Renormalization of configuration space into phase space (Can nonrealizable Hamiltonians be realized?)

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We show that gauge-type coupling can be the basis for new renormalization phenomena in which ordinary configuration-space coordinates become effectively canonically conjugate. This is demonstrated first for the case of a particle in a very strong magnetic field, where some of the experimental consequences can be surprising. Another example is presented which suggests a possible new approach to the quark-confinement problem.

This paper introduces an extension in an entirely new direction of previous work[1] on the use of simple models to describe renormalization phenomena. The original discussion was tied directly to mass renormalization in nonrelativistic electrodynamics. It was shown that the introduction of appropriately time-averaged observables results in the renormalization of the c-number commutation relation between position and velocity from $[x,\dot{x}]=i\hbar/m_0$ to $[x,\dot{x}]=i\hbar/m$, where m_0 and m are, respectively, the "bare" and "renormalized" masses. However, commutation relations between canonical observables are unaffected. In the following, we show that examples exist which lead to the renormalization of commutation relations between canonical observables with attendant unsuspected and interesting physical consequences.

Our basic Hamiltonian is $H=H_0+V(x,y)$, where H_0 describes the two-dimensional motion of a charged particle interacting with a constant, uniform magnetic field in the z direction, i. e.,

$$H_0=\tfrac{1}{2}\Omega[(p_x'-\tfrac{1}{2}y')^2+(p_y'+\tfrac{1}{2}x')^2]. \tag{1}$$

Here $\Omega=eB/mc$ is the Larmor frequency, and, for convenience, we have made the scale transformation $x'=x\sqrt{m\Omega}$, $p_x'=p_x(m\Omega)^{-1/2}$, $y'=y\sqrt{m\Omega}$, $p_y'=p_y\times(m\Omega)^{-1/2}$. Now introduce the splitting $x'=\bar{x}+\delta x$, $y'=\bar{y}+\delta y$, with

$$\begin{aligned}\bar{x}&=\tfrac{1}{2}x'-p_y', \quad \delta x=\tfrac{1}{2}x'+p_y'\\ \bar{y}&=\tfrac{1}{2}y'+p_x', \quad \delta y=\tfrac{1}{2}y'-p_x'.\end{aligned} \tag{2}$$

We then have $[\bar{x},\bar{y}]=i\hbar$, so that if $\bar{x}$ and $\bar{y}$ are indeed the relevant coordinates, we have a situation wherein a pair of canonical variables have had their commutator "renormalized" from zero to $i\hbar$. (It is obvious in this instance that, if $V=0$, δx and δy oscillate with the frequency Ω, while $\bar{x}$ and $\bar{y}$, which are the classical coordinates of the orbit center,[2] are constants of the motion.)

In order to justify the term "relevant" for $\bar{x}$ and $\bar{y}$, we start with a simple choice for V. Consider the case where

$$V(x',y')=\tfrac{1}{2}\alpha(x'^2+y'^2). \tag{3}$$

The Hamiltonian H can be written as

$$\begin{aligned}H=&\tfrac{1}{2}(\Omega+\alpha)[(\delta x)^2+(\delta y)^2]+\tfrac{1}{2}\alpha(\bar{x}^2+\bar{y}^2)\\&+\alpha(\bar{x}\delta x+\bar{y}\delta y),\end{aligned} \tag{4}$$

where there is no ambiguity in the last term because $\bar{x}$ and $\bar{y}$ commute with δx and δy. The first and second terms on the right in Eq. (4) represent two independent oscillators with frequencies $\Omega+\alpha$ and α. Their eigenstates $|N\rangle|n\rangle$ [with associated energies $(N+\frac{1}{2})\hbar(\Omega+\alpha)+(n+\frac{1}{2})\hbar\alpha$] provide a basis system in Hilbert space. Suppose now that Ω is so large (with α fixed) that any finite excitation of the system necessarily involves $N=0$. Then the third term on the right in Eq. (4), looked upon as a perturbation, can be disregarded if we neglect effects of order α/Ω, since its only nonvanishing matrix elements are between different values of N. Thus the finite excitations of the system are those corresponding to a one-dimensional oscillator with a Hamiltonian of the form $\frac{1}{2}\alpha(p^2+q^2)$.

The effect of the high-frequency mode associated with H_0 is to "renormalize" the potential $V(x',y')$ from an ordinary interaction in a two-dimensional configuration space to a one-dimensional Hamiltonian in phase space. That this feature is quite general becomes more apparent from the following choice for V:

$$V(x',y')=\tfrac{1}{2}\alpha y'^2+\lambda\cos\beta x'. \tag{5}$$

The Hamiltonian $H=H_0+V$ is then

$$H=\tfrac{1}{2}\Omega[(\delta x)^2+(\delta y)^2]+\tfrac{1}{2}\alpha(\delta y)^2+\tfrac{1}{2}\alpha\bar{y}^2 + \alpha\bar{y}\,\delta y + \lambda\cos\beta(\bar{x}+\delta x), \quad (6)$$

where the second term on the right causes a negligible shift in H_0, and the fourth term may also be neglected as per the discussion above. The last term may be analyzed by writing

$$e^{i\beta(\bar{x}+\delta x)} = e^{i\beta\bar{x}}e^{i\beta\delta x} = e^{i\beta\bar{x}}[e^{i\beta\delta x} - \langle e^{i\beta\delta x}\rangle] + e^{i\beta\bar{x}}\langle e^{i\beta\delta x}\rangle, \quad (7)$$

where the expectation value is taken in the state $N=0$ so that, in fact, $\langle e^{i\beta\delta x}\rangle = e^{-\hbar\beta^2/4}$. The term in square brackets in Eq. (7) obviously has zero expectation value in the $N=0$ state; it is also a bounded operator so that, regarded as a perturbation, it gives corrections of order $1/\Omega$, and can thus also be neglected. The Hamiltonian (6) then effectively becomes

$$H=H_0+\tfrac{1}{2}\alpha\bar{y}^2+\lambda'\cos\beta\bar{x}, \quad (8)$$

with $\lambda' = \lambda e^{-\hbar\beta^2/4}$.[3] Thus, once again, the high-frequency mode represented by H_0 plays the catalytic role of renormalizing the potential $V(x',y')$ into a one-dimensional Hamiltonian, in this case, of the Bloch type. The finite excitations of the system will therefore have the familiar band structure spectrum.

Up until now we have restricted ourselves to potential functions which renormalize to Hamiltonians quadratic in the "momentum," i. e., standard type Hamiltonians. It should be apparent from the above discussion that this limitation is by no means necessary. Indeed, the way is now open to consideration of a host of new phenomena corresponding to more general Hamiltonians. Of particular interest is the case where $V(x',y')$ is periodic in both variables, since an operator of the form $\cos(\lambda p/\hbar)$ generates finite translations and is truly nonlocal.[4] It is to be emphasized that just as a Hamiltonian for a particle with "bare" mass m_0 can underlie the dynamics of a particle of renormalized mass m, so can a "respectable" Hamiltonian, quadratic in the momenta, serve to generate dynamics associated with Hamiltonians of essentially arbitrary form.

A specific example of the consequences of the effective nonlocality is the following: Imagine a slab of nonmagnetic material placed perpendicular to the magnetic field, i. e., with its surface in the xy plane. Suppose also, for simplicity, that the xy dependence of the potential due to the slab is of the form $V=V_0\cos\beta x'$. Consider now the backscattering of an electron which impinges on the slab. The incoming electron is assumed to have sharp momentum in the z direction with an associated kinetic energy less than $\hbar\Omega$; it is also maximally focused in the xy plane so that its wave function, on the basis in which $\bar{y}$ and δy are diagonal, has the form[5] $\exp[ikz-\bar{y}^2/2\hbar-(\delta y)^2/2\hbar]$. The potential V is renormalized to $V_0\lambda'\cos\beta\bar{x}$ in accordance with the discussion above, so that to first order in the potential, the backscattered electron has a wave function of the form

$$\exp[-ikz-(\delta y)^2/2\hbar]\{\exp[-(1/2\hbar)(\bar{y}+\hbar\beta)^2] + \exp[-(1/2\hbar)(\bar{y}-\hbar\beta)^2]\}.$$

For sufficiently large β,[6] this scattered wave will thus consist of two transversally displaced, maximally focused "pencils."

Other examples of the consequences of the nonlocality will be considered elsewhere. Straightforward order-of-magnitude considerations indicate that the lowest temperatures and strongest magnetic fields experimentally available will make the observations of such effects conceivable, though very difficult. One can, of course, seek to observe the effects indirectly in naturally occurring magnetic fields of the requisite strength, for example, in neutron stars.[7]

Finally, we remark that a straightforward extension of the magnetic-field model suggests a new type of binding with some possibly far reaching conceptual consequences. The idea involved is, briefly, the following: The Hamiltonian H_0 of Eq. (1) is replaced by the three-dimensional, two-particle Hamiltonian[8]

$$H_0=\frac{1}{2m_0}[(\vec{p}_1+\lambda\vec{r}_2)^2+(\vec{p}_2-\lambda\vec{r}_1)^2]. \quad (9)$$

Solution of Hamilton's equations for this Hamiltonian alone gives for the coordinates,

$$\vec{r}_1(t)=\vec{r}_1(0)+\frac{\vec{V}_1(0)}{\Omega}\sin\Omega t-\frac{\vec{V}_2(0)}{\Omega}(\cos\Omega t-1),$$
$$\vec{r}_2(t)=\vec{r}_2(0)+\frac{\vec{V}_2(0)}{\Omega}\sin\Omega t+\frac{\vec{V}_1(0)}{\Omega}(\cos\Omega t-1), \quad (10)$$

with $\Omega=2\lambda/m_0$. Once again, if Ω is sufficiently large, the relevant coordinates are those of Eq. (10) time averaged over an interval $\gg 1/\Omega$ so that the oscillatory terms are removed. In component form, looking at the x component, say, these satisfy $[\bar{x}_2,\bar{x}_1]=ih/2\lambda$ or

$$\left[(\bar{x}_2-\bar{x}_1),\left(\frac{\bar{x}_1+\bar{x}_2}{2}\right)\right]=\frac{ih}{2\lambda}. \quad (11)$$

Thus the relative coordinates become conjugate to the center-of-mass coordinates.

If, now, we augment the Hamiltonian (9) by a potential function $V(|\vec{r}_2-\vec{r}_1|)$, the implication of Eq.

(11) and our above discussion is that the renormalized V becomes a function of the momentum conjugate to the center of mass, and thus acts as a free-particle Hamiltonian for the center-of-mass motion. The choice $V(|\vec{r}_2 - \vec{r}_1|) = \frac{1}{2}\alpha|\vec{r}_2 - \vec{r}_1|^2$ then obviously gives the standard nonrelativistic free-particle Hamiltonian with (center-of-mass) velocity having the components $\alpha(\bar{x}_2 - \bar{x}_1)$, etc.

What we have here, in effect, is a model wherein the "bare" motion, i.e., that which is seen in observations over very short time intervals, is that of a two-particle system with the coupling as described, while the "renormalized" motion represents a single free particle whose Hamiltonian depends on the original bare potential $V(|\vec{r}_2 - \vec{r}_1|)$.

Since we have discussed the model in nonrelativistic terms, the question of Galilean invariance arises. This invariance is ensured by the transformation property of the vector potential appearing in H_0, namely, $\vec{A} = -\lambda(\vec{r}_2 - \vec{r}_1)$.[8] A Galilean boost therefore introduces a fourth component to the potential, which results in an additive term in H proportional to the boost velocity and to $(\vec{r}_2 - \vec{r})$. This, in turn, results in the required shift in the center-of-mass velocity.[9]

The speculative extension of this model to quarks naturally suggests itself. Three basic problems are involved: (i) the extension to systems consisting of three or more constituent particles, (ii) the interaction between such systems, and (iii) the relativistic generalization of the whole picture. Preliminary investigation indicates that all three problems can be handled. Of particular interest is the fact, which should be obvious from the above discussion, that $V(|\vec{r}_2 - \vec{r}_1|)$ must be asymptotically linear in the particle separation. Thus this model, if meaningful, provides a new justification for a linear confinement potential.

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[1]Y. Aharonov, E. Lerner, and T. Banks, Lett. Nuovo Cimento 13, 305 (1975).

[2]M. H. Johnson and B. A. Lippmann, Phys. Rev. 76, 828 (1949).

[3]The parameter β can be chosen to make the exponential of the order of unity without vitiating the general argument. Physically, this exponential term is related to a tunneling phenomenon, as will be discussed in a future publication.

[4]Y. Aharonov, H. Pendleton, and A. Petersen, Int. J. Theor. Phys. 2, 213 (1969).

[5]The transverse part of this wave function is a ground state of the Hamiltonian of Eq. (1) for which the center of the orbit is maximally localized in both x and y (see Ref. 2).

[6]The choice of β is rather delicate here if one thinks in practical terms. It must satisfy the criterion of making the transverse displacement of the electron observable without resulting in λ' of Eq. (8) being so small that first-order scattering will not predominate.

[7]M. Ruderman, Phys. Rev. Lett. 27, 1306 (1971).

[8]Note that a simple gauge transformation brings H_0 to the form $(2m_0)^{-1}[(\vec{p}_1 - \vec{A})^2 + (\vec{p}_2 - \vec{A})^2]$, with $\vec{A} = -\lambda(\vec{r}_2 - \vec{r}_1)$.

[9]There is still the subtle point that time-independent vector potentials single out as a preferred reference frame that one in which the fourth component of the potential vanishes. The significance of this will be discussed in a future publication.