} e^{ikx}+re^{-ikx} & x<0 \\ te^{ikx} & x>L_n , \end{cases} \tag{8}$$

and we see that $t=M_{11}^{-1}$, so

$$t=M_{11}^{-1}\approx\frac{\beta_1^{-1}\cdots\beta_n^{-1}}{\prod_{i=2}^{n}(1-z_i)}=\beta_1^{-1}\cdots\beta_n^{-1}\frac{\prod_{i=2}^{n}z_i^{-1/2}}{\prod_{i=2}^{n}(z_i^{-1/2}-z_i^{1/2})}$$

$$=\frac{\prod\beta_i^{-1}}{(-2i)^{n-1}}\frac{e^{-ikL_n}}{\prod_{i=2}^{n}\sin[k(L_i-L_{i-1})]}. \tag{9}$$

The stationary phase formula for the delay time τ_g

$$\tau_g\equiv\hbar\frac{\partial}{\partial E}\arg(t), \tag{10}$$

yields the value $\tau_g=-mL/\hbar k=-L/v(k)$ for the delay, which cancels the time for a free particle, and we get an overall zero time for tunneling. Since this is true for all k, it should be true for an arbitrary wave packet, *as long as the stationary phase approximation holds.* The condition for that is derived in the following section.

B. The condition for superluminal tunneling of a packet

Restated for wave packets, our results so far can be summarized as

$$\Psi(x,t) = \begin{cases} \int g(k)[e^{ikx}+r(k)e^{-ikx}]e^{-i\omega(k)t}dk, & x<0 \\ \int (-ik)C(k)g(k)e^{ik(x-L_n)}e^{-i\omega(k)t}dk, & x>L_n, \end{cases} \tag{11}$$

and

$$C(k)=\frac{\prod \beta_i^{-1}}{-ik(-2i)^{n-1}\prod_{i=2}^{n}\sin[k(L_i-L_{i-1})]}. \tag{12}$$

When Δk is sufficiently small, the diffusion can be ignored and $C(k)$ can be considered constant (as will be shown shortly). Then we can again separate out the time dependence of the wave function and the spatial part can be written as

$$\psi(x)=\begin{cases} \phi(x) & x<0 \\ C\phi'(x-L_n) & x>L_n, \end{cases} \tag{13}$$

where $\phi(x)$ in the two regions is related through analytic continuation.

If $\phi(x)=R(x)e^{iS(x)}$ where $R(x)$ is large and slowly varying in the region $|x-x_0|<\Delta x$ and $S(x)$ goes through a few cycles there, then the time of arrival distribution of the transmitted packet will be approximately that of the incoming one, shifted by $-L/\langle v\rangle$. Note also that this is also true for a mixed state that can be decomposed into various pure states with this property.

Let us now find the explicit condition for $C(k)$ to be approximately constant, for the case where $L_j=(j/n)L$, $\alpha_j=\alpha$, and as before, $n<|\beta|=m\alpha/k$. In this case, we have

$$C(k)=\frac{\left(\frac{ik}{m\alpha}\right)^n}{-2ik(2i\sin kL/n)^{n-1}}. \tag{14}$$

Using the fact that $x/\sin x=1+x^2/6+O(x^4)$, we get

$$C(k)=-\frac{1}{m\alpha}\left(\frac{n-1}{2Lm\alpha}\right)^{n-1}\left[1+\frac{1}{6}\left(\frac{kL}{\sqrt{n-1}}\right)^2 + O\left(\left(\frac{kl}{n-1}\right)^3\right)\right]. \tag{15}$$

Thus, $C(k)$ will be approximately constant if the spectrum of the wave packet is limited to k such that $|k|\approx\sqrt{n-1}/L$. In other words, $\Delta k\approx\sqrt{n-1}/L$, or $\Delta x\approx L/\sqrt{n-1}$. This means that the length of the barrier can be arbitrarily longer than the "length" of the tunneling packet as usually defined (standard deviation of the x coordinate), the penalty paid being an exponential suppression of the amplitude. In passing notice the function $F(k)$ in Eqs. (1) and (2) displays a similar behavior.

C. Calculation of the dwell time

It is interesting to compare the "group delay" (which is zero in the low k limit) with the dwell time. For the sake of simplicity, we shall deal with the case $n=2$. A direct calculation of the dwell time of the transmitted component can be made by calculating the transmission coefficient after adding a potential that is constant over the region between the δ spikes, and vanishing outside it. We get

$$t\simeq\beta^{-1}\frac{e^{-ikL}}{-2i\sin k'L},$$

where $k'=\sqrt{2m(E-V_0)/\hbar}$ and V_0 is the value of the potential between the δ's. Clearly, $\partial\arg(t)/\partial V_0=0$, and the (conditional) dwell time is zero as expected.

III. SUPERLUMINALITY AND ITS RELATION TO INTERFERENCE IN THE TAIL OF THE WAVE FUNCTION

The calculation of the transmission coefficient t can also be done in a way more suggestive of superoscillations. Let us explain this for the case of 2-δ-functions [the $n=2$ case in Eq. (3), "Fabry Perot interferometer"].

Suppose a quasimonochromatic wave packet with wave number k arrives at the first δ spike. The transmitted component is the same as the original wave, except for an attenuation and phase that are independent of k. At the second δ spike, the wave splits into a (approximately unattenuated) reflected wave and a transmitted one, which is apart from a k-independent multiplicative constant the same as the impinging wave. The reflected component is again reflected at the first δ, and arrives at the second δ with an additional phase of $2kL$, but with approximately the same amplitude as the original transmitted wave. In a similar manner, one gets additional transmitted waves with additional phases of $2nkL, n=2,3,\ldots$, and amplitudes that decrease very slowly. Thus we get the following formal sum for the resulting amplitude of the wave (up to a multiplicative constant):

$$\sum_n e^{ikx}(e^{ik2L})^n=\frac{e^{ikx}}{1-e^{2ikL}}=e^{ikx}\frac{e^{-ikL}}{-2i\sin kL}, \tag{16}$$

which is in agreement with our previous calculation. This is an example of superoscillations since a sum of positive wave vectors results in a negative one (or, equivalently, a sum of positive shifts that results in a negative shift) [18]. This is true in the following sense: for $|k|\ll 1/L$ the denominator of the right-hand side can be considered constant. However, in such a small interval the function does not really oscillate, so it really does not have a well-defined frequency. To really speak about superoscillations we need to have a large number of δ functions. The sum for the case $n>2$ factors into $n-1$ sums of the above form, in the low kinetic-energy limit, since to leading order in β^{-1}, the only contributions are from waves that tunnel through each δ only once, but may be reflected any number of times between consecutive deltas. We then reproduce the results of Sec. II B, where we had the weaker condition $k<\sqrt{n-1}/L$, allowing the function o complete many oscillations in the region. Note the similarity to the situation described by Eqs. (1) and (2).

A different calculation of the (conditional) dwell time than that of Sec. II C assumes a "clock" that is activated by the presence of the particle in the region $[0,L]$, i.e., a degree of freedom τ with an interaction Hamiltonian $H_{\mathrm{Int}} = \theta_{[0,L]}(x) p_\tau$, where p_τ is the canonical momentum conjugate to τ. If the initial state of the clock has small enough Δp_τ, one gets for the final phase of τ the same expression as Eq. (16), with x,k replaced by the clock's coordinates. This can be interpreted as the sum appropriate for a weak measurement of $\theta_{[0,L]}(x)$, as shown in the following section. Note that in this case, not only do the dwell and phase times coincide, but they are also described by the same mechanism.

The group delay in tunneling through a thick barrier follows from the fact that under the barrier, no phase accumulates, and the entire phase shift comes from the boundaries and is practically independent of the thickness. For cases where interference with a delayed wave takes place, a few authors [4–10] have suggested a different mechanism. In Chiao and Steinberg's words [3]: "If destructive interference is set up between part of the wave traveling unimpeded and part that has suffered a delay Δt due to multiple reflections, one has $\Psi_{out}(t) = \Psi_{in}(t) - \xi \Psi(t-\Delta t) \approx (1-\xi)\Psi_{in}(t) + \xi \Delta t d\Psi_{in}(t)/dt \approx (1-\xi)\Psi_{in}[t + \xi\Delta t/(1-\xi)]$, which is already a linear extrapolation into the future. In cases where the dispersion is sufficiently flat, as in a band-gap medium, the extrapolation is, in fact, surprisingly better than this first-order approximation. As was suggested by Steinberg [6] and recently discussed more rigourously by Lee and Lee [8] and Lee [10], this implies that even a simple Fabry-Perot interferometer exhibits superluminality when excited off resonance" [presumably, $\xi \ll 1/\Delta t \Psi'(t)$]. Another physical interpretation of the mode reshaping process is also suggested in Ref. [7]. We would like to explain this "better than first-order" approximation. Let us instead look at the momentum wave function. A spatial shift corresponds to a linear shift in this function. A positive spatial delay would correspond to a linear shift steeper than one, and the converse for a negative delay. In the Taylor expansion of the transmission coefficient for the momentum wave function, the zero term is insignificant, the second corresponds, as just explained, to the spatial shift, and the higher give the distortion. When many waves with large and evenly distributed shifts interfere, their sum is for a wide range of momenta, zero, and, in particular, momentum independent. In other words, the momentum wave function is flat for a wide band of frequencies. This corresponds to a much better than first-order approximation of the spatial wave function, as can be seem in the special case of the system of Sec. III of this paper.

IV. THE DWELL TIME AS A WEAK VALUE

We would like to calculate the expectation value of the time measured by a "clock" consisting of an auxiliary system that interacts weakly with our particle as long as it stays in a given region. Furthermore, we would like to restrict the calculation only to the subensemble of particles that ultimately end up on the right of the barrier. The simplest interaction is perhaps the one defined by the Hamiltonian,

$$H_{\mathrm{int}} = P_\tau X_{(0,L)},$$

where τ is the degree of freedom of the clock and P_τ is its conjugate momentum, and

$$X_{(0,L)}(x) = \begin{cases} 1 & \text{if } 0<x<L \\ 0 & \text{otherwise.} \end{cases}$$

This is the effective form, for example, of the potential, seen by a particle in an S_z eigenstate, in the Stern-Gerlach experiment [τ being the z coordinate, and $(0,L)$ the region of the magnetic field]. Assuming the clock to have at large negative times the expectation value 0, a perturbation calculation shows that following the interaction with the particle and the subsequent postselection of the particle state to be $|f\rangle$, the expectation value of τ at large positive times is given by the formula:

$$E(\tau, t\to\infty | i,f) = \frac{\int_{-\infty}^{\infty} dt \int_0^L dx \Psi_f^*(x,t)\Psi_i(x,t)}{\int_{\infty}^{\infty} dx \Psi_f^*(x,0)\Psi_i(x,0)} \tag{17}$$

Steinberg [11,12] has arrived at this formula under similar assumptions by a somewhat different line of reasoning. He introduced the term conditional (quantum) probability for the probability distribution of a system following postselection, and we use the notation of conditional expectation in the formula above, in the same spirit. As noted by Steinberg, the last equation is a special case of a weak value.

This formula is valid when τ and p_τ do not appear in additional terms in the full Hamiltonian, but the generalization is straightforward. To prove the formula, let us work in the interaction picture. Denote the initial state of τ by $|\phi_\tau(t)\rangle$ [and the corresponding wave function by $\phi(\tau,t)$], and the initial (preselected) and final (postselected) states of the tunneling particle by $|i(t)\rangle, |f(t)\rangle$ [$\Psi_i(x,t), \Psi_f(x,t)$], respectively. Using the interaction picture and expanding to first order in P_τ [23],

$$\begin{aligned} |\phi_\tau,t\rangle|i(t)\rangle &= T\exp\left(-\frac{i}{\hbar}\int_{-\infty}^{t} H_I(t')dt'\right)|\phi_\tau, t\to-\infty\rangle \\ &\quad \times |i(t\to-\infty)\rangle \simeq \left(1-\frac{i}{\hbar}\int_{-\infty}^{t} H_I(t')dt'\right) \\ &\quad \times |\phi_\tau, t\to-\infty\rangle|i(t\to-\infty)\rangle, \end{aligned} \tag{18}$$

where

$$H_I(t) = T\exp\left(\frac{i}{\hbar}\int_{-\infty}^{t} H_0(t')\right) H_{\mathrm{int}} \exp\left[-\frac{i}{\hbar}\int_{-\infty}^{t} H_0(t')\right].$$

After making the postselection of state $|f\rangle$ for the particle, the clock is left in the state given by the above expression, multiplied on the left by $\langle f(t\to-\infty)|/\langle f,-\infty|i,-\infty\rangle$. The expression we get after writing out the explicit form of H_{int} is

$$\phi(\tau,t\to+\infty)\simeq\exp(-i\langle X_{(0,L)}\rangle_W P_\tau/\hbar)\,\phi(\tau,t\to-\infty)$$
$$=\phi(\tau+\langle X_{(0,L)}\rangle_W,t\to-\infty), \qquad (19)$$

where

$$\langle X_{(0,L)}\rangle_W=\frac{\int_{-\infty}^{\infty}dt\int_0^L dx\Psi_f^*(x,t)\Psi_i(x,t)}{\int_{\infty}^{\infty}dx\Psi_f^*(x,0)\Psi_i(x,0)}, \qquad (20)$$

(the integral in the denominator is evaluated at $t=0$ merely for convenience—it is of course, time invariant). The wave functions in the integral in the numerator can be taken in the Schrödinger representation. And the expectation for τ at large positive times is shifted (from its initial value of 0) by this value, as claimed. In contrast to the familiar eigenvalue spectrum of a physical operator, its weak values can take any complex values [24].

ACKNOWLEDGMENTS

The research was supported in part by Grant No. 62/01-1 of the Israel Science Foundation, established by the Israel Academy of Sciences and Humanities, NSF Grant No. PHY-9971005, and ONR Grant No. N00014-00-0383.

[1] E.H. Hauge and J.A. Støvneng, Rev. Mod. Phys. **61**, 917 (1989).
[2] R. Landauer and Th. Martin, Rev. Mod. Phys. **66**, 217 (1994).
[3] R.Y. Chiao and A.M. Steinberg, Prog. Opt. **XXXVII**, 345 (1997).
[4] R.Y. Chiao, P.G. Kwiat, and A.M. Steinberg, Sci. Am. **269** (2), 52 (1993).
[5] A.M. Steinberg, J. Phys. I **4**, 1813 (1994).
[6] A.M. Steinberg, Recherche **281**, 46 (1995).
[7] Y.P. Wang and D.L. Zhang, Phys. Rev. A **52**, 2597 (1995).
[8] B. Lee and W. Lee, Superlattices Microstruct. **18**, 277 (l995).
[9] G. Diener, Phys. Lett. A **223**, 327 (1996).
[10] B. Lee, OSA Annual Meeting Abstracts, 1996 (unpublished).
[11] A.M. Steinberg, Phys. Rev. A **52**, 32 (1995).
[12] A.M. Steinberg, Phys. Rev. Lett. **74**, 2405 (1995).
[13] Y. Aharonov, A. Casher, D. Albert, and L. Vaidman Phys. Lett. A **124**, 199 (1987).
[14] Y. Aharonov, D. Albert and L. Vaidman, Phys. Rev. Lett. **60**, 1351 (1988).
[15] M.V. Berry, J. Phys. A **27**, L391 (1999); *Quantum Coherence and Reality*, edited by J. A. Anandan and J. Safko (World Scientific, Singapore, 1994).
[16] V.S. Olkhovsky, E. Recami, and G. Salesi, Europhys. Lett. **57**, 879 (2002); e-print quant-ph/0002022.
[17] We follow the terminology of Chiao and Steinberg [3]. The group delay is defined as $\tau_g\equiv\hbar(\partial/\partial E)\arg(t)$, where t is the transmission coefficient for tunneling through the region. Sometimes, however, the same name is used for the difference between this value and the time it would take the same packet to traverse an equal distance in free propagation (i.e., for the additional delay introduced by the barrier). Other names for it are: phase-Wigner- and stationary phase-time. In the following we refer to the forward conditional dwell time [11,12], as the dwell time.
[18] The sum in Eq. (16) actually diverges, the physical reason being that we have neglected the attenuation of the amplitude, in order to maintain consistency with the low kinetic-energy approximation we have used so far. For the case $n=2$ it is easy to evaluate Eq. (6) without resort to that approximation, and the resulting transmission amplitude is

$$t(k)=\frac{\beta^{-2}}{\left(1+\frac{2}{\beta}+\frac{1}{\beta^2}\right)-e^{2ikL}}. \qquad (21)$$

Similarly, the sum on the left of Eq. (16) should be replaced by the exact one:

$$(1+\beta)^{-2}\sum_{j=0}^{\infty}\left[\left(\frac{\beta}{1+\beta}\right)^2 e^{2ikL}\right]^j=\frac{\beta^{-2}}{\left(1+\frac{2}{\beta}+\frac{1}{\beta^2}\right)-e^{2ikL}}. \qquad (22)$$

.
[19] Y. Aharonov, P.G. Bergman, and J.L. Lebowitz Phys. Rev. **134**, B1410 (1964).
[20] Y. Aharonov J. Anandan, S. Popescu, and L. Vaidman, Phys. Rev. Lett. **64**, 2965 (1990).
[21] Y. Aharonov and L. Vaidman, J. Phys. A **24**, 2315 (1991).
[22] B. Reznik and Y. Aharonov, Phys. Rev. A **52**, 2538 (1995).
[23] We can satisfy the condition of small perturbation by choosing an initial clock wave function concentrated about $P_\tau=0$ with small enough uncertainty.
[24] To see this, let us develop the initial and final states of the particle in terms of the eigenfunctions of the operator to be measured, A:

$$|i\rangle=\sum_k \alpha_k|a_k\rangle, \quad |f\rangle=\sum_k \beta k|a_k\rangle, \quad (A|a_k\rangle=a_k|a_k\rangle). \qquad (23)$$

Then, $A_W=\langle f|A|i\rangle/\langle f|i\rangle=\Sigma\beta_k^*\alpha_k a_k/\Sigma\beta_k^*\alpha_k$. Suppose A is nontrivial, i.e., it has more than one eigenvalue. Assume $k=1,2$ correspond to two of these, and take $\beta_1=\beta_2=1/\sqrt{2}$, then the two equations, $A_W=\alpha_1 a_1+\alpha_2 a_2/\alpha_1+\alpha_2=z$ and $|\alpha_1|^2+|\alpha_2|^2=1$, are three real equations in four unknowns. They can be solved for any value of z, as can be verified easily.